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Chen's Inequalities for Totally Real Submanifolds in Complex Space Forms with a Semi-Symmetric Metric Connection

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Abstract In this paper, we obtain Chen's inequalities for totally real submanifolds in complex space forms with a semi-symmetric metric connection. Also, some results of A. Mihai and C. Özgür's paper have been modified.

Keywords Chen's inequalities; complex space forms; totally real submanifolds; semi-symmetric metric connection.

MR(2010) Subject Classification 53C40; 53D12

1. Introduction

One of the basic problems in the submanifold theory is to find simple relations between the intrinsic and extrinsic curvatures of a submanifold. Related with famous Nash embedding theorem [9], Chen introduced a new type of Riemannian invariant, known as Chen's invariant δ_M (see [4]), which is given by

$$\delta_M(x) = \tau(x) - \inf\{K(\pi) \mid \pi \subset T_xM^n, \dim \pi = 2\},$$

where M^n is a Riemannian manifold, $K(\pi)$ is the sectional curvature of M^n associated with a 2-plane section and the scalar curvature τ at x is defined by $\tau(x) = \sum_{1\le i<j\le n} K(e_i \wedge e_j)$. For $n = 2$, this invariant vanishes trivially. The author's original motivation was to provide answers to a question raised by Chern concerning the existence of minimal isometric immersions into Euclidean space [14]. Therefore, Chen obtained a necessary condition for the existence of minimal isometric immersion from a given Riemannian manifold into Euclidean space and established inequalities for submanifolds in real space forms in terms of the sectional curvature, the scalar curvature and the mean curvature [4]. These inequalities are sharp, and many nice classes of submanifolds realize equality in all above inequalities. Afterwards, many papers studied similar problems for different submanifolds in various ambient space, like complex space forms [5], Sasakian space forms [8], Lorentzian manifolds [13] and quaternionic space forms [7]. In [5],

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Chen established an inequality for a totally real submanifold in a complex space form. Later, Oprea improved this inequality in Lagrangian case[16] by using optimization techniques applied in the setup of Riemmanian geometry [17]. Since then several authors have studied the equality case of this improved inequality, see for instance [10, 11].

Recently, Mihai and Özgür proved Chen's inequalities for submanifolds of real space forms, complex space forms and Sasakian space forms with semi-symmetric metric connections [1, 2]. In this paper, we obtain Chen's inequalities for totally real submanifolds in complex space forms with a semi-symmetric metric connection. We also show that a result of Mihai and Özgür [1, Theorem 4.1] is incorrect and the Corollary 4.2 from [1] is not ideal. For the sake of correcting the results, we establish Chen-Ricci inequalities for submanifolds of real space forms with a semi-symmetric metric connection at the end of Section 3.

2. Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of a Riemannian manifold endowed with a semi-symmetric metric connection are briefly presented.

Let $\overline{M}$ be an m-dimensional Riemannian manifold with Riemannian metric g, the linear connection $\overline{\nabla}$ and the Riemannian connection $\hat{\overline{\nabla}}$. For the vector fields $\overline{X}, \overline{Y}$ on $\overline{M}$ the torsion tensor field $\overline{T}$ of the linear connection $\overline{\nabla}$ is defined by $\overline{T}(\overline{X}, \overline{Y}) = \overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla}_{\overline{Y}}\overline{X} - [\overline{X}, \overline{Y}]$. A liner connection $\overline{\nabla}$ is said to be a semi-symmetric connection if the torsion tensor $\overline{T}$ of the connection $\overline{\nabla}$ satisfies $\overline{T}(\overline{X}, \overline{Y}) = \phi(\overline{Y})\overline{X} - \phi(\overline{X})\overline{Y}$, where ϕ is a 1-form on $\overline{M}$. Further, if $\overline{\nabla}$ satisfies $\overline{\nabla} g = 0$, then $\overline{\nabla}$ is called a semi-symmetric metric connection [12]. In [12], Yano obtained a relation between the semi-symmetric metric connection $\overline{\nabla}$ and the Riemannian connection $\hat{\overline{\nabla}}$ which is given by $\overline{\nabla}_{\overline{X}}\overline{Y} = \hat{\overline{\nabla}}_{\overline{X}}\overline{Y} + \phi(\overline{Y})\overline{X} - g(\overline{X}, \overline{Y})P$, where P is a vector field given by $g(P, \overline{X}) = \phi(\overline{X})$ for any vector field $\overline{X}$ on $\overline{M}$.

Let N^{n+p} be a complex space form of constant holomorphic sectional curvature $4c$ and of complex dimension $n + p$. A submanifold M^n of real dimension n in N^{n+p} is called totally real if the complex structure J of N^{n+p} carries each tangent space TM of M^n into its corresponding normal space $T^{\perp}M$ (see [3]). In particular, for $p = 0$, M^n is Lagrangian.

Let M^n be an n-dimensional totally real submanifold of an $(n+p)$-dimensional complex space form $N^{n+p}(4c)$ with the semi-symmetric metric connection $\overline{\nabla}$ and the Riemannian connection $\hat{\overline{\nabla}}$. On M^n we consider the induced semi-symmetric metric connection denoted by ∇ and the induced Levi-Civita connection denoted by $\hat{\nabla}$. We denote by R and $\hat{R}$ the curvature tensors associated to ∇ and $\hat{\nabla}$, respectively.

The Gauss formulas with respect to ∇, respectively $\hat{\nabla}$, can be written as follows

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \hat{\overline{\nabla}}_X Y = \hat{\nabla}_X Y + \hat{h}(X, Y),$$

for any vector field $\overline{X}$ on N^{n+p}, where h is a $(0, 2)$ symmetric tensor on M^n and $\hat{h}$ is the second fundamental form associated to Riemaniann connection $\hat{\nabla}$ (see [18]). According to the formula (7) from [18] h is also symmetric.

The curvature tensor $\hat{\overline{R}}$ with respect to the Levi-Civita connection $\hat{\overline{\nabla}}$ on $N^{n+p}(4c)$ is expressed by [3]

$$\hat{\overline{R}}(\overline{X},\overline{Y},\overline{Z},\overline{W}) = c\{g(\overline{X},\overline{Z})g(\overline{Y},\overline{W}) - g(\overline{X},\overline{W})g(\overline{Y},\overline{Z}) + g(J\overline{X},\overline{Z})g(J\overline{Y},\overline{W}) - g(J\overline{X},\overline{W})g(J\overline{Y},\overline{Z}) + 2g(\overline{X},J\overline{Y})g(\overline{Z},J\overline{W})\}, \tag{2.1}$$

and the curvature tensor $\overline{R}$ with respect to the semi-symmetric metric connection $\overline{\nabla}$ on N^{n+p} can be written as [15]

$$\overline{R}(\overline{X},\overline{Y},\overline{Z},\overline{W}) = \hat{\overline{R}}(\overline{X},\overline{Y},\overline{Z},\overline{W}) + \alpha(\overline{Y},\overline{Z})g(\overline{X},\overline{W}) - \alpha(\overline{X},\overline{Z})g(\overline{Y},\overline{W}) + \alpha(\overline{X},\overline{W})g(\overline{Y},\overline{Z}) - \alpha(\overline{Y},\overline{W})g(\overline{X},\overline{Z}), \tag{2.2}$$

for any vector fields $\overline{X},\overline{Y},\overline{Z},\overline{W}$ on N^{n+p}, where α is a $(0,2)$-tensor field defined by

$$\alpha(\overline{X},\overline{Y}) = (\hat{\overline{\nabla}}_{\overline{X}}\phi)\overline{Y} - \phi(\overline{X})\phi(\overline{Y}) + \frac{1}{2}\phi(P)g(\overline{X},\overline{Y}).$$

From (2.1) and (2.2) it follows that the curvature tensor $\overline{R}$ can be expressed as

$$\overline{R}(\overline{X},\overline{Y},\overline{Z},\overline{W}) = c\{g(\overline{X},\overline{Z})g(\overline{Y},\overline{W}) - g(\overline{X},\overline{W})g(\overline{Y},\overline{Z}) + g(J\overline{X},\overline{Z})g(J\overline{Y},\overline{W}) - g(J\overline{X},\overline{W})g(J\overline{Y},\overline{Z}) + 2g(\overline{X},J\overline{Y})g(\overline{Z},J\overline{W})\} + \alpha(\overline{Y},\overline{Z})g(\overline{X},\overline{W}) - \alpha(\overline{X},\overline{Z})g(\overline{Y},\overline{W}) + \alpha(\overline{X},\overline{W})g(\overline{Y},\overline{Z}) - \alpha(\overline{Y},\overline{W})g(\overline{X},\overline{Z}). \tag{2.3}$$

For any vector fields X,Y,Z,W on M, the Gauss equation with respect to the semi-symmetric metric connection is [18]

$$R(X,Y,Z,W) = \overline{R}(X,Y,Z,W) + g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z)). \tag{2.4}$$

In N^{n+p} we can choose a local orthonormal frame

$$e_1,\ldots,e_n,e_{n+1},\ldots,e_{n+p},$$

$$e_{1^*} = Je_1,\ldots,e_{n^*} = Je_n, e_{(n+1)^*} = Je_{n+1},\ldots,e_{(n+p)^*} = Je_{n+p}, \tag{2.5}$$

such that, restricting to M^n, $e_1,e_2,\ldots,e_n$ are tangent to M^n. If we write $\hat{h}^{\beta}_{ij} = g(\hat{h}(e_i,e_j),e_\beta)$, we have [3]

$$\hat{h}^{m^*}_{ij} = \hat{h}^{j^*}_{im} = \hat{h}^{i^*}_{jm}. \tag{2.6}$$

Similarly, we write $h^{\beta}_{ij} = g(h(e_i,e_j),e_\beta)$. We denote $\lambda = \sum_{i=1}^{n}\alpha(e_i,e_i)$.

We use the following convention on the ranges of indices unless otherwise stated

$$\beta = n+1,\ldots,n+p,(n+1)^*,\ldots,(n+p)^*; \quad i,j,m = 1,2,\ldots,n.$$

If $\hat{h}^{\beta}_{ij} = k^{\beta}g_{ij}$, where k^r are real-valued functions on M, then M is said to be totally umbilical with respect to Levi-Civita connection. Similarly, if $h^{\beta}_{ij} = k^{\beta}g_{ij}$, then M is said to be totally umbilical with respect to semi-symmetric metric connection [18].

The squared length of h is $\| h \|^2 = \sum_{1\le i<j\le n} g(h(e_i,e_j),h(e_i,e_j))$ and the mean curvature vector of M associated to ∇ is $\zeta = \frac{1}{n}\sum_{i=1}^{n} h(e_i,e_i)$, denoting by H the mean curvature of M^n associated to ∇. Similarly, the mean curvature vector of M^n associated to $\hat{\nabla}$ is $\hat{\zeta} = \frac{1}{n}\sum_{i=1}^{n}\hat{h}(e_i,e_i)$, denoting by $\hat{H}$ the mean curvature of M associated to Riemannian connection $\hat{\nabla}$.

Let $\pi \subset T_xM$ and $\pi^\perp \subset T_x^\perp M$ be plane sections for any x in M^n and $K(\pi)$ the sectional curvature of M^n associated to the induced semi-symmetric metric connection ∇. The scalar curvature τ at x is defined by

$$\tau(x) = \sum_{1\le i<j\le n} K_{ij} = \sum_{1\le i<j\le n} K(e_i \wedge e_j).$$

According to the formula (7) from [18] we have

Lemma 2.1 ([18]) *If P is a tangent vector field on M^n, we have $h = \hat{h}, \zeta = \hat{\zeta}$.*

Lemma 2.2 ([18, Theorem 3]) *A submanifold M of a Riemannian manifold N is totally umbilical if and only if it is totally umbilical with respect to the semi-symmetric metric connection.*

From Lemma 2.1 and (2.6), we immediately have

Lemma 2.3 *If P is a tangent vector field on M^n, then $h_{ij}^{m^*} = h_{im}^{j^*} = h_{jm}^{i^*}$.*

In Section 3, we use a simple way to obtain the relation between the Ricci curvature and the spared mean curvature. We need the following lemma.

Lemma 2.4 *Let $f(x_1, x_2, \dots, x_n)$ be a function in R^n defined by*

$$f(x_1, x_2, \dots, x_n) = x_1 \sum_{i=2}^n x_i.$$

If $x_1 + x_2 + \cdots + x_n = 2\varepsilon$, then we have

$$f(x_1, x_2, \dots, x_n) \le \varepsilon^2$$

with the equality holding if and only if $x_1 = x_2 + x_3 + \cdots + x_n = \varepsilon$.

Proof From $x_1 + x_2 + \cdots + x_n = 2\varepsilon$, we have

$$\sum_{i=2}^n x_i = 2\varepsilon - x_1.$$

It follows that

$$f(x_1, x_2, \dots, x_n) = x_1(2\varepsilon - x_1) = -(x_1 - \varepsilon)^2 + \varepsilon^2.$$

Lemma 2.4 is completed. □

3. Chen-Ricci inequalities

Chen established a relationship between the Ricci curvature and mean curvature for totally real submanifolds of complex space forms [6]. In this paper, we obtain an inequality between the Ricci curvature and mean curvature in the direction of a unit tangent vector X and the mean curvature with respect to the semi-symmetric metric connection, as an answer of the basic problem in submanifold theory which we have mentioned in the introduction.

Theorem 3.1 *Let $M^n, n \ge 2$, be an n-dimensional totally real submanifold of an $(n+p)$-dimensional complex space form $N^{n+p}(4c)$, endowed with a semi-symmetric metric connection*

$\overline{\nabla}$. *For each unit vector X in T_xM we have*

$$\mathrm{Ric}(X) \leq (n-1)c - (n-2)\alpha(X,X) - \lambda + \frac{n^2}{4}H^2, \tag{3.1}$$

where H and α are the mean curvature and characteristic tensor of M^n respect to the semi-symmetric metric connection, respectively. The equality case of (3.1) holds for all unit tangent vectors at x if and only if either

(1) $n \neq 2$, $h^\beta_{ij} = 0$, $\forall i,j,\beta$ or

(2) $n = 2$, $h^\beta_{11} = h^\beta_{22}$, $h^\beta_{12} = 0$, $\forall i,j,\beta$,

where h is a $(0,2)$ symmetric tensor on M^n.

Proof Let $X \in T_xM$ be a unit tangent vector at x. We choose the local field of orthonormal frames (2.5) at x such that $e_1 = X$. From the equations (2.3) and (2.4) it follows that

$$R_{ijij} = c - \alpha(e_i,e_i) - \alpha(e_j,e_j) + \sum_{\beta=n+1}^{(n+p)^*}[h^\beta_{ii}h^\beta_{jj} - (h^\beta_{ij})^2]. \tag{3.2}$$

Using (3.2), one derives

$$\begin{aligned}\mathrm{Ric}(X) &= \sum_{i=2}^{n} R_{1i1i} = (n-1)c - (n-1)\alpha(X,X) - \sum_{i=2}^{n}\alpha(e_i,e_i) + \sum_{\beta=n+1}^{(n+p)^*}\sum_{i=2}^{n}[h^\beta_{11}h^\beta_{ii} - (h^\beta_{1i})^2] \\ &\leq (n-1)c - (n-2)\alpha(X,X) - \lambda + \sum_{\beta=n+1}^{(n+p)^*}\sum_{i=2}^{n} h^\beta_{11}h^\beta_{ii}. \end{aligned} \tag{3.3}$$

Let us consider the quadratic forms $f_\beta : \mathbb{R}^n \to \mathbb{R}$, defined by

$$f_\beta(h^r_{11}, h^r_{22}, \ldots, h^r_{nn}) = \sum_{i=2}^{n} h^\beta_{11}h^\beta_{ii}.$$

We consider the problem $\max f_\beta$, subject to $\Xi : h^\beta_{11} + h^\beta_{22} + \cdots + h^\beta_{nn} = k^\beta$, where k^β is a real constant. From Lemma 2.4, we can see that the solution $(h^\beta_{11}, h^\beta_{22}, \ldots, h^\beta_{nn})$ of the problem in question must satisfy

$$h^\beta_{11} = \sum_{i=2}^{n} h^\beta_{ii} = \frac{k^\beta}{2}, \tag{3.4}$$

which implies

$$f_\beta \leq \frac{(k^\beta)^2}{4}. \tag{3.5}$$

From (3.3) and (3.5) we have

$$\begin{aligned}\mathrm{Ric}(X) &\leq (n-1)a - (n-2)\alpha(X,X) - \lambda + \sum_{\beta=n+1}^{(n+p)^*}\frac{(k^\beta)^2}{4} \\ &= (n-1)a - (n-2)\alpha(X,X) - \lambda + \frac{n^2}{4}H^2.\end{aligned}$$

Next, we shall study the equality case.

If the equality case of inequality (3.1) holds for all unit tangent vectors at x, noting that X is arbitrary, by computing $\mathrm{Ric}(e_j), j=2,3,\ldots,n$ and combining (3.3) and (3.4) we have

$$h_{ij}^{\beta}=0,\quad i\neq j,\quad \forall\beta,$$

$$h_{11}^{\beta}+h_{22}^{\beta}+\cdots+h_{nn}^{\beta}-2h_{ii}^{\beta}=0,\quad \forall i,\beta.$$

We can distinguish two cases:

(1) $n\neq 2$, $h_{ij}^{\beta}=0$, $\forall i,j,\beta$ or

(2) $n=2$, $h_{11}^{\beta}=h_{22}^{\beta}$, $h_{12}^{\beta}=0$, $\forall i,j,\beta$.

The converse is trivial. □

Remark 3.2 In [2], Mihai and Özgür did not establish Chen-Ricci inequalities for submanifolds of complex space forms equipped with a semi-symmetric metric connection. Thus, our main result is not covered by [2].

Theorem 3.3 *If the equality case of inequality (3.1) holds for all unit tangent vectors of M^n, then M^n is a totally umbilical submanifold. Moreover, we have*

(i) The equality case of inequality (3.1) holds for all unit tangent vectors of M^2 if and only if M^2 is a totally umbilical submanifold.

(ii) If P is a tangent vector field on M^n and $n\geq 3$, the equality case of (3.1) holds for all unit tangent vectors of M^n if and only if M^n is a totally geodesic submanifold.

Proof For $n=2$, the equality case of inequality (3.1) holds for all unit tangent vectors of M^2 if and only if M^2 is a totally umbilical submanifold with respect to the semi-symmetric metric connection. Then from Lemma 2.2, M^2 is a totally umbilical submanifold with respect to the Levi-Civita connection. For $n\geq 3$, from Theorem 3.1 the the equality case of inequality (3.1) holds for all unit tangent vectors of M^n if and only if $h_{ij}^{\beta}=0$, $\forall i,j,\beta$. According to the formula (7) from [18], we have $\hat{h}_{ij}^{\beta}=h_{ij}^{\beta}+k^{\beta}g_{ij}$, where k^{β} are real-valued functions on M. Thus, we have $\hat{h}_{ij}^{\beta}=k^{\beta}g_{ij}$, which implies M^n is a totally umbilical submanifold.

If P is a tangent vector field on M^n, from Lemma 2.1 we have $\hat{h}=h$. For $n\geq 3$, from Theorem 3.1 the the equality case of inequality (3.1) holds for all unit tangent vectors of M^n if and only if $h_{ij}^{\beta}=0$, $\forall i,j,\beta$. Thus we have $\hat{h}_{ij}^{\beta}=0$, $\forall i,j,\beta$, which implies M^n is a totally geodesic submanifold. □

Remark 3.4 It is very odd that the coefficients of $\alpha(X,X)$ and λ of inequality (3.1) are different from (4.1) in [1]. By simple calculation, we can show that the inequality (4.1) from [1] is incorrect. In the proof of Theorem 4.1 in [1], they wrote

$$\begin{aligned} n^2\parallel H\parallel^2 &\geq \frac{1}{2}n^2\parallel H\parallel^2+2\big(\tau-\sum_{2\leq i<j\leq n}\big)K_{ij}+2\sum_{r=n+1}^{n+p}\sum_{j=2}^{n}(h_{1j}^{r})^2\\ &=-2(n-1)c+2(2n-3)\lambda-2(n-2)\alpha(e_1,e_1),\end{aligned}$$

but according to (4.2) and (4.3) in [1], one gets

$$\begin{aligned} n^2 \parallel H \parallel^2 &\geq \frac{1}{2}n^2 \parallel H \parallel^2 + 2\big(\tau - \sum_{2\leq i<j\leq n}\big)K_{ij} + 2\sum_{r=n+1}^{n+p}\sum_{j=2}^{n}(h_{1j}^r)^2 \\ &= -2(n-1)c + 2\lambda + 2(n-2)\alpha(e_1, e_1). \end{aligned}$$

This is the reason why they made a mistake.

Remark 3.5 The Corollary 4.2 from [1] is not ideal because Mihai and Özgür only classified submanifolds in real space forms endowed with a semi-symmetric metric connection satisfying the equality case of (4.1) from [1] in the case that P is tangent to the submanifold. From Theorem 3.3, we know that, without the condition that P is tangent to M, we can also classify totally real submanifolds in complex space forms endowed with a semi-symmetric metric connection satisfying the equality case of (3.1).

Under these circumstances it becomes necessary to give a theorem, which could present a sharp Chen-Ricci inequality for submanifolds in real space forms endowed with a semi-symmetric metric connection. The equality case should be considered.

Theorem 3.6 *Let M^n be an n-dimensional submanifold of a real space form $N^{n+p}(c)$ of constant curvature c and of real dimension $n+p$ equipped with a semi-symmetric metric connection. Then:*

(i) For each unit vector X in T_xM we have

$$\mathrm{Ric}(X) \leq (n-1)c - (n-2)\alpha(X,X) - \lambda + \frac{n^2}{4}H^2. \tag{3.6}$$

(ii) If the equality case of inequality (3.6) holds for all unit tangent vectors of M^n, then M^n is a totally umbilical submanifold. Moreover, we have

(1) The equality case of inequality (3.6) holds for all unit tangent vectors of M^2 if and only if M^2 is a totally umbilical submanifold.

(2) If P is a tangent vector field on M^n and $n \geq 3$, the equality case of (3.6) holds for all unit tangent vectors of M^n if and only if M^n is a totally geodesic submanifold.

Remark 3.7 We omit the proof of Theorem 3.6 since it is essentially similar to the proofs of Theorems 3.1 and 3.3.

4. Chen like inequalities relating δ_M

Let $M^n, n \geq 3$, be an n-dimensional submanifold of a complex space form $N^{n+p}(4c)$ of constant holomorphic sectional curvature $4c$ and of complex dimension $n+p$. For any tangent vector field X to M^n, denote $JX = QX + FX$, where QX and FX are the tangential and normal components of JX, respectively. We put $\parallel Q \parallel^2 = \sum_{i,j=1}^{n} g^2(Je_i, e_j)$. Following [2], we denote by $\Theta^2(\pi) = g^2(Qe_1, e_2) = g^2(Je_1, e_2)$, where $\{e_1, e_2\}$ is an orthonormal basis of a 2-plane section π. $\Theta^2(\pi)$ is a real number in $[0, 1]$, independent of choice of e_1, e_2. Denote by $K(\pi)$ the sectional curvature of M^n with respect to the induced semi-symmetric metric connection ∇. For any orthonormal basic $\{e_1, \ldots, e_n\}$ of the tangent space T_xM, the scalar curvature τ at x is defined by $\tau(x)=\sum_{1\le i<j\le n}K(e_i\wedge e_j)$. Mihai and Özgür proved the following

Theorem 4.1 ([2, Theorem 3.1]) *Let M^n, $n\ge 3$, be an n-dimensional submanifold of an $(n+p)$-dimensional complex space form $N^{n+p}(4c)$, endowed with a semi-symmetric metric connection $\overline{\nabla}$. Then we have*

$$\tau(x)-K(\pi)\le \frac{(n-2)(n+1)c}{2}+\frac{n^2(n-2)}{2(n-1)}H^2-(n-2)\lambda-\mathrm{trace}(\alpha\mid_{\pi^\perp})+\frac{3\parallel Q\parallel^2-6\Theta^2(\pi)}{2}c, \tag{4.1}$$

where H and α are the mean curvature and characteristic tensor of M^n respect to the semi-symmetric metric connection, respectively.

If P is tangent to M^n, for Lagrangian submanifolds, from Lemma 2.1 the inequality (4.1) becomes

Corollary 4.2 *Let M^n, $n\ge 3$, be an n-dimensional Lagrangian submanifold of an n-dimensional complex space form $N^n(4c)$, endowed with a semi-symmetric metric connection $\overline{\nabla}$. If P is tangent to M^n, we have*

$$\tau(x)-K(\pi)\le \frac{(n-2)(n+1)c}{2}+\frac{n^2(n-2)}{2(n-1)}\hat{H}^2-(n-2)\lambda-\mathrm{trace}(\alpha\mid_{\pi^\perp}). \tag{4.2}$$

We can obtain an improved inequality (4.2) as follows.

Theorem 4.3 *Let M^n, $n\ge 3$, be an n-dimensional Lagrangian submanifold of an n-dimensional complex space form $N^n(4c)$, endowed with a semi-symmetric metric connection $\overline{\nabla}$. If P is tangent to M^n, we have*

$$\tau(x)-K(\pi)\le \frac{(n-2)(n+1)c}{2}+\frac{n^2(2n-3)}{2(2n+3)}\hat{H}^2-(n-2)\lambda-\mathrm{trace}(\alpha\mid_{\pi^\perp}). \tag{4.3}$$

Proof We consider the point $x\in M^n$. Choose the local field of orthonormal frames (2.5) such that $\{e_1,e_2\}$ is an orthonormal frame in the 2-plane which minimize the sectional curvature at the point x. We remark that

$$\alpha(e_1,e_1)+\alpha(e_2,e_2)=\lambda-\mathrm{trace}(\alpha\mid_{\pi^\perp}). \tag{4.4}$$

Using (3.2) and (4.4), we have

$$\begin{aligned}\tau(x)-K(\pi)=\tau-R_{1212}=&\frac{(n+1)(n-2)c}{2}-\sum_{1\le i<j\le n}[\alpha(e_i,e_i)+\alpha(e_j,e_j)]+\lambda-\mathrm{trace}(\alpha\mid_{\pi^\perp})+\\ &\sum_{m=1}^{n}\Big(\sum_{1\le i<j\le n}h_{ii}^{m^*}h_{jj}^{m^*}-h_{11}^{m^*}h_{22}^{m^*}-\sum_{1\le i<j\le n}(h_{ij}^{m^*})^2+(h_{12}^{m^*})^2\Big)\\ =&\frac{(n+1)(n-2)c}{2}-(n-1)\lambda+\lambda-\mathrm{trace}(\alpha\mid_{\pi^\perp})+\\ &\sum_{m=1}^{n}\Big((h_{11}^{m^*}+h_{22}^{m^*})\sum_{3\le j\le n}h_{jj}^{m^*}+\sum_{3\le i<j\le n}h_{ii}^{m^*}h_{jj}^{m^*}-\sum_{3\le j\le n}(h_{1j}^{m^*})^2-\sum_{2\le i<j\le n}(h_{ij}^{m^*})^2\Big)\\ \le&\frac{(n+1)(n-2)c}{2}-\sum_{2\le i<j\le n}(h_{ij}^{i^*})^2-\sum_{2\le i<j\le n}(h_{ij}^{j^*})^2-(n-2)\lambda-\mathrm{trace}(\alpha\mid_{\pi^\perp})+\end{aligned}$$

$$\sum_{m=1}^{n}\Big((h_{11}^{m^*}+h_{22}^{m^*})\sum_{3\le j\le n}h_{jj}^{m^*}+\sum_{3\le i<j\le n}h_{ii}^{m^*}h_{jj}^{m^*}\Big)-\sum_{3\le j\le n}(h_{1j}^{1^*})^2-\sum_{3\le j\le n}(h_{1j}^{j^*})^2.$$

From Lemma 2.3, we have

$$\begin{aligned}\tau(x)-K(\pi)\le&\frac{(n+1)(n-2)c}{2}-\sum_{3\le j\le n}(h_{jj}^{1^*})^2-\sum_{2\le i\ne j\le n}(h_{jj}^{i^*})^2-(n-2)\lambda-\mathrm{trace}(\alpha\mid_{\pi^\perp})+\\&\sum_{m=1}^{n}\Big((h_{11}^{m^*}+h_{22}^{m^*})\sum_{3\le j\le n}h_{jj}^{m^*}+\sum_{3\le i<j\le n}h_{ii}^{m^*}h_{jj}^{m^*}\Big)-\sum_{3\le j\le n}(h_{11}^{j^*})^2.\end{aligned}\tag{4.5}$$

Let us consider the quadratic forms $f_{m^*}:R^n\to R$, $m=1,2,\dots,n$, defined by

$$f_{1^*}(h_{11}^{1^*},h_{22}^{1^*},\dots,h_{nn}^{1^*})=(h_{11}^{1^*}+h_{22}^{1^*})\sum_{3\le j\le n}h_{jj}^{1^*}+\sum_{3\le i<j\le n}h_{ii}^{1^*}h_{jj}^{1^*}-\sum_{3\le j\le n}(h_{jj}^{1^*})^2,$$

$$f_{2^*}(h_{11}^{2^*},h_{22}^{2^*},\dots,h_{nn}^{2^*})=(h_{11}^{2^*}+h_{22}^{2^*})\sum_{3\le j\le n}h_{jj}^{2^*}+\sum_{3\le i<j\le n}h_{ii}^{2^*}h_{jj}^{2^*}-\sum_{3\le j\le n}(h_{jj}^{2^*})^2,$$

$$\begin{aligned}f_{m^*}(h_{11}^{m^*},h_{22}^{m^*},\dots,h_{nn}^{m^*})=&(h_{11}^{m^*}+h_{22}^{m^*})\sum_{3\le j\le n}h_{jj}^{m^*}+\sum_{3\le i<j\le n}h_{ii}^{m^*}h_{jj}^{m^*}-(h_{11}^{m^*})^2-\\&\sum_{2\le j\le n,j\ne m}(h_{jj}^{m^*})^2,\quad m=3,4,\dots,n.\end{aligned}$$

Carefully reading the proof of Theorem 2 in [16], we can easily get

$$\sum_{m=1}^{n}f_{m^*}\le\frac{n^2(2n-3)}{2(2n+3)}\hat{H}^2.\tag{4.6}$$

Using (4.5) and (4.6) gives

$$\tau(x)-K(\pi)\le\frac{(n-2)(n+1)c}{2}+\frac{n^2(2n-3)}{2(2n+3)}\hat{H}^2-(n-2)\lambda-\mathrm{trace}(\alpha\mid_{\pi^\perp}).\ \Box$$

As an application of Theorem 4.3, we have the following result.

Theorem 4.4 *Let M^n, $n\ge 3$, be an n-dimensional Lagrangian submanifold of an n-dimensional complex space form $N^n(4c)$, endowed with a semi-symmetric metric connection $\overline{\nabla}$. If P is a tangent vector field on M^n, the equality case of inequality (4.2) holds if and only if M is minimal.*

Proof From (4.2) and (4.3), we have

$$\frac{(n-1)n^2}{2(n-1)}\hat{H}^2\le\frac{n^2(2n-3)}{2(2n+3)}\hat{H}^2,$$

which implies

$$\frac{n^2\hat{H}^2}{2}\Big(\frac{2n-3}{2n+3}-\frac{n-2}{n-1}\Big)\ge 0.$$

For any $n\ge 3$, $\frac{2n-3}{2n+3}-\frac{n-2}{n-1}<0$, so we get $\hat{H}=0$ which implies M is minimal. $\Box$

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