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The Triangle-Free Graphs with Rank 6

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Abstract The rank of a graph G is defined to be the rank of its adjacency matrix $A(G)$. In this paper we characterize all connected triangle-free graphs with rank 6.

Keywords graph; rank; nullity.

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1. Introduction

Throughout this paper we consider simple graphs. The *rank* of a graph $G=(V(G),E(G))$, denoted by $r(G)$, is defined to be the rank of its adjacency matrix; and the *nullity* of G, denoted by $\eta(G)$, is defined to be the multiplicity of zero eigenvalues of its adjacency matrix. It is easy to see that $r(G)+\eta(G)=|V(G)|$. The graph G is called *singular* or *nonsingular* if $\eta(G)>0$ or $\eta(G)=0$.

In chemistry, a conjugated hydrocarbon molecule can be modeled by its molecular graph. It was known in [1] or [2], if the molecule represented by a graph G is chemically stable, then G is nonsingular. In 1957, Collatz and Sinogowitz [3] posed the problem of characterizing all singular graphs. The problem has received a lot of attention, although it is very hard and only some particular results are known [4–7].

We review some known results related to this topic. In [8] and [9] the smallest rank among n-vertex trees in which no vertex has degree greater than a fixed value was determined, and the corresponding trees were constructed. The singular line graphs of trees were described in [10] and [11]. The rank set of bipartite graphs of fixed order and the bipartite graphs of rank 4 were determined in [12]. It was shown in [13] that all connected graphs with rank 2 (respectively, 3) are complete bipartite graphs (respectively, complete tripartite graphs). In [14] and [15], the authors proved that each connected graph with rank 4 (or 5) is obtained from one of the 8 (or 24) reduced graphs by multiplication of vertices.

Now a natural problem is left open: *To characterize graphs with rank* 6. Because a graph of rank 6 can be obtained by multiplication of vertices from its reduced form, the problem

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is equivalent to characterize reduced graphs with rank 6. We also note that regular bipartite graphs with rank 6 has been characterized in [12]. In the present paper, we focus our attention on connected reduced triangle-free graphs with rank 6, and characterize all such graphs.

2. Preliminaries

We first introduce two graph operations: *multiplication* and *reduction* of vertices. Let G be a graph on vertices $v_1, v_2, \ldots, v_n$, and let $m = (m_1, m_2, \ldots, m_n)$ be a list of positive integers. Denote by $G \circ m$ the graph obtained from G by replacing each vertex v_i of G with an independent set of m_i vertices $v_i^1, v_i^2, \ldots, v_i^{m_i}$ for $i = 1, 2, \ldots, n$, and joining v_i^s with v_j^t for $i = 1, 2 \ldots, m_i$ and $j = 1, 2, \ldots, m_j$ if and only if v_i and v_j are adjacent in G for $i, j = 1, 2, \ldots, n$. The resulting graph $G \circ m$ is said to be obtained from G by *multiplication of vertices* [14, 15].

Define a relation $\approx$ in $V(G)$ in the way that $x \approx y$ if and only if $N_G(x) = N_G(y)$, where $N_G(x)$ denotes the neighborhood of a vertex x in G. Obviously, the relation is an equivalence one, and each equivalence class $\bar{v} = \{x : N_G(x) = N_G(v)\}$ is an independent set. Now construct a new graph $R(G)$ obtained from G by taking each equivalence class $\bar{v}$ as a vertex and joining $\bar{v}$ with $\bar{u}$ if and only if v, u are adjacent in G. The graph $R(G)$ is called to be obtained from G by *reduction of vertices*, and is a *reduced form* of G. One can see the above two operations are inverse to each other, and preserve the rank of graphs [12].

A graph is called *reduced* if itself is a reduced form, i.e., the neighborhoods of distinct vertices are distinct. Denote $H \subseteq G$ if H is a subgraph of G, and $H \lhd G$ if H is an induced subgraph of G. For $H \subseteq G$ and $v \in V(G)$, denote by $N_H(v)$ the neighborhood of v in H. If two vertices u and v are adjacent in G, then we write $u \sim v$; otherwise write $u \nsim v$. The distance of two vertices u, v in G is denoted as $\mathrm{dist}_G(u, v)$. For $w \in V(G) \backslash V(H)$ and $H \lhd G$, the distance between w and H is denoted and defined by $\mathrm{dist}_G(w, H) = \min_{v \in V(H)}\{\mathrm{dist}_G(w, v)\}$. For later use we now introduce some basic results.

Lemma 2.1 ([1]) *Let G be a tree. Then $r(G) = 2\mu(G)$, where $\mu(G)$ is the matching number of G.*

Lemma 2.2 ([1]) *Let G be a graph containing a pendant vertex, and let H be the induced subgraph of G by deleting the pendant vertex and the vertices adjacent to it. Then $\eta(G) = \eta(H)$, or equivalently, $r(G) = r(H) + 2$.*

Lemma 2.3 ([12]) *Let B be a bipartite graph of order n. Then $\eta(B) = n - 2k$ for some $k \in \{0, 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$.*

Lemma 2.4 ([16]) *Let H be an induced subgraph of a reduced graph G for which $r(H) = r(G)$. If $u, v \in V(G) \backslash V(H)$ are not adjacent in G, then $N_H(u) \neq N_H(v)$.*

Lemma 2.5 ([16]) *Let H be an induced subgraph of a reduced graph G for which $r(H) = r(G)$, and let v be a vertex not in H. Then $N_H(v) \neq N_H(u)$ for any $u \in V(H)$.*

Lemma 2.6 ([16]) *Let H be a proper induced subgraph of a connected graph G for which $r(H) \geq r(G) - 1$. Then $dist_G(v, H) = 1$ for each vertex $v \in V(G) \backslash V(H)$.*

Lemma 2.7 *Let H be a nonsingular induced subgraph of a reduced graph G for which $r(H) = r(G)$. Assume G has exactly two distinct vertices u, v outside H. Let $\tilde{G}$ be obtained from G by adding a new edge uv if $uv \notin E(G)$ or deleting the edge uv otherwise. Then $r(\tilde{G}) = r(G) + 2$.*

Proof The adjacency matrix of G and $\tilde{G}$ can be written as:

$$A(G) = \begin{pmatrix} 0 & \theta & \alpha \\ \theta & 0 & \beta \\ \alpha^T & \beta^T & A(H) \end{pmatrix}, \ A(\tilde{G}) = \begin{pmatrix} 0 & 1-\theta & \alpha \\ 1-\theta & 0 & \beta \\ \alpha^T & \beta^T & A(H) \end{pmatrix},$$

where the first two rows of both matrices correspond to u, v, respectively, θ equals 1 or 0 if $uv \in E(G)$ or not, $A(H)$ is the adjacency matrix of H. Let

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -A(H)^{-1}\alpha^T & -A(H)^{-1}\beta^T & I_k \end{pmatrix},$$

where k is the order of H. Then

$$Q^T A(G) Q = \begin{pmatrix} -\alpha B^{-1}\alpha^T & \theta - \alpha B^{-1}\beta^T & 0 \\ \theta - \beta B^{-1}\alpha^T & -\beta B^{-1}\beta^T & 0 \\ 0 & 0 & A(H) \end{pmatrix},$$

and

$$Q^T A(\tilde{G}) Q = \begin{pmatrix} -\alpha B^{-1}\alpha^T & 1-\theta - \alpha B^{-1}\beta^T & 0 \\ 1-\theta - \beta B^{-1}\alpha^T & -\beta B^{-1}\beta^T & 0 \\ 0 & 0 & A(H) \end{pmatrix}.$$

Since $r(G) = r(H)$, the left-upper submatrix of order 2 of $Q^T A(G) Q$ is zero. So

$$Q^T A(\tilde{G}) Q = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & A(H) \end{pmatrix},$$

whose rank is $r(H) + 2$. $\square$

Corollary 2.8 *Let H be a nonsingular induced subgraph of a reduced graph G for which $r(H) = r(G)$. Let $\bar{G}$ be obtained from G by adding some edges uv within $V(G) \backslash V(H)$ if $uv \notin E(G)$ or deleting some edges uv within $V(G) \backslash V(H)$. Then $r(\bar{G}) \geq r(G) + 2$.*

Proof Let u, v be two specified vertices in $V(G) \backslash V(H)$ such that $uv \in E(G)$ but $uv \notin E(\bar{G})$ or $uv \notin E(G)$ but $uv \in E(\bar{G})$. Taking the subgraph G' of G and $\bar{G}'$ of $\bar{G}$ both induced by $V(H) \cup \{u, v\}$, we have $r(\bar{G}') \geq r(G') + 2$ by Lemma 2.7. The result follows as $r(\bar{G}) \geq r(\bar{G}')$ and $r(G') = r(G)$. $\square$

3. Characterizing reduced triangle-free graphs with rank 6

In this section we first give a characterization of the nonsingular triangle-free graphs of order 6, i.e., Theorem 3.1[1]. To get the rank of the graphs listed in Figures 2–8, one can use Lemma 2.2 or MATHEMATICA if it is necessary. Let $\mathbf{P}_k$ denote the path of order k, and let $\mathbf{C}_k$ denote the cycle of order k (to avoid the confusion of C_k used in Figure 5). The disjoint union of two graphs G and H is written as $G \cup H$. Denote by kH the disjoint union of k copies of H.

Theorem 3.1 *If G is a nonsingular triangle-free graph of order* 6, *then G is one graph in Figure 1.*

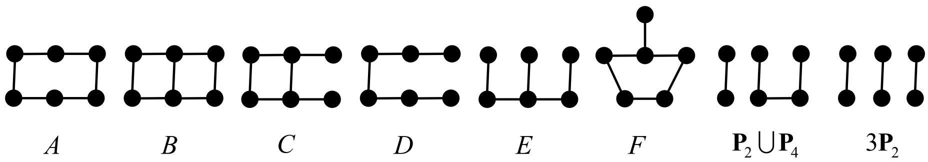

Figure 1 Eight nonsingular graphs $A, B, C, D, E, F, \mathbf{P}_2 \cup \mathbf{P}_4, 3\mathbf{P}_2$

Proof Denote by $g(G)$ the *girth* of a graph G (i.e., the shortest length of cycles in G). Suppose that G is a nonsingular triangle-free graph of order 6. Then either $g(G) = 4, 5, 6$, or G is a tree. If $g(G) = 6$, clearly $G = C_6$.

Case 1 G is disconnected. Observe that each component of G is nonsingular and G must be bipartite. By Lemma 2.3, the rank of each component of G must be 2 or 4. As the connected nonsingular bipartite graphs of order 2 must be $\mathbf{P}_2$, and connected nonsingular bipartite graphs of order 4 must be $\mathbf{P}_4$ (see [4] or [14] for details), G is $3\mathbf{P}_2$ or $\mathbf{P}_4 \cup \mathbf{P}_2$.

Case 2 G is connected and $g(G) = 5$. Let v be the unique vertex of G outside $\mathbf{C}_5$. If $|N_{C_5}(v)| \geq 2$, then $g(G) \leq 4$. Thus $|N_{\mathbf{C}_5}(v)| = 1$ and $G = F$.

Case 3 G is connected and $g(G) = 4$. Note that $r(C_4) = 2$, $r(G) = 6$, and deleting a vertex from G reduces $r(G)$ at most 2. There exists a graph $H \lhd G$ such that $|V(H)| = 5$ and $r(H) = 4$. As shown in [14], H is obtained from $\mathbf{C}_4$ by attaching a pendant vertex. For the reconstruction of G, one needs to add a vertex together with some edges. It is easy to check that G is the graph B or C.

Case 4 G is a tree. By Lemma 2.1, $\mu(G) = 3$. Thus $G \supseteq 3\mathbf{P}_2$. Since a tree of order 6 must contain 5 edges, we have $|E(G \backslash 3\mathbf{P}_2)| = 2$. One can easily see that G is the graph D or E. $\square$

Let G be a connected reduced triangle-free graph with rank 6. We know that G contains a nonsingular induced subgraph of order 6, which is one listed in Figure 1 by Theorem 3.1. In order to reconstruct G, we are left to consider how to add vertices to the graphs in Figure 1. The main result of this paper is the following Theorem 3.2. It is easy to check each graph in Figure 2 has rank 6. So the sufficiency of Theorem 3.2 follows. For the necessity we first discuss the case of G being bipartite in Section 3.1, and then the case of G being non-bipartite and triangle-free

[1] The result can be verified using SAGE by the command:
`GL=[G for G in list(graphs(6)) if G.is_triangle_free() and det(G.am())=0]`, where GL is the list containing all nonsingular triangle-free graphs of order 6.

in Section 3.2. The proof will be given at the end of this section before we show some lemmas.

Theorem 3.2 *Let G be a connected reduced triangle-free graph. Then $r(G) = 6$ if and only if there exist two graphs H_1 and H_2 such that $H_1 \lhd G \lhd H_2$, where H_1 is one graph in Figure 1 and H_2 is one graph in Figure 2.*

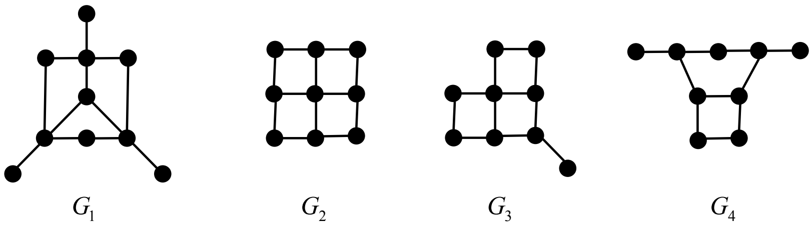

Figure 2 Four reduced graphs with rank 6: G_1, G_2, G_3, G_4

For $H \lhd G$, Denote by $H(k)$ the set of vertices $v \in V(G)\backslash V(H)$ such that $|N_H(v)| = k$.

3.1. Reduced bipartite graphs of rank 6

Lemma 3.3 *Let G be a connected reduced bipartite graph with rank* 6.

(1) If G contains $H_1 \in \{A, D, E\}$ as a subgraph, then $|N_{H_1}(v)| \le 3$ for each $v \in V(G)\backslash V(H_1)$;

(2) If G contains $H_2 \in \{B, C\}$ as a subgraph, then $|N_{H_2}(v)| \le 2$ for each $v \in V(G)\backslash V(H_2)$.

Proof If $G \supseteq H_1$, suppose that $v \in V(G)\backslash V(H_1)$ and $N_{H_1}(v) = \{v_1, v_2, \ldots, v_k\} \neq \emptyset$. Then $\mathrm{dist}_{H_1}(v_i, v_j)$ is even; otherwise G is not bipartite. Note that there exist at most three vertices whose pairwise distance is even in H_1, which implies that $k \le 3$.

If $G \supseteq H_2$, then $|N_{H_2}(v)| \le 3$ follows from (1). If $|N_{H_2}(v)| = 3$ for some $v \in V(G)\backslash V(H_2)$, then $N_{H_2}(v) = N_{H_2}(u)$, where u is a the vertex with degree 3 in H_2, thus G is not reduced by Lemma 2.5, implying that $|N_{H_2}(v)| \le 2$ for each $v \in V(G)\backslash V(H_2)$. $\square$

The rest of this section follows from a similar process, by which we obtain the connected reduced bipartite graphs with rank 6 from a nonsingular graph, say H with rank 6, by adding some new vertices, say $v_1, v_2 \ldots, v_k$ to it. By Lemmas 2.4–2.6, $|N_H(v_i)| \ge 1$; $N_H(v_i) \neq N_H(u)$ for any $u \in V(H)$ and $1 \le i \le k$; and $N_H(v_i) \neq N_H(v_j)$ for $1 \le i < j \le k$ if $v_i \not\sim v_j$.

Lemma 3.4 *Let G be a connected reduced bipartite graph with rank* 6, *which contains A (or* $\mathbf{C}_6$*) in Figure 1 as an induced subgraph. Then G is an induced subgraph of G_1 in Figure 2.*

Proof By Lemmas 2.6 and 3.3, $1 \le |N_A(v)| \le 3$ for $v \in V(G)\backslash V(A)$. If $A(2) \neq \emptyset$, let $v \in A(2)$ and v_1, v_2 be its two neighbors in A. If $\mathrm{dist}_A(v_1, v_2)$ is odd, then G contains an odd cycle. If $\mathrm{dist}_A(v_1, v_2) = 2$, then G is not reduced by Lemma 2.5. So $A(2) = \emptyset$. We divide the remaining proof into some cases:

Case 1 $A(3) = \emptyset$. If $|A(1)| = 1$, then $G \lhd G_1$ of Figure 2 obviously. Now we assume $|A(1)| \ge 2$. Suppose $v_1, v_2, \ldots, v_k \in A(1)$ and $v_1', v_2', \ldots, v_k'$ are their unique neighbors in A, respectively.

Observing the graph A_1, A_2 in Figure 3 both have rank 8, so $\mathrm{dist}_A(v'_i, v'_j) \neq 1$ for $i \neq j$, which implies that $\mathrm{dist}_A(v'_{i_1}, v'_{j_1}) = 2$ and $\mathrm{dist}_A(v'_{i_2}, v'_{j_2}) = 3$ cannot hold at the same time for some $v'_{i_1}, v'_{i_2}, v'_{j_1}, v'_{j_2}$. If $\mathrm{dist}_A(v'_i, v'_j) = 2$ for some $i \neq j$, then $|A(1)| \leq 3$. Since $\mathbf{C}_6 \lhd A_3 \lhd G_1$, we have $r(A_3) = 6$, where A_3 is listed in Figure 3. It follows that $v_i \nsim v_j$ in G; otherwise $r(G) \geq 8$ by Lemma 2.7. Hence $G \lhd A_3 \lhd G_1$. If $\mathrm{dist}_A(u'_i, v'_j) = 3$, then obviously $|A(1)| = 2$. However, one can check that $r(A_4) = r(A_5) = 8$.

Case 2 $A(1) = \emptyset$. Suppose $u, v \in A(3)$, $N_A(u) = \{u_1, u_2, u_3\}$ and $N_A(v) = \{v_1, v_2, v_3\}$. Then $\mathrm{dist}_A(u_i, u_j)$ and $\mathrm{dist}_A(v_i, v_j)$ must be even; otherwise G is not bipartite. So, if $N_A(v) \cap N_A(u) \neq \emptyset$, then $N_A(u) = N_A(v)$, which implies $|A(3)| \leq 2$. If $|A(3)| = 1$, then $G \lhd G_1$ of Figure 2. If $|A(3)| = 2$, let $u, v \in A(3)$. If $N_A(u) = N_A(v)$, then $u \sim v$ as G is reduced by Lemma 2.4; but in this case G would have triangles. So $N_A(u) \cap N_A(v) = \emptyset$. However, $r(A_6) = r(A_7) = 8$.

Case 3 $A(1) \neq \emptyset$ and $A(3) \neq \emptyset$. Assume $u \in A(3)$ and $v \in A(1)$. Noting that $r(A_8) = r(A_9) = 8$ while $r(G_1) = 6$, we have $\mathrm{dist}_G(u, v) = 2$. Then $G \lhd G_1$. $\square$

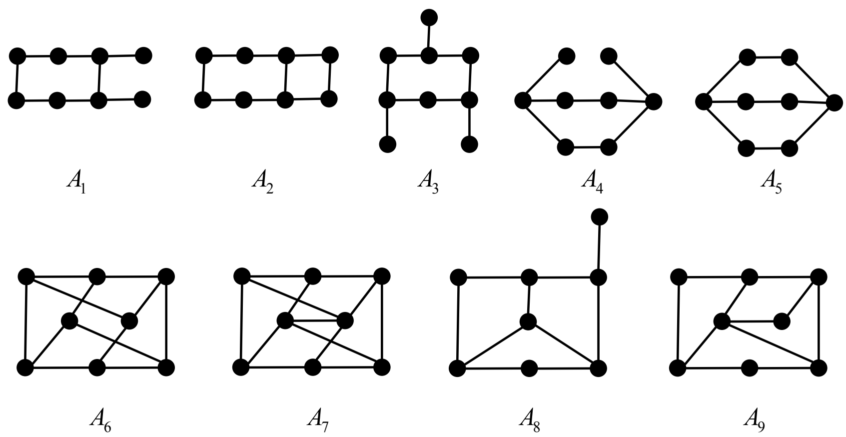

Figure 3 Illustration for Lemma 3.4, where $r(A_i) = 8$ for $i \in \{1, 2, 4, 5, 6, 7, 8\}$ and $r(A_3) = 6$

Lemma 3.5 *Let G be a connected reduced bipartite graph of rank 6, which contains B of Figure 1 as an induced subgraph. Then G is an induced subgraph of G_1, G_2 or G_3.*

Proof By Lemmas 2.6 and 3.3, $1 \leq |N_B(v)| \leq 2$ for $v \in V(G) \backslash V(B)$.

Case 1 $B(2) \neq \emptyset$. Let $v \in B(2)$ and let v_1 and v_2 be two neighbors of v in B. As G is bipartite, $\mathrm{dist}_B(v_1, v_2)$ should be even, which implies $\mathrm{dist}_B(v_1, v_2) = 2$. As G is reduced, $|N_B(v_1)| = |N_B(v_2)| = 2$ and $A \lhd G$. By Lemma 3.4, G is an induced subgraph of G_1.

Case 2 $B(2) = \emptyset$. One can check that $r(B_i) = 8$ for $i = 1, 2, 3, 4$, where B_i's are listed in Figure 4, which shows $|B(1)| \leq 3$. If $|B(1)| = 1$, the result holds obviously that $G \lhd G_2$.

If $|B(1)| = 2$, let $u, v \in B(1)$ and let u', v' be their unique neighbors in B, respectively. As $r(B_3) = r(B_4) = 8$, $\mathrm{dist}_B(u', v') \leq 2$. We have four subcases:

(a) $\mathrm{dist}_B(u', v') = 1$ and $|N_B(u')| = |N_B(v')| = 2$. As $r(G_3) = 6$, $u \nsim v$ by Lemma 2.7.

Then $G \lhd G_3$.

(b) $\mathrm{dist}_B(u', v') = 1$ and one of u', v' has 3 neighbors in B. As $r(B_1) = r(B_2) = 8$, exactly one of u', v' has 3 neighbors in B. As $r(G_3) = 6$, $u \sim v$ by Lemma 2.7. Then $G \lhd G_3$.

(c) $\mathrm{dist}_B(u', v') = 2$ and $|N_B(u')| = |N_B(v')| = 2$. As $r(B_5) = 6$, $u \nsim v$. Then $G \lhd B_5 \lhd G_1$.

(d) $\mathrm{dist}_B(u', v') = 2$ and exactly one of u', v' has 3 neighbors in B. As $r(G_3) = 6$, $u \nsim v$. Then $G \lhd G_3$.

If $|B(1)| = 3$, let $u, v, w \in B(1)$ and u', v', w' be their unique neighbors in B, respectively. There are three possible subcases:

(a) $u' \sim v'$ and $v' \sim w'$, $|N_B(v')| = 3$ and $|N_B(u')| = |N_B(w')| = 2$. As $r(G_2) = 6$, $G = G_2$ by Corollary 2.8.

(b) $u' \sim v'$ and $v' \sim w'$, $|N_B(u')| = 3$ and $|N_B(v')| = |N_B(w')| = 2$. As $r(G_3) = 6$, $G = G_3$ by Corollary 2.8.

(c) $G = B_5$. As $r(B_5) = 6$, $G \lhd G_1$. $\square$

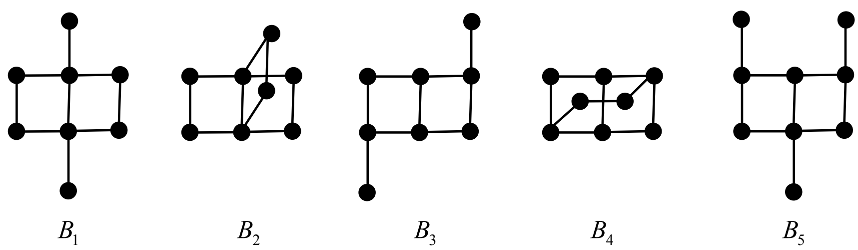

Figure 4 Illustration for Lemma 3.5, where $r(B_i) = 8$ for $i \in \{1, 2, 3, 4\}$ and $r(B_5) = 6$

Lemma 3.6 *Let G be a connected reduced bipartite graph with rank 6, which contains C of Figure 1 as an induced subgraph. Then G is an induced subgraph of G_1, G_2 or G_3.*

Proof We have $1 \le |N_C(v)| \le 2$ for each $v \in V(G)\backslash V(C)$.

Case 1 $C(2) \neq \emptyset$. Let $v \in C(2)$, and let v_1, v_2 be its two neighbors in C. As G is bipartite, $\mathrm{dist}_C(v_1, v_2)$ should be even, and $\mathrm{dist}_C(v_1, v_2) = 2$. As G is reduced, by Lemma 2.5 one of v_1 and v_2 is a pendant vertex of C. Thus $B \lhd G$ and the result follows from Lemma 3.5.

Case 2 $C(2) = \emptyset$. If $|C(1)| = 1$, the result holds obviously. Now we suppose $|C(1)| \ge 2$. If there exist $u, v \in C(1)$ such that $u \sim v$, letting u', v' be their unique neighbors in C, respectively, then $\mathrm{dist}_C(u', v')$ should be odd by the bipartiteness of G. Note that $|N_C(v')| \neq 3$ and $|N_C(u')| \neq 3$; otherwise G is not reduced or G is B_1 shown in Figure 4 and $r(G) = 8$. If $\mathrm{dist}_C(u', v') = 1$, then $B \lhd G$. If $\mathrm{dist}_C(v', u') = 3$, noting that $r(C_1) = r(C_2) = 8$, this case cannot occur, where C_1, C_2 are listed in Figure 5.

Now assume $v_1, v_2, \ldots, v_k \in C(1)$ and $v_i \nsim v_j$ for all $1 \le i < j \le k$. Similarly, none of them has a neighbor with degree 3 in C. Note that $r(C_3) = r(C_4) = 8$ and $r(C_5) = r(C_6) = 6$. We conclude that $G \lhd C_5 \lhd G_3$ or $G \lhd C_6 \lhd G_1$. $\square$

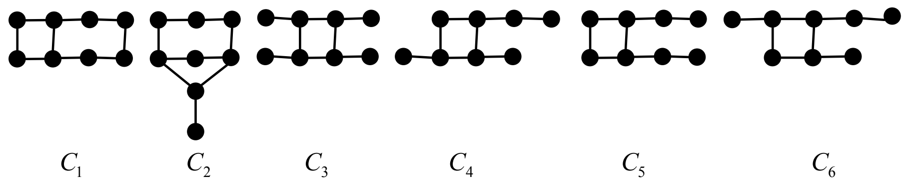

Figure 5 Illustration for Lemma 3.6, where $r(C_i) = 8$ for $i \in \{1, 2, 3, 4\}$ and $r(C_i) = 6$ for $i \in \{5, 6\}$

Lemma 3.7 *Let G be a connected reduced bipartite graph with rank 6, which contains D of Figure 1 as an induced subgraph. Then G is an induced subgraph of G_1, G_2 or G_3.*

Proof We have $1 \le |N_D(v)| \le 3$ for each $v \in V(G)\backslash V(D)$.

Case 1 $D(2) \neq \emptyset$. Let $v \in D(2)$, and let $N_D(v) = \{v_1, v_2\}$. Then $\mathrm{dist}_D(v_1, v_2)$ should be even. If $\mathrm{dist}_D(v_1, v_2) = 2$, then G is not reduced by Lemma 2.5. If $\mathrm{dist}_D(v_1, v_2) = 4$, then $A \lhd G$. The result follows from Lemma 3.4.

Case 2 $D(3) \neq \emptyset$. Let $v \in D(3)$, and let v_1, v_2 and v_3 be its three neighbors in D. Since $\mathrm{dist}_D(v_i, v_j)$ must be even, we have $B \lhd G$. The result follows from Lemma 3.5.

Case 3 $D(2) = D(3) = \emptyset$. Let $v_1, v_2, \ldots, v_k \in D(1)$ and let $v'_1, v'_2 \ldots, v'_k$ be their unique neighbors in D, respectively. As G is reduced, v'_i cannot be the quasi-pendant vertex of D (i.e., the vertex adjacent to a pendant vertex). If $k = 1$, then $G \lhd G_1$ and the result follows.

If $v_i \sim v_j$ for some $1 \le i < j \le k$, then $\mathrm{dist}_D(v'_i, v'_j)$ should be odd. We have the following subcases:

(a) If $\mathrm{dist}_D(v'_i, v'_j) = 1$, then v'_i and v'_j must be the two vertices in the middle of the path D. Thus $C \lhd G$ and the result follows.

(b) If $\mathrm{dist}_D(v'_i, v'_j) = 3$, then one of v'_i, v'_j must be the pendant vertex of D. Thus $A \lhd G$ and the result follows.

(c) If $\mathrm{dist}_D(v'_i, v'_j) = 5$, then $D_1 \lhd G$. If $G = D_1$, then $G \lhd G_2$. If $D_2 \lhd G$, then $r(G) \ge 8$. If $D_3 \lhd G$, then G is not reduced. If $D_4 \lhd G$, then $A \lhd G$ and the result follows.

If $v_i \nsim v_j$ for any $1 \le i < j \le k$. Noting that $r(D_5) = 6$ and $r(D_6) = r(D_7) = r(D_8) = 8$, we have $G \lhd D_5 \lhd G_1$, where D_5, D_6, D_7, D_8 are listed in Figure 6. $\square$

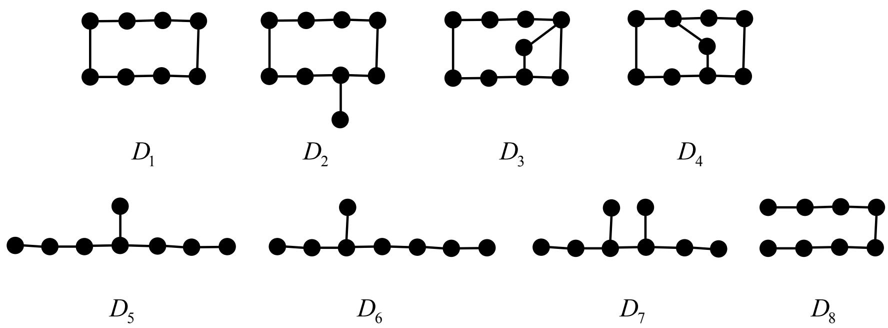

Figure 6 Illustration for Lemma 3.7, where $r(D_i) = 6$ for $i \in \{1, 5\}$ and $r(D_i) = 8$ for $i \in \{2, 6, 7, 8\}$

Lemma 3.8 *Let G be a connected reduced bipartite graph with rank* 6*, which contains E of Figure 1 as an induced subgraph. Then G is an induced subgraph of G_1, G_2 or G_3.*

Proof We have $1 \le |N_E(v)| \le 3$ for each $v \in V(G)\backslash V(E)$.

Case 1 $E(2) \ne \emptyset$. Let $v \in E(2)$ and let v_1 and v_2 be its two neighbors in E. The $\mathrm{dist}_E(v_1, v_2)$ is even. If $\mathrm{dist}_E(v_1, v_2) = 4$, then $A \lhd G$. Suppose $\mathrm{dist}_E(v_1, v_2) = 2$. Noting that G is reduced, by Lemma 2.5, at least one of v_1 and v_2 has degree 2 in E, and hence $C \lhd G$. The result follows from Lemmas 3.6.

Case 2 $E(3) \ne \emptyset$. Let $v \in E(3)$ and let v_1, v_2 and v_3 be the neighbors of v in E. The $\mathrm{dist}_E(v_i, v_j)$ is even for $1 \le i < j \le 3$. By Lemma 2.5, two of v_1, v_2, v_3 are pendant in E and the other one has degree 3 in E. Thus $B \lhd G$ and the result follows.

Case 3 $E(2) = E(3) = \emptyset$. For each $v \in E(1)$, it must be adjacent to a pendant vertex of E by Lemma 2.5, which implies $|E(1)| \le 3$ by Lemma 2.4. Let $v \in E(1)$, and let v' be its unique neighbor in E (i.e. a pendant vertex of E). If $|E(1)| = 1$, then $G \lhd G_1$ and the result follows. If $|E(1)| \ge 2$, noting that $r(E_1) = r(E_2) = 8$ and $r(E_3) = 6$, we have $G \lhd E_3 \lhd G_1$, where E_1, E_2, E_3 are listed in Figure 7. $\square$

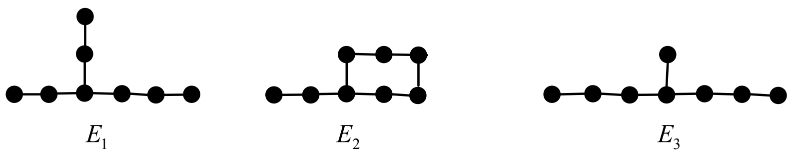

Figure 7 Illustration for Lemma 3.8, where $r(E_1) = r(E_2) = 8$ and $r(E_3) = 6$

Lemma 3.9 *Let G be a connected reduced bipartite graph with rank* 6*, which contains $\mathbf{P}_4 \cup \mathbf{P}_2$ of Figure 1 as an induced subgraph. Then G is an induced subgraph of G_1, G_2 or G_3.*

Proof Note that G is connected and bipartite, and $\mathrm{dist}_G(v, \mathbf{P}_4 \cup \mathbf{P}_2) = 1$ for each vertex $v \in V(G)\backslash V(\mathbf{P}_4 \cup \mathbf{P}_2)$. We have two cases:

Case 1 There exists a vertex $v \in V(G)\backslash V(\mathbf{P}_4 \cup \mathbf{P}_2)$ such that $\mathrm{dist}_G(v, \mathbf{P}_4) = \mathrm{dist}_G(v, \mathbf{P}_2) = 1$. Then $|N_{\mathbf{P}_2}(v)| = 1$ and $|N_{\mathbf{P}_4}(v)| \le 2$. If $|N_{\mathbf{P}_4}(v)| = 1$, then $D \lhd G$; if $|N_{\mathbf{P}_4}(v)| = 2$, then $C \lhd G$. The result follows by Lemmas 3.6 and 3.7.

Case 2 There exist two vertices $u, v \in V(G)\backslash V(\mathbf{P}_4 \cup \mathbf{P}_2)$ such that $\mathrm{dist}_G(u, \mathbf{P}_4) = \mathrm{dist}_G(v, \mathbf{P}_2) = 1$ and $u \sim v$. Then $D \lhd G$ and the result also follows. $\square$

Lemma 3.10 *Let G be a connected reduced bipartite graph with rank* 6*, which contains $3\mathbf{P}_2$ of Figure 1 as an induced subgraph. Then G is an induced subgraph of G_1, G_2 or G_3.*

Proof We denote the three disjoint paths as $\mathbf{P}_2^1, \mathbf{P}_2^2, \mathbf{P}_2^3$, respectively. Then we have two cases:

Case 1 There exist two vertices $u, v \in V(G)\backslash V(3\mathbf{P}_2)$ such that $\mathrm{dist}_G(u, \mathbf{P}_2^1) = \mathrm{dist}_G(v, \mathbf{P}_2^2) = 1$ and $u \sim v$. Then $D \lhd G$ and the result follows by Lemma 3.7.

Case 2 There exists a vertex $v \in V(G)\backslash V(3\mathbf{P}_2)$ such that $\mathrm{dist}_G(v, \mathbf{P}_2^1) = \mathrm{dist}_G(v, \mathbf{P}_2^2) = 1$. If $\mathrm{dist}_G(v, \mathbf{P}_2^3) = 1$, then $E \lhd G$; if $\mathrm{dist}_G(v, \mathbf{P}_2^3) \geq 2$, then $(\mathbf{P}_4 \cup \mathbf{P}_2) \lhd G$. The result follows by Lemmas 3.8 and 3.9. $\Box$

3.2. Reduced triangle-free and non-bipartite graphs with rank 6

Observe that if a graph G contains an induced odd cycle $\mathbf{C}_{2k+1}$, then $r(G) \geq r(\mathbf{C}_{2k+1}) = 2k+1$. So, if G is triangle-free and non-bipartite, and G has rank 6, then G contains an induced $\mathbf{C}_5$.

Lemma 3.11 *Let G be a triangle-free and non-bipartite graph with rank* 6*. Then $F \lhd G$ of Figure 1.*

Proof As discussed above, G contains an induced cycle $\mathbf{C}_5$. Let v be an arbitrary vertex not in $\mathbf{C}_5$. Note that $r(\mathbf{C}_5) = 5$, thus $|N_{\mathbf{C}_5}(v)| \geq 1$ by Lemma 2.6. If $|N_{\mathbf{C}_5}(v)| = 1$, then $F \lhd G$ and we are done. If $|N_{\mathbf{C}_5}(v)| \geq 3$, then G contains triangles; a contradiction.

Now suppose $|N_{\mathbf{C}_5}(v)| = 2$ for each $v \notin V(\mathbf{C}_5)$, and the distance between two neighbors of v in $\mathbf{C}_5$ is 2. Let $V(G)\backslash V(\mathbf{C}_5) = \{v_1, v_2, \ldots, v_k\}$. If $v_{i_0} \sim v_{j_0}$ for some i_0, j_0, then $N_{\mathbf{C}_5}(v_{i_0}) \cap N_{\mathbf{C}_5}(v_{j_0}) = \emptyset$; otherwise G contains triangles. If $v_{i_0} \not\sim v_{j_0}$ for some i_0, j_0, then $N_{\mathbf{C}_5}(v_{i_0}) \cap N_{\mathbf{C}_5}(v_{j_0}) \neq \emptyset$; otherwise $r(G) \geq 7$. Thus for any $i \neq j$, $v_i \sim v_j$ if and only if $N_{\mathbf{C}_5}(v_i) \cap N_{\mathbf{C}_5}(v_j) = \emptyset$. We will prove by induction on k that G is obtained from $\mathbf{C}_5$ by several steps of multiplication of vertices and $r(G) = r(\mathbf{C}_5) = 5$, which leads to an contradiction.

If $k = 1$, the results follow easily since $N_G(v_1) = N_G(u)$ for some $u \in V(\mathbf{C}_5)$. Assume that the result holds for $k \leq l$ $(l \geq 1)$. Now consider the case of $k = l+1$. Let $V(G)\backslash V(\mathbf{C}_5) = \{v_1, v_2, \ldots, v_{l+1}\}$. Let $H = G - v_{l+1}$, the graph obtained from G by deleting the vertex v_{l+1} together with the edges incident to it. By the assumption, H is not reduced, say $N_H(v_l) = N_H(u_2)$, where $u_2 \in \mathbf{C}_5$ and $N_{C_5}(u_2) = \{u_1, u_3\}$.

If $N_{\mathbf{C}_5}(v_{l+1}) \cap N_{\mathbf{C}_5}(v_l) \neq \emptyset$, then $v_{l+1} \not\sim u_2$ and $v_{l+1} \not\sim v_l$. Thus $N_G(v_l) = N_G(u_2)$ and G is obtained from $G - v_l$ by multiplication of u_2 and addition of v_l. Note that $G - v_l$ is obtained from $\mathbf{C}_5$ by several steps of multiplication of vertices by the induction, so is G.

If $N_{\mathbf{C}_5}(v_{l+1}) \cap N_{\mathbf{C}_5}(v_l) = \emptyset$, then $v_{l+1} \sim v_l$ and $v_{l+1} \sim u_2$, thus $N_G(v_l) = N_G(u_2)$ and the result follows again by a similar discussion as the above. $\Box$

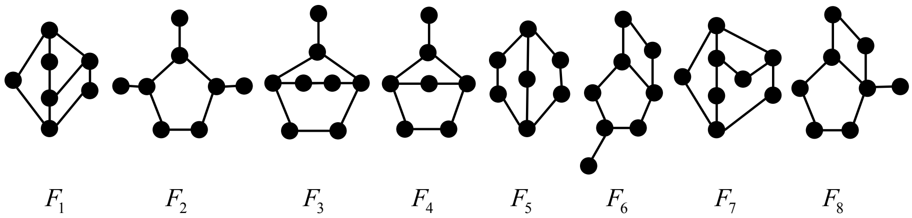

Figure 8 Illustration for Lemma 3.11, where $r(F_i) \geq 7$ for $i \in \{1, 2, \ldots, 8\}$

Lemma 3.12 *Let G be a connected reduced triangle-free graph with rank* 6*, which contains F of Figure 1 as an induced subgraph. Then G is an induced subgraph of G_4 of Figure 2.*

Proof If $|N_F(v)| \geq 4$ for some $v \in V(G)\backslash V(F)$, then $C_3 \lhd G$. If $|N_F(v)| = 3$ for some $v \in V(G)\backslash V(F)$, say $N_F(v) = \{v_1, v_2, v_3\}$, then $\mathrm{dist}_F(v_i, v_j) \geq 2$ for $1 \leq i < j \leq 3$. Thus one of v_1, v_2, v_3 is a pendant vertex in F. So, $F_1 \lhd G$ or G is obtained from F by multiplication of the vertex with degree 3, where F_1 is listed in Figure 8. However, $r(F_1) = 7$. So we conclude that $1 \leq |N_F(v)| \leq 2$ for $v \in V(G)\backslash V(F)$.

Case 1 $F(2) = \emptyset$. Let $F(1) = \{v_1, v_2, \ldots, v_k\}$, and $N_F(v_i) = \{u_i\}$ for $i = 1, 2, \ldots, k$. Then u_i cannot be adjacent to the pendant or the quasi-pendant vertex of F; otherwise $r(G) \geq 7$ by Lemma 2.2 or G is not reduced by Lemma 2.5. Note that $r(F_2) = r(F_3) = 8$, then $|F(1)| \leq 3$. Since $r(G_4) = 6$, by Corollary 2.5 we have $G \lhd G_4$.

Case 2 $F(2) \neq \emptyset$. Let w_1 be the vertex with maximum degree in F, and $N_F(w_1) = \{w_2, w_3, w_4\}$, where w_2 is a pendant vertex of F. Let $F(2) = \{v_1, v_2, \ldots, v_k\}$ and $N_F(v_i) = \{u_i^1, u_i^2\}$ for $i = 1, 2, \ldots, k$. Since G is triangle-free, we have $2 \leq \mathrm{dist}_F(u_i^1, u_i^2) \leq 3$. If $u_i^1 = w_3$ and $u_i^2 = w_4$ for some i, then $r(G) \geq r(F_4) = 7$, where F_4 is listed in Figure 8. As G is a reduced graph with rank 6, one of u_i^1, u_i^2 must be w_2. Set $F(2) = F(2)_1 \cup F(2)_2$, where $F(2)_1 = \{v_i \mid \mathrm{dist}_F(u_i^1, u_i^2) = 2\}$ and $F(2)_2 = \{v_i \mid \mathrm{dist}_F(u_i^1, u_i^2) = 3\}$. As $r(F_5) = 7$, $F(2)_2 = \emptyset$, where F_5 is listed in Figure 8. Note that $|F(2)_1| \leq 2$. Suppose that $F(2)_1 = \{v_1, v_2\}$. We have $N_F(v_1) = \{w_2, w_3\}$ and $N_F(v_2) = \{w_2, w_4\}$. If $v_1 \sim v_2$, $C_3 \lhd G$; otherwise $C_7 \lhd G$, which implies $r(G) \geq 7$. Thus $|F(2)_1| \leq 1$. If $F(1) = \emptyset$, then $G \lhd G_4$. If $F(1) \neq \emptyset$, noting that $r(F_i) = 8$ for $i = 6, 7, 8$, we have $G \lhd G_4$ again. $\Box$

With the above lemmas in hands, we are now able to give a complete proof for Theorem 3.2.

Proof for Theorem 3.2 Note that each graph in Figures 1 and 2 has rank 6 and is triangle-free. So if $H_1 \lhd G \lhd H_2$, where H_1 comes from Figure 1 and H_2 comes from Figure 2, we have $r(G) = 6$. Thus the sufficiency follows.

For the necessity, we know if $r(G) = 6$, then G contains a nonsingular graph of order 6 shown in Figure 1. By Lemmas 3.4–3.10, we conclude that if G is a bipartite reduced graph with rank 6, then G is an induced subgraph of G_1, G_2 or G_3. By Lemma 3.12, if G is a non-bipartite reduced graph with rank 6, then G is an induced subgraph of G_4. $\Box$

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