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Finite Groups with Some Subgroups Weakly s-Permutably Embedded

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Abstract Let P be a Sylow p-subgroup of a group G with the smallest generator number d, where p is a prime. Denote by $\mathcal{M}_d(P)=\{P_1,P_2,\ldots,P_d\}$ a set of maximal subgroups of P such that $\Phi(P)=\cap_{n=1}^{d}P_n$. In this paper, we investigate the structure of a finite group G under the assumption that the maximal subgroups in $\mathcal{M}_d(P)$ are weakly s-permutably embedded in G, some interesting results are obtained which generalize some recent results. Finally, we give some further results in terms of weakly s-permutably embedded subgroups.

Keywords finite groups; weakly s-permutably embedded subgroups; p-nilpotent groups; p-supersolvable groups; supersolvable groups.

MR(2010) Subject Classification 20D10; 20D20

1. Introduction

All groups considered in this paper will be finite. G always denotes a finite group, $|G|$ is the order of G, $\pi(G)$ denotes the set of all primes dividing $|G|$, G_p a Sylow p-subgroup of G. Let $H\leq G$, H^G and H_G denote the normal closure and the core of H in G, respectively; H_{sG} denotes the maximal s-permutable subgroup of G contained in H, H_{eG} an s-permutably embedded subgroup of G contained in H. For convenience, a semidirect product of a group A by a group B is denoted by $A{:}B$.

Two subgroups H and K of G are said to be permutable if $HK=KH$. It is easy to see that two subgroups of G, H and K, permute if and only if the set HK is a subgroup of G. A subgroup H of G is said to be permutable in G if it permutes with every subgroup of G. H is called s-permutable in G if it permutes with every Sylow subgroup of G (see [5]). This definition is generalized by Skiba to be weakly s-permutable subgroup [10]. H is said to be weakly s-permutable in G if there is a subnormal subgroup K of G such that $G=HK$ and $H\cap K\leq H_{sG}$. H is said to be s-permutably embedded in G provided that every Sylow subgroup

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of H is a Sylow subgroup of some s-permutable subgroup of G (see [2]). In [7], H is said to be weakly s-permutably embedded in G if there are a subnormal subgroup T and an s-permutably embedded subgroup H_{eG} of G contained in H such that $G = HT$ and $H \cap T \leq H_{eG}$. Weakly s-permutably embedding property covers both s-permutably embedding property and weakly s-permutability.

Srinivasan [11] proved that, if all maximal subgroups of all Sylow subgroups of G are normal (or permutable) in G, then G is supersolvable. Asaad, Ramadan and Shaalan [1] generalized Srinivasan's result: if all maximal subgroups of all Sylow subgroups of G are s-permutable in G, then G is supersolvable. Recently, Ballester-Bolinches and Pedraza-Aquilera [2] went further: if all maximal subgroup of all Sylow subgroups of G are s-permutably embedded in G, then G is supersolvable. This result is generalized in [7, Theorem 3.4] through replacing s-permutably embedded subgroups by weakly s-permutably embedded subgroups. In [7], the authors also get some criteria for the p-nilpotency of finite groups under the assumption that certain subgroups are weakly s-permutably embedded. The following example indicates that even a supersolvable group could not ensure that all its maximal subgroups of all Sylow subgroups are weakly s-permutably embedded in the whole group. Hence we could reduce the number of restricted maximal subgroups of Sylow subgroups of G to get the structural results.

Example 1.1 Let $G = \langle a, b, c |\ a^5 = b^5 = c^2 = 1, a^c = a, b^c = b^{-1}, ab = ba\rangle = Z_5 \times D_{10}$. Clearly, G is supersolvable. But the maximal subgroup $\langle ab\rangle$ of a Sylow 5-subgroup of G is not weakly s-permutably embedded in G.

Notation 1.2 In [6], authors defined the following set. Let d be the smallest generator number of a p-group P. Denote by $\mathcal{M}_d(P) = \{P_1, P_2, \ldots, P_d\}$ some maximal subgroups of P such that $\Phi(P) = \cap_{n=1}^{d} P_n$.

It is easy to figure out that d is fairly smaller than the number of all the maximal subgroups of P. We know, for a p-subgroup P, there are many sets $\mathcal{M}_d(P)$ such that $\mathcal{M}_d(P) = \{P_1, P_2, \ldots, P_d\}$ and $\Phi(P) = \cap_{n=1}^{d} P_n$. In this paper, when $\mathcal{M}_d(P)$ appears in the statement of the theorem, it is always assumed that $\mathcal{M}_d(P)$ is some fixed one of these sets.

In [6], authors proved: if each member in $\mathcal{M}_d(P)$ is s-permutably embedded in the whole group G, $P \in \mathrm{Syl}_p(G)$, then G is p-nilpotent. Recently, in [9], authors proved the following reuslt: if every member in $\mathcal{M}_d(P)$ is c^*-normal in G, then G is p-supersolvable. Recall here that a subgroup H of a group G is called c^*-normal in G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K$ is s-permutably embedded in G.

The main purpose of this paper is to investigate the structure of G through assuming that some families of subgroups of G are weakly s-permutably embedded in G. Some results in [6, 9, etc] are generalized.

2. Preliminaries

Below we give some lemmas that will be used in our proofs.

Lemma 2.1 ([5]) *(a) An s-permutable subgroup of G is subnormal in G;*

(b) If $H \leq K \leq G$ and H is s-permutable in G, then H is s-permutable in K;

(c) If H is a subnormal Hall subgroup of G, then $H \lhd G$;

(d) Let $K \lhd G$. If H is s-permutable in G, then HK/K is s-permutable in G/K;

(e) If P is an s-permutable p-subgroup of G for some prime p, then $N_G(P) \geq O^p(G)$.

Lemma 2.2 ([8, Lemma 2.3]) *Suppose that H is s-permutable in G, P a Sylow p-subgroup of H, where p is a prime. If $H_G = 1$, then P is s-permutable in G.*

Lemma 2.3 ([8, Lemma 2.4]) *Suppose P is a p-subgroup of G contained in $O_p(G)$. If P is s-permutably embedded in G, then P is s-permutable in G.*

Now we list some basic properties of weakly s-permutably embedded subgroups.

Lemma 2.4 ([7, Lemma 2.5]) *Let U be a weakly s-permutably embedded subgroup of G and N a normal subgroup of G. Then*

(a) If $U \leq H \leq G$, then U is weakly s-permutably embedded in H.

(b) If $N \leq U$, then U/N is weakly s-permutably embedded in G/N.

(c) Let π be a set of primes, U a π-subgroup and N a π'-subgroup. Then $(UN)/N$ is weakly s-permutably embedded in G/N.

(d) Suppose U is a p-group for some prime p and U is not s-permutable embedded in G. Then G has a normal subgroup M such that $|G : M| = p$ and $G = MU$.

(e) Suppose U is a p-group contained in $O_p(G)$ for some prime p, then U is weakly s-permutable in G.

Lemma 2.5 *Let N be an elementary abelian normal p-subgroup of a group G. Assume that N has a subgroup D such that $1 < |D| < |N|$ and every subgroup H of N satisfying $|H| = |D|$ is weakly s-permutably embedded in G. Then some maximal subgroup of N is normal in G.*

Proof Since $N \leq O_p(G)$, by Lemma 2.4(e), H is weakly s-permutable in G. By [10, Lemma 2.11], the result holds. □

Lemma 2.6 ([4, Lemma 2.6]) *Let G be a group. Assume that $1 < N \lhd G$ and $N \cap \Phi(G) = 1$. Then the Fitting subgroup $F(N)$ of N lies in $\mathrm{Soc}(G)$ and therefore $F(N)$ is the direct product of the minimal normal subgroups of G contained in $F(N)$.*

The following result is well known which is due to Gaschütz.

Lemma 2.7 *Let G be a finite group, N an abelian normal subgroup of G. Suppose that $N \leq M \leq G$ and $(|N|, |G : M|) = 1$. Then N has a complement in G if N has a complement in M.*

3. Main results

Theorem 3.1 *Let G be a p-solvable group and $P \in \mathrm{Syl}_p(G)$ for some $p \in \pi(G)$. If the subgroups* *in $\mathcal{M}_d(P)$ are weakly s-permutably embedded in G, then G is p-supersolvable.*

Proof Let G be a minimal counter-example. We will derive a final contradiction in several steps.

By Lemma 2.4, Steps 1–2 are obvious.

Step 1. $O_{p'}(G) = 1$.

Step 2. $\Phi(P)_G = 1$. Further, $O_p(G)$ is elementary abelian.

Step 3. $|N| = p$ if N is a minimal normal subgroup of G contained in P.

Let N be a minimal normal subgroup of G contained in P. If $N \not\leq O^p(G)$, then $N \cong NO^p(G)/O^p(G)$ has order p. Thus we may suppose that $N \leq O^p(G)$. Since $\Phi(P)_G = 1$, we have $N \not\leq \Phi(P)$. Thus there exists a maximal subgroup in $\mathcal{M}_d(P)$, say P_1, such that $N \not\leq P_1$. By hypothesis, P_1 is weakly s-permutably embedded in G. Then there is a subnormal subgroup T_1 of G such that $G = P_1T_1$ and $P_1 \cap T_1 \leq (P_1)_{eG}$, where $(P_1)_{eG} \in \mathrm{Syl}_p(S)$ and S is s-permutable in G. Clearly, $N \leq O^p(G) \leq T_1$. Then $P_1 \cap N = (P_1)_{eG} \cap N = N \cap S$ which is s-permutable in G. Then $N_G(P_1 \cap N) = N_G((P_1)_{eG} \cap N) = N_G(N \cap S) = G$. It follows that $P_1 \cap N = 1$, and so $|N| = p$, as desired. This proves Step 3.

Step 4. Finishing the proof.

By Step 1, we have $O_p(G) \neq 1$ since G is p-solvable. By Step 2, $O_p(G)$ is elementary ableian. By Step 1, there holds $\Phi(G) \leq O_p(G) = F(G)$. Suppose that $\Phi(G) \neq 1$. Pick a minimal normal subgroup N of G contained in $\Phi(G)$. By Step 3, we have N has order p. Since $\Phi(P)_G = 1$, N is not contained in $\Phi(P)$. Then N has a complement in P. By Lemma 2.7, we have N has a complement in G, contrary to the choice of N. Thus $\Phi(G) = 1$. Lemma 2.6 implies that $O_p(G)$ can be decomposed as $O_p(G) = N_1 \times N_2 \times \cdots \times N_s$ with $|N_i| = p$ for $1 \leq i \leq s$. We know that $C_G(O_p(G)) \leq O_p(G)$ since G is p-solvable and $O_{p'}(G) = 1$. Since $O_p(G)$ is abelian, we have $C_G(O_p(G) = O_p(G)$. Then $G/O_p(G) = G/C_G(O_p(G)) \leq G/C_G(N_1) \times \cdots \times G/C_G(N_s)$ which is abelian. This means G is supersolvable, in particular, G is p-supersolvable.

This completes the proof of the theorem. □

Remark G is "p-solvable" in Theorem 3.1 could not be removed, any non-abelian simple group with a prime-order Sylow p-subgroup is a counterexample.

Theorem 3.2 *Let G be a group and let P be a Sylow p-subgroup of G, $p \in \pi(G)$. Suppose that the elements of $\mathcal{M}_d(P)$ are weakly s-permutably embedded in G. Then G is p-nilpotent if one of the following holds:*

1). $N_G(P)$ is p-nilpotent;

2). p is the minimal prime divisor of $|G|$.

Proof Let G be a minimal counter-example. We will derive a contradiction in several steps. Let $\mathcal{M}_d(P) = \{P_1, P_2, \ldots, P_d\}$. For each P_i, there is a subnormal subgroup $T_i \lhd \lhd G$ such that $G = P_iT_i$ and $P_i \cap T_i \leq (P_i)_{eG}$. Clearly, it is easy to get Steps 1–3.

Step 1. $O_{p'}(G) = 1$.

Step 2. If $P \leq H < G$, then H is p-nilpotent.

Step 3. $\Phi(P)_G = 1$. In particular, $O_p(G)$ is elementary abelian.

Step 4. $|N| = p$ for any minimal normal subgroup N of G contained in P.

Let N be a minimal normal subgroup of G contained in P. If $N \nleq O^p(G)$, then $N \cong NO^p(G)/O^p(G)$ has order p. Thus we may suppose that $N \leq O^p(G)$. Since $\Phi(P)_G = 1$, it follows that $N \nleq \Phi(P)$. Thus there exists a maximal subgroup in $\mathcal{M}_d(P)$, say P_1, such that $N_p \nleq P_1$. By hypothesis, P_1 is weakly s-permutably embedded in G. Then there is a subnormal subgroup T_1 such that $G = P_1T_1$ and $P_1 \cap T_1 \leq (P_1)_{eG}$, where $(P_1)_{eG} \in \mathrm{Syl}_p(S)$ and S is s-permutable in G. Clearly, $N \leq O^p(G) \leq T_1$. Then $P_1 \cap N = (P_1)_{eG} \cap N = S \cap N$ which is s-permutable in G. Then $N_G(P_1 \cap N) = N_G((P_1)_{eG} \cap N) = N_G(N \cap S) = G$. It follows that $P_1 \cap N = 1$, and so $|N| = p$. Step 4 is proved.

Step 5. Every minimal normal subgroup of G is contained in P.

Let N be a minimal normal subgroup of G. If $NP < G$, by Step 2, NP is p-nilpotent. Then N is a p-subgroup by Step 1. Thus we may suppose that $NP = G$. Then $N = O^p(G)$, and $N_p := N \cap P \nleq \Phi(P)$. This yields that N is the unique minimal normal subgroup of G. Since $N_p \nleq \Phi(P)$, it follows that there exists a maximal subgroup in $\mathcal{M}_d(P)$, say P_1, such that $N_p \nleq P_1$. By hypothesis, P_1 is weakly s-permutably embedded in G. Then there is a subnormal subgroup T_1 such that $G = P_1T_1$ and $P_1 \cap T_1 \leq (P_1)_{eG}$, where $(P_1)_{eG} \in \mathrm{Syl}_p(S)$ and S is s-permutable in G. Clearly, $N = O^p(G) \leq T_1$. Then $P_1 \cap N = (P_1)_{eG} \cap N$. If $S_G > 1$, then $N \leq S_G$, and so $N_p \leq (P_1)_{eG} \leq P_1$, a contradiction. Let $S_G = 1$. Then $(P_1)_{eG}$ is s-permutable in G by Lemma 2.2. It follows that $P_1 \cap N = (P_1)_{eG} \cap N$ which is s-permutable in G. This yields that $((P_1)_{eG} \cap N)^G = ((P_1)_{eG} \cap N)^{O^p(G)P} = ((P_1)_{eG} \cap N)^P = (P_1 \cap N)^P = P_1 \cap N \leq N$. Consequently, $((P_1)_{eG} \cap N)^G = 1$ or N since N is minimal normal in G. If $((P_1)_{eG} \cap N)^G = N$, then $N_p \leq N \leq P_1$, contrary to the choice of P_1. If $((P_1)_{eG} \cap N)^G = 1$, then $N \cap P_1 = (P_1)_{eG} \cap N \leq ((P_1)_{eG} \cap N)^G = 1$. It follows that N_p has order p. We know that $P \leq N_G(N_p)$. If $N_G(N_p) = G$, then $N = N_p$, we are done. We suppose that $N_G(N_p) < G$. Then $N_G(N_p)$, also $N_N(N_p)$, is p-nilpotent by Step 2. Thus $N_N(N_p) = C_N(N_p)$. By Burnside's theorem, N is p-nilpotent, and so $N = N_p$. Thus Step 5 holds.

Step 6. Finishing the proof.

By Step 5, $O_p(G) \neq 1$. Step 3 implies that $O_p(G)$ is elementary ableian. Let N be a minimal normal subgroup of G. By Steps 4–5, N has order p. Since $\Phi(P)_G = 1$, N is not contained in $\Phi(P)$. Then N has a complement in P. By Lemma 2.7, N has a complement in G. It follows that $O_p(G) \cap \Phi(G) = 1$. By Lemma 2.6 and Step 4, $O_p(G)$ can be decomposed as $O_p(G) = N_1 \times N_2 \times \cdots \times N_s$ with $|N_i| = p$ for $1 \leq i \leq s$. Then G has presentation $G = O_p(G) : M$ for some suitable subgroup M. Clearly, $C := C_M(O_p(G)) \lhd G$. If $C \neq 1$, then C has some non-trivial subgroup contained in $O_p(G)$ by Step 5, contrary to the fact that $C \leq M$. Thus $C = 1$, and $C_G(O_p(G)) = O_p(G)$. Thus $M \cong G/O_p(G) \leq G/C_G(N_1) \times \cdots \times G/C_G(N_s)$. Then M is abelian, and $O_p(G) = P$. Thus G is p-nilpotent if either $G = N_G(P)$ is p-nilpotent or p is the minimal prime divisor of $|G|$.

This completes the proof of the theorem. □

Theorem 3.3 *Let G be a group. Suppose that the elements of $\mathcal{M}_d(P)$ are weakly s-permutably embedded in G for each Sylow subgroup P of G. Then G is supersolvable.*

Proof By Theorem 3.2, we have G has a Sylow Tower of supersolvable type. Let $P \in \mathrm{Syl}_p(G)$, where p is the maximal prime divisor of $|G|$. Then $P \lhd G$. By induction, G/P is supersolvable. Then G is solvable. Consider the group pair (G, P), by Theorem 3.1, we have G is p-supersolvable. This means that G is supersolvable. The proof is completed. $\square$

Below we show some further results in terms of weakly s-permutably embedded subgroups. The following theorem was proved in [3].

Theorem 3.4 *Let p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of G. Then G is p-nilpotent if P has m_p maximal subgroups which are s-permutable in G, where d_p is the smallest generator number of P and*

$$m_p > \begin{cases} 1, \ d_p = 2; \\ (p^{d_p-2} - 1)/(p-1), \ d_p > 2. \end{cases}$$

Similarly, the result holds for the subnormal case.

Theorem 3.4' *Let p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of G. Then G is p-nilpotent if P has m_p maximal subgroups which are subnormal in G, where*

$$m_p > \begin{cases} 1, \ d_p = 2; \\ (p^{d_p-2} - 1)/(p-1), \ d_p > 2. \end{cases}$$

However, the result does not hold for the weakly s-permutably embedded case.

Example Let $G = S_4 \times Z_2$, and $P = D_8 \times Z_2$. Clearly $p = 2$ and $d_p = 3$. Then $(2^{3-2} - 1)/(2-1) = 1$. Then there are at least two maximal subgroups of P which are normally embedded, of course weakly s-permutably embedded, in G. But G is not 2-nilpotent.

The above example shows that $(p^{d_p-2} - 1)/(p-1)$ is "too small" to guarantee the p-nilpotency of G under the assumption that some suitable subgroups are weakly s-permutably embedded. Thus we have the following results.

Theorem 3.5 *Let p be the smallest prime dividing $|G|$ and $P \in \mathrm{Syl}_p(G)$. Then G is p-nilpotent if and only if P has m_p maximal subgroups which are weakly s-permutably embedded in G, where d_p is the smallest generator number of P and*

$$m_p > \begin{cases} 1, \ d_p = 2; \\ (p^{d_p-1} - 1)/(p-1), \ d_p > 2. \end{cases}$$

Proof Let G be a p-nilpotent group. Then $G = G_{p'} : P$, and so each maximal subgroup of P is normally embedded, also weakly s-permutably embedded, in G. Thus the necessary part of the theorem holds.

Below, we prove the sufficient part. Let G be a minimal counterexample. Steps 1–3 are obvious.

Step 1. $O_{p'}(G) = 1$.

Step 2. $\Phi(P)_G = 1$. In particular, $O_p(G)$ is elementary abelian.

Step 3. If $P \le H < G$, then H is p-nilpotent.

Step 4. All minimal normal subgroups of G are contained in P.

Let N be a minimal normal subgroup of G. By induction, $G = NP$, and $N = O^p(G)$ which is unique. If $N \cap P \le \Phi(P)$, N is p-nilpotent by Tate's result, and so $N \le O_p(G)$. Thus we may suppose that $N_p := N \cap P \nleq \Phi(P)$. The number of maximal subgroups of P containing N_p is at most $(p^{d_p-1}-1)/(p-1)$. By hypothesis, there exists a maximal subgroup P_1 of P such that $N_p \nleq P_1$, and P_1 is weakly s-permutably embedded in G. Then there is a subnormal subgroup T_1 of G such that $G = P_1T_1$ and $P_1 \cap T_1 \le (P_1)_{eG}$, where $(P_1)_{eG} \in Syl_p(S)$ and S is s-permutable in G. Clearly, $N = O^p(G) \le T_1$. Then $P_1 \cap N = (P_1)_{eG} \cap N$. If $S_G > 1$, then $N \le S_G$ according to the uniqueness of N, and so $N_p \le (P_1)_{eG} \le P_1$, a contradiction. Suppose that $S_G = 1$. Then $(P_1)_{eG}$ is s-permutable in G by Lemma 2.2. It follows that $P_1 \cap N = (P_1)_{eG} \cap N$ which is s-permutable in G. Then $((P_1)_G \cap N)^G = ((P_1)_{eG} \cap N)^{O^p(G)P} = ((P_1)_{eG} \cap N)^P = (P_1 \cap N)^P = P_1 \cap N \le N$. It follows that $((P_1)_{eG} \cap N)^G = 1$ or N. If $((P_1)_{eG} \cap N)^G = N$, then $N \le P_1$, a contradiction. If $((P_1)_{eG} \cap N)^G = 1$, then $N \cap P_1 = (P_1)_{eG} \cap N \le ((P_1)_{eG} \cap N)^G = 1$. This means that N_p has order p and N is p-nilpotent, and so $N_p = N$, as desired. Thus Step 4 holds.

Step 5. Each minimal normal subgroups of G has order p.

Let N be a minimal normal subgroup of G contained in P. If $N \nleq O^p(G)$, then $N \cong NO^p(G)/O^p(G)$ has order p. Thus we may suppose that $N \le O^p(G)$. Since $\Phi(P)_G = 1$, $N \nleq \Phi(P)$. The number of maximal subgroups of P containing N_p is at most $(p^{d_p-1}-1)/(p-1)$. By hypothesis of theorem, there exists a maximal subgroup P_1 of P such that $N_p \nleq P_1$, and P_1 is weakly s-permutably embedded in G. Then there is a subnormal subgroup T_1 of G such that $G = P_1T_1$ and $P_1 \cap T_1 \le (P_1)_{eG}$, where $(P_1)_{eG} \in \mathrm{Syl}_p(S)$ and S is s-permutable in G. Clearly, $N \le O^p(G) \le T_1$. Then $P_1 \cap N = (P_1)_{eG} \cap N = N \cap S$ which is s-permutable in G. Then $N_G(P_1 \cap N) = N_G((P_1)_{eG} \cap N) = N_G(N \cap S) = G$. It follows that $P_1 \cap N = 1$. Then $|N| = p$. This proves Step 5.

Step 6. Final contradiction.

By Steps 2 and 4, we can choose a subgroup M of G such that $G = O_p(G) : M$. By Step 5, since p is minimal, $G = O_p(G) : M = O_p(G) \times M$. Again, by Step 4, each minimal normal subgroup of M is contained in $O_p(G)$ if $M \ne 1$, a contradiction. Thus $M = 1$ and $G = O_p(G)$, this is a final contradiction.

The final contradiction completes the proof. □

Theorem 3.6 *Suppose that P is a Sylow p-subgroup of G for some $p \in \pi(G)$. Then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent and P has m_p maximal subgroups which are weakly s-permutably embedded in G, where d_p is the smallest generator number of P and*

$$m_p > \begin{cases} 1, \ d_p = 2; \\ (p^{d_p-1}-1)/(p-1), \ d_p > 2. \end{cases}$$

Proof We just need to prove the sufficient part. Let G be a minimal counterexample. Steps 1–3 are obvious.

Step 1. $O_{p'}(G) = 1$.

Step 2. $\Phi(P)_G = 1$. Further, $O_p(G)$ is elementary abelian.

Step 3. If $P \leq H < G$, then H is p-nilpotent.

Step 4. Each minimal normal subgroup of G is contained in P.

Let N be a minimal normal subgroup of G. Using the same argument as the proof of Step 4 in Theorem 3.5, we could obtain that N is abelian or N_p has order p. If N is abelian, then we are done. Let N_p have order p. Clearly, $P \leq N_G(N_p)$. If $N_G(N_p) = G$, then $N = N_p$, we are done. Suppose that $N_G(N_p) < G$. Then $N_G(N_p)$, also $N_N(N_p)$, is p-nilpotent by Step 3. Thus $N_N(N_p) = C_N(N_p)$. By Burnside's theorem, N is p-nilpotent, and so $N = N_p$ which is contained in P. This proves Step 4.

Using the same argument as the proof of Step 5 in Theorem 3.5, we have:

Step 5. Each minimal normal subgroup of G has order p.

Step 6. Finishing the proof.

With similar argument used in Step 6 of Theorem 3.2, we have $O_p(G) = N_1 \times \cdots \times N_s$, each N_i has prime order. Then G can be written as $G = O_p(G) : M$ for some suitable subgroup M. Clearly, $C := C_M(O_p(G)) \lhd G$. If $C \neq 1$, C has some non-trivial subgroup contained in $O_p(G)$ by Step 4. Thus $M \cong G/O_p(G) \leq G/C_G(N_1) \times \cdots \times G/C_G(N_s)$ which is abelian by Step 5. Then $O_p(G) = P$ since $G = O_p(G) : M$ and M is abelian. By hypothesis, $G = N_G(O_p(G)) = N_G(P)$ is p-nilpotent, a final contradiction.

This completes the proof of the theorem. $\Box$

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