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The Nilpotent-Centralizer Methods

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Abstract An $n\times n$ complex sign pattern (ray pattern) $\mathcal{S}$ is said to be spectrally arbitrary if for every monic nth degree polynomial $f(\lambda)$ with coefficients from $\mathbb{C}$, there is a complex matrix in the complex sign pattern class (ray pattern class) of $\mathcal{S}$ such that its characteristic polynomial is $f(\lambda)$. We derive the Nilpotent-Centralizer methods for spectrally arbitrary complex sign patterns and ray patterns, respectively. We find that the Nilpotent-Centralizer methods for three kinds of patterns (sign pattern, complex sign pattern, ray pattern) are the same in form.

Keywords complex sign pattern; ray pattern; spectrally arbitrary pattern; nilpotence.

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1. Introduction

A sign pattern $\mathcal{A}$ of order n is a matrix whose entries are in the set $\{+,-,0\}$. Its sign pattern class is

$$Q(\mathcal{A})=\{A \mid A\in M_n(\mathbb{R}) \text{ and } \operatorname{sgn}(A)=\mathcal{A}\}.$$

For $n\times n$ sign patterns $\mathcal{A}=[a_{kl}]$ and $\mathcal{B}=[b_{kl}]$, the matrix $\mathcal{S}=\mathcal{A}+\mathrm{i}\mathcal{B}$ is called a complex sign pattern of order n, where $\mathrm{i}^2=-1$ (see [1]). Clearly, the (k,l)-entry of $\mathcal{S}$ is $a_{kl}+\mathrm{i}b_{kl}$ for $k,l=1,2,\ldots,n$. Associated with an $n\times n$ complex sign pattern $\mathcal{S}=\mathcal{A}+\mathrm{i}\mathcal{B}$ is a class of complex matrices, called the complex sign pattern class of $\mathcal{S}$, defined by

$$Q_c(\mathcal{S})=\{C=A+\mathrm{i}B \mid A \text{ and } B \text{ are } n\times n \text{ real matrices, and } \operatorname{sgn}(A)=\mathcal{A},\operatorname{sgn}(B)=\mathcal{B}\}.$$

A ray pattern $\mathcal{S}=[s_{jk}]$ of order n is a matrix with entries $s_{jk}\in\{e^{\mathrm{i}\theta}\mid 0\le\theta<2\pi\}\cup\{0\}$, where $\mathrm{i}^2=-1$. Its ray pattern class is

$$Q_r(\mathcal{S})=\{A=[a_{jk}]\in M_n(\mathbb{C})\mid a_{jk}=r_{jk}s_{jk},\ \text{where } r_{jk}\in\mathbb{R}^+ \text{ for } 1\le j,k\le n\}.$$

Let $\mathcal{A}$ be a sign pattern of order $n\ge 2$. If for any given real monic polynomial $f(\lambda)$ of degree n, there is a real matrix $A\in Q(\mathcal{A})$ having characteristic polynomial $f(\lambda)$, then $\mathcal{A}$ is a spectrally arbitrary sign pattern (SAP).

Let $\mathcal{S}=\mathcal{A}+\mathrm{i}\mathcal{B}$ be a complex sign pattern of order $n\ge 2$. If for every monic nth degree polynomial $f(\lambda)$ with coefficients from $\mathbb{C}$, there is a complex matrix in $Q_c(\mathcal{S})$ such that its

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characteristic polynomial is $f(\lambda)$, then $\mathcal{S}$ is said to be a spectrally arbitrary complex sign pattern (SAC) (see [2]).

A ray pattern $\mathcal{S}$ of order n ($n \geq 2$) is said to be spectrally arbitrary (SAR) if each monic polynomial of degree n with coefficients from $\mathbb{C}$ is the characteristic polynomial of some matrix in $Q_r(\mathcal{S})$ (see [3]).

For two $n \times n$ sign patterns $\mathcal{A} = [a_{kl}]$ and $\mathcal{B} = [b_{kl}]$, if $a_{kl} = b_{kl}$ whenever $b_{kl} \neq 0$, then $\mathcal{A}$ is a superpattern of $\mathcal{B}$, and $\mathcal{B}$ is a subpattern of $\mathcal{A}$. Note that each sign pattern is a superpattern and a subpattern of itself.

For two $n \times n$ complex sign patterns $\mathcal{S}_1 = \mathcal{A}_1 + \mathrm{i}\mathcal{B}_1$ and $\mathcal{S}_2 = \mathcal{A}_2 + \mathrm{i}\mathcal{B}_2$, if $\mathcal{A}_1$ is a superpattern of $\mathcal{A}_2$, and $\mathcal{B}_1$ is a superpattern of $\mathcal{B}_2$, then $\mathcal{S}_1$ is a superpattern of $\mathcal{S}_2$, and $\mathcal{S}_2$ is a subpattern of $\mathcal{S}_1$.

A ray pattern $\mathcal{S}_1 = [p_{jk}]$ is a superpattern of a ray pattern $\mathcal{S}_2 = [s_{jk}]$ if $p_{jk} = s_{jk}$ whenever $s_{jk} \neq 0$. And $\mathcal{S}_2$ is a subpattern of $\mathcal{S}_1$ if $\mathcal{S}_1$ is a superpattern of $\mathcal{S}_2$.

A sign pattern $\mathcal{A}$ (complex sign pattern $\mathcal{S}$, ray pattern $\mathcal{S}$) is said to be potentially nilpotent (PN) if there exists a matrix $B \in Q(\mathcal{A})$ ($B \in Q_c(\mathcal{S})$, $B \in Q_r(\mathcal{S})$) such that $B^k = 0$ for some positive integer k.

For a complex sign pattern $\mathcal{S} = \mathcal{A} + \mathrm{i}\mathcal{B}$, the sign patterns $\mathcal{A}$ and $\mathcal{B}$ are the real part pattern and imaginary part pattern of $\mathcal{S}$, respectively. Let C be an $n \times n$ complex matrix. We use $\Re(C)_{lk}$ (respectively, $\Im(C)_{lk}$) to denote the real part (respectively, imaginary part) of the (l, k) entry of C.

Let C be an $n \times n$ real or complex matrix. If there is some positive integer k such that $C^k = 0$, then C is said to be a nilpotent matrix. The smallest such k is called the nilpotence index (simply, index) of C.

The problem of classifying the spectrally arbitrary sign patterns was introduced in [4] by Drew et al. In their article, they developed the Nilpotent-Jacobian method for showing that a sign pattern and all its superpatterns are spectrally arbitrary. In [2, 3], the concepts of spectrally arbitrary ray patterns and spectrally arbitrary complex sign patterns were introduced, and the two articles extended the Nilpotent-Jacobian method for sign patterns to ray patterns and complex sign patterns, respectively. Work on spectrally arbitrary patterns (sign patterns, ray patterns, and complex sign patterns) has continued in several articles including [2–9].

In [6], Garnett and Shader derived the Nilpotent-Centralizer method to prove a sign pattern to be spectrally arbitrary.

Theorem 1.1 (Nilpotent-Centralizer Method) ([6]) *Let $\mathcal{A}$ be an $n \times n$ sign pattern and A be a nilpotent realization of index n of $\mathcal{A}$. If the only matrix in the centralizer of A satisfying $B \circ A^T = 0$ is the zero matrix, then the pattern $\mathcal{A}$ and each of its superpatterns is a spectrally arbitrary pattern.*

Motivated by [6], in this work, we give the Nilpotent-Centralizer methods for spectrally arbitrary complex sign patterns and ray patterns, respectively. We find that the Nilpotent-Centralizer methods for three kinds of patterns (sign pattern, complex sign pattern, ray pattern) are the same in form.

2. The Nilpotent-Centralizer method for spectrally arbitrary complex sign patterns

In [2], a means to show that a complex sign pattern and all its superpatterns are spectrally arbitrary was established.

Theorem 2.1 (Nilpotent-Jacobian Method) ([2]) *Let $\mathcal{S} = \mathcal{A} + \mathrm{i}\mathcal{B}$ be a complex sign pattern of order $n \geq 2$, and suppose that there exists some nilpotent complex matrix $C = A + \mathrm{i}B \in Q_c(\mathcal{S})$, where $A \in Q(\mathcal{A})$, $B \in Q(\mathcal{B})$, and A and B have at least $2n$ nonzero entries, say $a_{i_1j_1}, \ldots, a_{i_{n_1}j_{n_1}}, b_{i_{n_1+1}j_{n_1+1}}, \ldots, b_{i_{2n}j_{2n}}$. Let X be the complex matrix obtained by replacing these entries in C by variables $x_1, \ldots, x_{2n}$, and the characteristic polynomial of X be*

$$\begin{aligned}|\lambda I - X| =& \lambda^n + (f_1(x_1, x_2, \ldots, x_{2n}) + \mathrm{i}g_1(x_1, x_2, \ldots, x_{2n}))\lambda^{n-1} + \cdots + \\ & (f_{n-1}(x_1, x_2, \ldots, x_{2n}) + \mathrm{i}g_{n-1}(x_1, x_2, \ldots, x_{2n}))\lambda + \\ & (f_n(x_1, x_2, \ldots, x_{2n}) + \mathrm{i}g_n(x_1, x_2, \ldots, x_{2n})).\end{aligned}$$

If the Jacobian matrix $J = \frac{\partial(f_1,\ldots,f_n,g_1,\ldots,g_n)}{\partial(x_1,x_2,\ldots,x_{2n})}$ is nonsingular at

$$(x_1, \ldots, x_{2n}) = (a_{i_1j_1}, \ldots, a_{i_{n_1}j_{n_1}}, b_{i_{n_1+1}j_{n_1+1}}, \ldots, b_{i_{2n}j_{2n}}),$$

then the complex sign pattern $\mathcal{S}$ is spectrally arbitrary, and every superpattern of $\mathcal{S}$ is a spectrally arbitrary complex sign pattern.

In the following, we show a slight generalization of the Nilpotent-Jacobian method for spectrally arbitrary complex sign pattern.

Let $C = A + \mathrm{i}B$ be an $n \times n$ nilpotent matrix with m nonzero entries $a_{i_1j_1}, \ldots, a_{i_{n_1}j_{n_1}}$, $b_{i_{n_1+1}j_{n_1+1}}, \ldots, b_{i_mj_m}$, and $C(x_1, x_2, \ldots, x_m)$ be the matrix obtained from C by replacing those nonzero entries by x_k for $k = 1, 2, \ldots, m$, where $x_1, x_2, \ldots, x_m$ are distinct indeterminates. Then there exist polynomials $\beta_1 = f_1 + \mathrm{i}g_1, \beta_2 = f_2 + \mathrm{i}g_2, \ldots, \beta_n = f_n + \mathrm{i}g_n$, where f_i, g_i are in $\mathbb{R}[x_1, x_2, \ldots, x_m]$, such that the characteristic polynomial of $C(x_1, x_2, \ldots, x_m)$ is $x^n + \beta_1 x^{n-1} + \cdots + \beta_{n-1}x + \beta_n$. Define $\hbar : \mathbb{R}^m \to \mathbb{R}^{2n}$ by

$$\begin{aligned}&\hbar(x_1, x_2, \ldots, x_m) \\ &\quad = (f_1(x_1, x_2, \ldots, x_m), \ldots, f_n(x_1, x_2, \ldots, x_m), g_1(x_1, x_2, \ldots, x_m), \ldots, g_n(x_1, x_2, \ldots, x_m)).\end{aligned}$$

We call $\hbar$ the polynomial map of C. The Jacobian of $\hbar$, denoted $\mathrm{Jac}(\hbar)$, is defined as

$$\mathrm{Jac}(\hbar) = \frac{\partial(f_1, f_2, \ldots, f_n, g_1, g_2, \ldots, g_n)}{\partial(x_1, x_2, \ldots, x_m)},$$

and set

$$\mathrm{Jac}(\hbar)|_C = \mathrm{Jac}(\hbar)|_{(x_1,x_2,\ldots,x_m)=(a_{i_1j_1},\ldots,a_{i_{n_1}j_{n_1}},b_{i_{n_1+1}j_{n_1+1}},\ldots,b_{i_mj_m})}.$$

Now, we will try to give the Nilpotent-Centralizer method to show that a complex sign pattern is spectrally arbitrary. Some results are similar to the corresponding results in [6].

Theorem 2.2 (Extended Nilpotent-Jacobian Method) *Let $\mathcal{S}$ be a complex sign pattern of order n, $C \in Q_c(\mathcal{S})$ be a nilpotent complex matrix and $\hbar$ be the polynomial map of C. If $\mathrm{Jac}(\hbar)|_C$ has rank $2n$, then the complex sign pattern $\mathcal{S}$, and each superpattern of $\mathcal{S}$, is a SAC.*

Proof Assume that $\mathrm{Jac}(\hbar)|_C$ has rank $2n$. Then $\mathrm{Jac}(\hbar)|_C$ has $2n$ linearly independent columns. Let P be the set of entries of C corresponding to these columns. By applying the Nilpotent-Jacobian method, the resulting Jacobian matrix evaluated at the initial points will have rank $2n$, therefore is nonsingular. So by Theorem 2.1, the complex sign pattern $\mathcal{S}$ and each superpattern of $\mathcal{S}$ is a SAC. The theorem follows.

Lemma 2.3 *Let C be an $n \times n$ complex matrix, $Y = [y_{kl}] = [p_{kl} + \mathrm{i}q_{kl}]$ be a complex matrix with the same complex sign pattern as C and whose nonzero entries are distinct indeterminates. Let the characteristic polynomial of Y be*

$$c_Y = \det(xI - Y) = x^n + (f_1 + \mathrm{i}g_1)x^{n-1} + \cdots + (f_{n-1} + \mathrm{i}g_{n-1})x + (f_n + \mathrm{i}g_n),$$

where $f_m = f_m(p_{11}, p_{12}, \ldots, p_{nn}, q_{11}, q_{12}, \ldots, q_{nn})$, and $g_m = g_m(p_{11}, p_{12}, \ldots, p_{nn}, q_{11}, q_{12}, \ldots, q_{nn})$ for $m = 1, 2, \ldots, n$. Then for (k, l) with $p_{kl} \neq 0$, we have

$$\begin{cases} \dfrac{\partial f_m}{\partial p_{kl}}\Big|_{Y=C} = \text{the real part of the coefficient of } x^{n-m} \text{ in } -\mathrm{adj}(xI - C)_{lk}, \\ \dfrac{\partial g_m}{\partial p_{kl}}\Big|_{Y=C} = \text{the imaginary part of the coefficient of } x^{n-m} \text{ in } -\mathrm{adj}(xI - C)_{lk}, \end{cases}$$

and for (k, l) with $q_{kl} \neq 0$, we have

$$\begin{cases} -\dfrac{\partial f_m}{\partial q_{kl}}\Big|_{Y=C} = \text{the imaginary part of the coefficient of } x^{n-m} \text{ in } -\mathrm{adj}(xI - C)_{lk}, \\ \dfrac{\partial g_m}{\partial q_{kl}}\Big|_{Y=C} = \text{the real part of the coefficient of } x^{n-m} \text{ in } -\mathrm{adj}(xI - C)_{lk}. \end{cases}$$

Proof Let E_{kl} be the $n \times n$ matrix which has 1 in the (k, l)-entry and 0's elsewhere.

For (k, l) with $p_{kl} \neq 0$, we have

$$\begin{aligned} \frac{\partial c_Y}{\partial p_{kl}} &= \lim_{h \to 0} \frac{\det(xI - (Y + hE_{kl})) - \det(xI - Y)}{h} \\ &= \lim_{h \to 0} \frac{\det(xI - Y) - h(-1)^{k+l} \det((xI - Y)(k, l)) - \det(xI - Y)}{h} \\ &= -\mathrm{adj}(xI - Y)_{lk}. \end{aligned}$$

Hence

$$\begin{aligned} -\mathrm{adj}(xI - Y)_{lk} = &\frac{\partial c_Y}{\partial p_{kl}} = \Big(\frac{\partial f_1}{\partial p_{kl}} + \mathrm{i}\frac{\partial g_1}{\partial p_{kl}}\Big)x^{n-1} + \Big(\frac{\partial f_2}{\partial p_{kl}} + \mathrm{i}\frac{\partial g_2}{\partial p_{kl}}\Big)x^{n-2} + \\ &\cdots + \Big(\frac{\partial f_{n-1}}{\partial p_{kl}} + \mathrm{i}\frac{\partial g_{n-1}}{\partial p_{kl}}\Big)x + \Big(\frac{\partial f_n}{\partial p_{kl}} + \mathrm{i}\frac{\partial g_n}{\partial p_{kl}}\Big), \end{aligned}$$

and so

$$\begin{cases} \dfrac{\partial f_m}{\partial p_{kl}}\Big|_{Y=C} = \text{the real part of the coefficient of } x^{n-m} \text{ in } -\mathrm{adj}(xI-C)_{lk}, \\ \dfrac{\partial g_m}{\partial p_{kl}}\Big|_{Y=C} = \text{the imaginary part of the coefficient of } x^{n-m} \text{ in } -\mathrm{adj}(xI-C)_{lk}. \end{cases}$$

Similarly, for (k,l) with $q_{kl} \neq 0$, we have

$$\begin{aligned} \frac{\partial c_Y}{\partial q_{kl}} &= \lim_{h\to 0} \frac{\det(xI-(Y+\mathrm{i}hE_{kl})) - \det(xI-Y)}{h} \\ &= \lim_{h\to 0} \frac{\det(xI-Y) - \mathrm{i}h(-1)^{k+l}\det((xI-Y)(k,l)) - \det(xI-Y)}{h} \\ &= -\mathrm{i}\ \mathrm{adj}(xI-Y)_{lk}. \end{aligned}$$

Then

$$\begin{aligned} -\mathrm{i}\ \mathrm{adj}(xI-Y)_{lk} = \frac{\partial c_Y}{\partial q_{kl}} =& \Big(\frac{\partial f_1}{\partial q_{kl}} + \mathrm{i}\frac{\partial g_1}{\partial q_{kl}}\Big)x^{n-1} + \Big(\frac{\partial f_2}{\partial q_{kl}} + \mathrm{i}\frac{\partial g_2}{\partial q_{kl}}\Big)x^{n-2} + \\ & \cdots + \Big(\frac{\partial f_{n-1}}{\partial q_{kl}} + \mathrm{i}\frac{\partial g_{n-1}}{\partial q_{kl}}\Big)x + \Big(\frac{\partial f_n}{\partial q_{kl}} + \mathrm{i}\frac{\partial g_n}{\partial q_{kl}}\Big), \end{aligned}$$

equivalently,

$$\begin{aligned} -\mathrm{adj}(xI-Y)_{lk} =& \Big(\frac{\partial g_1}{\partial q_{kl}} - \mathrm{i}\frac{\partial f_1}{\partial q_{kl}}\Big)x^{n-1} + \Big(\frac{\partial g_2}{\partial q_{kl}} - \mathrm{i}\frac{\partial f_2}{\partial q_{kl}}\Big)x^{n-2} + \\ & \cdots + \Big(\frac{\partial g_{n-1}}{\partial q_{kl}} - \mathrm{i}\frac{\partial f_{n-1}}{\partial q_{kl}}\Big)x + \Big(\frac{\partial g_n}{\partial q_{kl}} - \mathrm{i}\frac{\partial f_n}{\partial q_{kl}}\Big). \end{aligned}$$

Thus

$$\begin{cases} -\dfrac{\partial f_m}{\partial q_{kl}}\Big|_{Y=C} = \text{the imaginary part of the coefficient of } x^{n-m} \text{ in } -\mathrm{adj}(xI-C)_{lk}, \\ \dfrac{\partial g_m}{\partial q_{kl}}\Big|_{Y=C} = \text{the real part of the coefficient of } x^{n-m} \text{ in } -\mathrm{adj}(xI-C)_{lk}. \end{cases}$$

The lemma follows.

Lemma 2.4 ([6]) *If the $n \times n$ complex matrix C is nilpotent, then*

$$\mathrm{adj}(xI-C)_{lk} = (x^{n-1}I + x^{n-2}C + x^{n-3}C^2 + \cdots + C^{n-1})_{lk} = \sum_{m=0}^{n-1} x^{n-m-1}(C^m)_{lk}.$$

Lemmas 2.3 and 2.4 imply the following theorem.

Theorem 2.5 *Let C be an $n \times n$ nilpotent complex matrix, $Y = [y_{kl}] = [p_{kl} + \mathrm{i}q_{kl}]$ be a complex matrix with the same complex sign pattern as C and whose nonzero entries are distinct indeterminates. Let the characteristic polynomial of Y be*

$$c_Y = \det(xI-Y) = x^n + (f_1+\mathrm{i}g_1)x^{n-1} + \cdots + (f_{n-1}+\mathrm{i}g_{n-1})x + (f_n+\mathrm{i}g_n),$$

where $f_m = f_m(p_{11}, p_{12}, \ldots, p_{nn}, q_{11}, q_{12}, \ldots, q_{nn})$, and $g_m = g_m(p_{11}, p_{12}, \ldots, p_{nn}, q_{11}, q_{12}, \ldots, q_{nn})$

for $m = 1, 2, \ldots, n$. Then for (k, l) with $p_{kl} \neq 0$, we have

$$\begin{cases} \dfrac{\partial f_m}{\partial p_{kl}}\Big|_{Y=C} = -\Re(C^{m-1})_{lk}, \\ \dfrac{\partial g_m}{\partial p_{kl}}\Big|_{Y=C} = -\Im(C^{m-1})_{lk} = -\Re(-\mathrm{i}C^{m-1})_{lk}, \end{cases}$$

and for (k, l) with $q_{kl} \neq 0$, we have

$$\begin{cases} \dfrac{\partial f_m}{\partial q_{kl}}\Big|_{Y=C} = \Im(C^{m-1})_{lk}, \\ \dfrac{\partial g_m}{\partial q_{kl}}\Big|_{Y=C} = -\Re(C^{m-1})_{lk} = \Im(-iC^{m-1})_{lk}. \end{cases}$$

For two complex patterns $\mathcal{S}_1 = \mathcal{A}_1 + \mathrm{i}\mathcal{B}_1$ and $\mathcal{S}_2 = \mathcal{A}_2 + \mathrm{i}\mathcal{B}_2$, the operating $\mathcal{S}_1 \circ \mathcal{S}_2$ is defined as $\mathcal{S}_1 \circ \mathcal{S}_2 = \mathcal{A}_1 \circ \mathcal{A}_2 + \mathrm{i}(\mathcal{B}_1 \circ \mathcal{B}_2)$.

Example 2.6 Consider the following 3×3 complex sign pattern

$$\mathcal{S} = \begin{bmatrix} 1-\mathrm{i} & 1 & 0 \\ 1+\mathrm{i} & 0 & -1 \\ 1 & 0 & -1+\mathrm{i} \end{bmatrix}.$$

The complex matrix

$$C = \begin{bmatrix} 1-\mathrm{i}\sqrt{3} & 1 & 0 \\ 2+\mathrm{i}2\sqrt{3} & 0 & -1 \\ 8 & 0 & -1+\mathrm{i}\sqrt{3} \end{bmatrix} \in Q_c(\mathcal{S})$$

is nilpotent (see Page 689 in [2]), and

$$C^0 = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, C^1 = \begin{bmatrix} 1-\mathrm{i}\sqrt{3} & 1 & 0 \\ 2+\mathrm{i}2\sqrt{3} & 0 & -1 \\ 8 & 0 & -1+\mathrm{i}\sqrt{3} \end{bmatrix}, C^2 = \begin{bmatrix} 0 & 1-\mathrm{i}\sqrt{3} & -1 \\ 0 & 2+\mathrm{i}2\sqrt{3} & 1-\mathrm{i}\sqrt{3} \\ 0 & 8 & -2-\mathrm{i}2\sqrt{3} \end{bmatrix}.$$

Set

$$Y = \begin{bmatrix} p_{11}+\mathrm{i}q_{11} & p_{12} & 0 \\ p_{21}+\mathrm{i}q_{21} & 0 & p_{23} \\ p_{31} & 0 & p_{33}+\mathrm{i}q_{33} \end{bmatrix},$$

and

$$c_Y = \det(xI - Y) = x^3 + (f_1 + \mathrm{i}g_1)x^2 + (f_2 + \mathrm{i}g_2)x + (f_3 + \mathrm{i}g_3).$$

Then

$$\begin{cases} f_1 = -p_{11} - p_{33}, \\ f_2 = -p_{12}p_{21} + p_{11}p_{33} - q_{11}q_{33}, \\ f_3 = -p_{12}p_{23}p_{31} + p_{12}p_{21}p_{33} - p_{12}q_{21}q_{33}, \\ g_1 = -q_{11} - q_{33}, \\ g_2 = p_{11}q_{33} + p_{33}q_{11} - p_{12}q_{21}, \\ g_3 = p_{12}p_{33}q_{21} + p_{12}p_{21}q_{33}. \end{cases}$$

Calculations show that

$$\mathrm{Jac}(\hbar)=\frac{\partial(f_1,f_2,f_3,g_1,g_2,g_3)}{\partial(p_{11},p_{12},p_{21},p_{23},p_{31},p_{33},q_{11},q_{21},q_{33})}$$

$$=\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ p_{33} & -p_{21} & -p_{12} & 0 & 0 & p_{11} & -q_{33} & 0 & -q_{11} \\ 0 & p_{21}p_{33}-p_{23}p_{31}-q_{21}q_{33} & p_{12}p_{33} & -p_{12}p_{31} & -p_{12}p_{23} & p_{12}p_{21} & 0 & -p_{12}q_{33} & -p_{12}q_{21} \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ q_{33} & -q_{21} & 0 & 0 & 0 & q_{11} & p_{33} & -p_{12} & p_{11} \\ 0 & p_{33}q_{21}+p_{21}q_{33} & p_{12}q_{33} & 0 & 0 & p_{12}q_{21} & 0 & p_{12}p_{33} & p_{12}p_{21} \end{bmatrix},$$

and

$$\mathrm{Jac}(\hbar)|_C=\left[\begin{array}{cccccc|ccc} -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & -2 & -1 & 0 & 0 & 1 & -\sqrt{3} & 0 & \sqrt{3} \\ 0 & 0 & -1 & -8 & 1 & 2 & 0 & -\sqrt{3} & -2\sqrt{3} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ \sqrt{3} & -2\sqrt{3} & 0 & 0 & 0 & -\sqrt{3} & -1 & -1 & 1 \\ 0 & 0 & \sqrt{3} & 0 & 0 & 2\sqrt{3} & 0 & -1 & 2 \end{array}\right].$$

Let $\widehat{\mathrm{Jac}}(\hbar)|_C$ be the matrix obtained from $\mathrm{Jac}(\hbar)|_C$ by changing the signs of the first six columns, that is,

$$\widehat{\mathrm{Jac}}(\hbar)|_C=\left[\begin{array}{cccccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & -1 & -\sqrt{3} & 0 & \sqrt{3} \\ 0 & 0 & 1 & 8 & -1 & -2 & 0 & -\sqrt{3} & -2\sqrt{3} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ -\sqrt{3} & 2\sqrt{3} & 0 & 0 & 0 & \sqrt{3} & -1 & -1 & 1 \\ 0 & 0 & -\sqrt{3} & 0 & 0 & -2\sqrt{3} & 0 & -1 & 2 \end{array}\right].$$

Theorem 2.5 states that the entries in a column of $\widehat{\mathrm{Jac}}(\hbar)|_C$ are corresponding to the relevant entries of matrices

$$C^0, C^1, C^2, -\mathrm{i}C^0, -\mathrm{i}C^1, -\mathrm{i}C^2, \tag{2.1}$$

where,

$$C^0=\begin{bmatrix} 1 & 0+\mathrm{i}0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad C^1=\begin{bmatrix} 1-\mathrm{i}\sqrt{3} & 1+\mathrm{i}0 & 0 \\ 2+\mathrm{i}2\sqrt{3} & 0 & -1 \\ 8 & 0 & -1+\mathrm{i}\sqrt{3} \end{bmatrix},$$

$$C^2=\begin{bmatrix} 0 & 1-\mathrm{i}\sqrt{3} & -1 \\ 0 & 2+\mathrm{i}2\sqrt{3} & 1-\mathrm{i}\sqrt{3} \\ 0 & 8 & -2-\mathrm{i}2\sqrt{3} \end{bmatrix},\quad -\mathrm{i}C^0=\begin{bmatrix} 0-\mathrm{i} & 0+\mathrm{i}0 & 0 \\ 0 & -\mathrm{i} & 0 \\ 0 & 0 & -\mathrm{i} \end{bmatrix},$$

$$-\mathrm{i}C^1=\begin{bmatrix} -\sqrt{3}-\mathrm{i} & 0-\mathrm{i} & 0 \\ 2\sqrt{3}-\mathrm{i}2 & 0 & \mathrm{i} \\ -\mathrm{i}8 & 0 & \sqrt{3}+\mathrm{i} \end{bmatrix},\quad -\mathrm{i}C^2=\begin{bmatrix} 0 & -\sqrt{3}-\mathrm{i} & \mathrm{i} \\ 0 & 2\sqrt{3}-\mathrm{i}2 & -\sqrt{3}-\mathrm{i} \\ 0 & -\mathrm{i}8 & -2\sqrt{3}+\mathrm{i}2 \end{bmatrix}.$$

The following facts are clear.

(1) The entries of the first six columns of $\widehat{\mathrm{Jac}}(\hbar)|_C$ are corresponding to the real parts of the relevant entries of matrices in (2.1). For example, the entries of the first column of $\widehat{\mathrm{Jac}}(\hbar)|_C$ are the real parts of (1,1) entries of matrices in (2.1), respectively; the entries of the third column of $\widehat{\mathrm{Jac}}(\hbar)|_C$ are the real parts of (1,2) entries of matrices in (2.1), respectively.

(2) The entries of the last three columns of $\widehat{\mathrm{Jac}}(\hbar)|_C$ are corresponding to the imaginary parts of the relevant entries of matrices in (2.1). For example, the entries of the eighth column of $\widehat{\mathrm{Jac}}(\hbar)|_C$ are the imaginary parts of (1,2) entries of matrices in (2.1), respectively.

Note that $\mathrm{rank}(\mathrm{Jac}(\hbar)|_C) < 6 \Longleftrightarrow \mathrm{rank}(\widehat{\mathrm{Jac}}(\hbar)|_C) < 6 \Longleftrightarrow$ the row vectors of $\widehat{\mathrm{Jac}}(\hbar)|_C$ are linearly dependent $\Longleftrightarrow$ there exist real numbers $a_m (m = 1, 2, \dots, 6)$, not all 0, such that

$$(a_1C^0 + a_2C^1 + a_3C^2 - \mathrm{i}a_4C^0 - \mathrm{i}a_5C^1 - \mathrm{i}a_6C^2) \circ C^T = 0. \tag{2.2}$$

We can represent (2.2) as

$$((a_1 - \mathrm{i}a_4)C^0 + (a_2 - \mathrm{i}a_5)C^1 + (a_3 - \mathrm{i}a_6)C^2) \circ C^T = 0.$$

So $\mathrm{rank}(\mathrm{Jac}(\hbar)|_C) < 6 \Longleftrightarrow$ there exist complex numbers $c_m (m = 1, 2, 3)$, not all 0, such that

$$(c_1C^0 + c_2C^1 + c_3C^2) \circ C^T = 0. \tag{2.3}$$

Now we get the following theorem.

Theorem 2.7 *Let C be an $n \times n$ nilpotent complex matrix, and let $\hbar$ be its polynomial map. Then $\mathrm{Jac}(\hbar)|_C$ has rank less than $2n$ if and only if there exists a nonzero polynomial $v(x) \in \mathbb{C}[x]$ of degree at most $n - 1$ such that $v(C) \circ C^T = 0$.*

Proof $\mathrm{Jac}(\hbar)|_C$ has rank less than $2n$ if and only if there exist real numbers $a_m (m = 1, 2, \dots, 2n)$, not all 0, such that

$$(a_1C^0 + a_2C^1 + \cdots + a_nC^{n-1} - \mathrm{i}a_{n+1}C^0 - \mathrm{i}a_{n+2}C^1 - \cdots - \mathrm{i}a_{2n}C^{n-1}) \circ C^T = 0. \tag{2.4}$$

We can represent (2.4) as

$$((a_1 - \mathrm{i}a_{n-1})C^0 + (a_2 - \mathrm{i}a_{n+2})C^1 + \cdots + (a_n - \mathrm{i}a_{2n})C^{n-1}) \circ C^T = 0. \tag{2.5}$$

Therefore $\mathrm{Jac}(\hbar)|_C$ has rank less than $2n$ if and only if there is a nonzero polynomial $v(x) \in \mathbb{C}[x]$ (namely, $v(x) = (a_1 - \mathrm{i}a_{n-1}) + (a_2 - \mathrm{i}a_{n+2})x + \cdots + (a_n - \mathrm{i}a_{2n})x^{n-1}$) of degree at most $n - 1$ such that $v(C) \circ C^T = 0$. The theorem follows.

In the following context, the proofs of Lemma 2.8 and Theorem 2.9 are the same as Lemma 3.6 and Theorem 3.7 in [6], respectively. We will omit them.

Lemma 2.8 *Let C be an $n \times n$ nilpotent complex matrix of index n. Then there exists a nonzero polynomial $v(x) \in \mathbb{C}[x]$ of degree at most $n - 1$ such that $v(C) \circ C^T = 0$ if and only if there exists a nonzero complex matrix H in the centralizer of C such that $H \circ C^T = 0$.*

Theorem 2.9 (Nilpotent-Centralizer Method for spectrally arbitrary complex sign patterns) *Let $\mathcal{S}$ be a complex sign pattern of order n, and $C \in Q_c(\mathcal{S})$ be a nilpotent complex matrix of* *index n. If the only complex matrix H in the centralizer of C satisfying $H \circ C^T = 0$ is the zero matrix, then the complex sign pattern $\mathcal{S}$ and each of its superpatterns is spectrally arbitrary.*

3. The Nilpotent-Centralizer method for spectrally arbitrary ray patterns

In [3], a means to show that a ray pattern and all its superpatterns are spectrally arbitrary was established as follows.

(1) Find a nilpotent matrix in the given ray pattern class.

(2) Change $2n$ of the positive coefficients (denoted $r_1, r_2, \ldots, r_{2n}$) of the $e^{\mathrm{i}\theta_{ij}}$ in this nilpotent matrix to variables $t_1, t_2, \ldots, t_{2n}$.

(3) Express the characteristic polynomial of the resulting matrix as:

$$\lambda^n + (f_1(t_1, t_2, \ldots, t_{2n}) + \mathrm{i}g_1(t_1, t_2, \ldots, t_{2n}))\lambda^{n-1} + \cdots +$$
$$(f_{n-1}(t_1, t_2, \ldots, t_{2n}) + \mathrm{i}g_{n-1}(t_1, t_2, \ldots, t_{2n}))\lambda + (f_n(t_1, t_2, \ldots, t_{2n}) + \mathrm{i}g_n(t_1, t_2, \ldots, t_{2n})).$$

(4) Find the Jacobian matrix

$$J = \frac{\partial(f_1, f_2, \ldots, f_n, g_1, g_2, \ldots, g_n)}{\partial(t_1, t_2, \ldots, t_{2n})}.$$

(5) If the determinant of J, evaluated at $(t_1, t_2, \ldots, t_{2n}) = (r_1, r_2, \ldots, r_{2n})$ is nonzero, then the given ray pattern and all of its superpatterns are spectrally arbitrary.

In the following, we show a slight generalization of the Nilpotent-Jacobian method for spectrally arbitrary ray patterns.

Let $C = [c_{jk}]$ be an $n \times n$ nilpotent matrix with m nonzero entries $c_{i_1j_1}, c_{i_2j_2}, \ldots, c_{i_mj_m}$, where $c_{i_kj_k} = r_k e^{\mathrm{i}\theta_k}$, $k = 1, 2, \ldots, m$. Let $C(x_1, x_2, \ldots, x_m)$ be the matrix obtained from C by replacing r_k by x_k for $k = 1, \ldots, m$, where $x_1, x_2, \ldots, x_m$ are distinct indeterminates. Then there exist polynomials $\beta_1 = f_1 + \mathrm{i}g_1, \beta_2 = f_2 + \mathrm{i}g_2, \ldots, \beta_n = f_n + \mathrm{i}g_n$, where f_i, g_i is in $\mathbb{R}[x_1, x_2, \ldots, x_m]$, such that the characteristic polynomial of $C(x_1, x_2, \ldots, x_m)$ is $x^n + \beta_1 x^{n-1} + \cdots + \beta_{n-1}x + \beta_n$. Define $\tilde{h} : \mathbb{R}^m \to \mathbb{R}^{2n}$ by

$$\tilde{h}(x_1, x_2, \ldots, x_m)$$
$$= (f_1(x_1, x_2, \ldots, x_m), \ldots, f_n(x_1, x_2, \ldots, x_m), g_1(x_1, x_2, \ldots, x_m), \ldots, g_n(x_1, x_2, \ldots, x_m)).$$

We call $\tilde{h}$ the polynomial map of C. The Jacobian of $\tilde{h}$ is defined as

$$\mathrm{Jac}(\tilde{h}) = \frac{\partial(f_1, f_2, \ldots, f_n, g_1, g_2, \ldots, g_n)}{\partial(x_1, x_2, \ldots, x_m)}.$$

Theorem 3.1 (Extended Nilpotent-Jacobian Method) *Let $\mathcal{S}$ be a ray pattern of order n, $C \in Q_r(\mathcal{S})$ be a nilpotent complex matrix and $\tilde{h}$ be the polynomial map of C. If*

$$\mathrm{Jac}(\tilde{h})|_C = \mathrm{Jac}(\tilde{h})|_{(x_1, x_2, \ldots, x_m) = (r_1, r_2, \ldots, r_m)}$$

has rank $2n$, then the ray pattern $\mathcal{S}$, and each superpattern of $\mathcal{S}$, is a SAR.

The proof of Theorem 3.1 is similar to Theorem 2.2, and we omit it.

Now, we will try to give the Nilpotent-Centralizer method to show that a ray pattern is spectrally arbitrary.

Lemma 3.2 *Let $\mathcal{S}=[s_{kl}]$ be a ray pattern of order n and $C\in Q_r(S)$. Take $Y=[r_{kl}s_{kl}]\in Q_r(S)$. Let the characteristic polynomial of Y be*

$$c_Y=\det(xI-Y)=x^n+(f_1+\mathrm{i}g_1)x^{n-1}+\cdots+(f_{n-1}+\mathrm{i}g_{n-1})x+(f_n+\mathrm{i}g_n),$$

where $f_m=f_m(r_{11},r_{12},\ldots,r_{nn})$ and $g_m=g_m(r_{11},r_{12},\ldots,r_{nn})$ for $m=1,2,\ldots,n$. Then for (k,l) with $s_{kl}\neq 0$, we have

$$\begin{cases}\frac{\partial f_m}{\partial r_{kl}}\Big|_{Y=C}=\textit{the real part of the coefficient of } x^{n-m} \textit{ in } -s_{kl}\mathrm{adj}(xI-C)_{lk},\\ \frac{\partial g_m}{\partial r_{kl}}\Big|_{Y=C}=\textit{the imaginary part of the coefficient of } x^{n-m} \textit{ in } -s_{kl}\mathrm{adj}(xI-C)_{lk},\end{cases}$$

Proof For (k,l) with $s_{kl}\neq 0$, we have

$$\begin{aligned}\frac{\partial c_Y}{\partial r_{kl}}&=\lim_{h\to 0}\frac{\det(xI-(Y+hs_{kl}E_{kl}))-\det(xI-Y)}{h}\\&=\lim_{h\to 0}\frac{\det(xI-Y)-hs_{kl}(-1)^{k+l}\det((xI-Y)(k,l))-\det(xI-Y)}{h}\\&=-s_{kl}\mathrm{adj}(xI-Y)_{lk}.\end{aligned}$$

Hence

$$\begin{aligned}-s_{kl}\mathrm{adj}(xI-Y)_{lk}=&\Big(\frac{\partial f_1}{\partial r_{kl}}+\mathrm{i}\frac{\partial g_1}{\partial r_{kl}}\Big)x^{n-1}+\Big(\frac{\partial f_2}{\partial r_{kl}}+\mathrm{i}\frac{\partial g_2}{\partial r_{kl}}\Big)x^{n-2}+\\&\cdots+\Big(\frac{\partial f_{n-1}}{\partial r_{kl}}+\mathrm{i}\frac{\partial g_{n-1}}{\partial r_{kl}}\Big)x+\Big(\frac{\partial f_n}{\partial r_{kl}}+\mathrm{i}\frac{\partial g_n}{\partial r_{kl}}\Big),\end{aligned}$$

and so

$$\begin{cases}\frac{\partial f_m}{\partial r_{kl}}\Big|_{Y=C}=\text{the real part of the coefficient of } x^{n-m} \text{ in } -s_{kl}\mathrm{adj}(xI-C)_{lk},\\ \frac{\partial g_m}{\partial r_{kl}}\Big|_{Y=C}=\text{the imaginary part of the coefficient of } x^{n-m} \text{ in } -s_{kl}\mathrm{adj}(xI-C)_{lk}.\end{cases}$$

The lemma follows.

Lemmas 3.2 and 2.4 imply the following theorem.

Theorem 3.3 *Let $\mathcal{S}=[s_{kl}]$ be a ray pattern of order n, and $C\in Q_r(S)$ be a nilpotent complex matrix. Take $Y=[r_{kl}s_{kl}]\in Q_r(S)$. Let the characteristic polynomial of Y be*

$$c_Y=\det(xI-Y)=x^n+(f_1+\mathrm{i}g_1)x^{n-1}+\cdots+(f_{n-1}+\mathrm{i}g_{n-1})x+(f_n+\mathrm{i}g_n),$$

where $f_m=f_m(r_{11},r_{12},\ldots,r_{nn})$ and $g_m=g_m(r_{11},r_{12},\ldots,r_{nn})$ for $m=1,2,\ldots,n$. Then for (k,l) with $s_{kl}\neq 0$, we have

$$\begin{cases}\frac{\partial f_m}{\partial r_{kl}}\Big|_{Y=C}=-\Re(s_{kl}C^{m-1})_{lk}=-\Re(C^{m-1}\circ\mathcal{S}^T)_{lk},\\ \frac{\partial g_m}{\partial r_{kl}}\Big|_{Y=C}=-\Im(s_{kl}C^{m-1})_{lk}=-\Re(-iC^{m-1}\circ\mathcal{S}^T)_{lk}.\end{cases}$$

Theorem 3.3 implies that the entries in a column of $\mathrm{Jac}(\tilde{h})|_C$ are corresponding to the negative of real parts of relevant entries of matrices

$$C^0 \circ \mathcal{S}^T, C^1 \circ \mathcal{S}^T, \ldots, C^{m-1} \circ \mathcal{S}^T, -\mathrm{i}(C^0 \circ \mathcal{S}^T), -\mathrm{i}(C^1 \circ \mathcal{S}^T), \ldots, -\mathrm{i}(C^{m-1} \circ \mathcal{S}^T). \tag{3.1}$$

For example, the entries of $\frac{\partial(f_1,f_2,\ldots,f_n,g_1,g_2,\ldots,g_n)}{\partial(r_{kl})}|_C$ are corresponding to the negative of real parts of (l,k) entries of matrices as shown in (3.1), respectively.

By a similar approach to Theorem 2.7, we can prove the following theorem, and we omit the proof.

Theorem 3.4 *Let $\mathcal{S} = [s_{jk}]$ be a ray pattern of order n, $C \in Q_r(\mathcal{S})$ be an nilpotent matrix, and $\tilde{h}$ be its polynomial map. Then $\mathrm{Jac}(\tilde{h})|_C$ has rank less than $2n$ if and only if there exists a nonzero polynomial $v(x) \in \mathbb{C}[x]$ of degree at most $n-1$ such that $v(C) \circ \mathcal{S}^T = 0$.*

Similarly as in the case of complex sign patterns, we have following Lemma 3.5 and Theorem 3.6 about ray patterns.

Lemma 3.5 *Let $\mathcal{S} = [s_{jk}]$ be a ray pattern of order n, and $C \in Q_r(\mathcal{S})$ be an nilpotent complex matrix of index n. Then there exists a nonzero polynomial $v(x) \in \mathbb{C}[x]$ of degree at most $n-1$ such that $v(C) \circ S^T = 0$ if and only if there exists a nonzero complex matrix H in the centralizer of C such that $H \circ \mathcal{S}^T = 0$.*

Theorem 3.6 (Nilpotent-Centralizer Method for spectrally arbitrary ray patterns) *Let $\mathcal{S}$ be a ray pattern of order n, and $C \in Q_r(\mathcal{S})$ be a nilpotent complex matrix of index n. If the only complex matrix H in the centralizer of C satisfying $H \circ \mathcal{S}^T = 0$ is the zero matrix, then the ray pattern $\mathcal{S}$ and each of its superpatterns is spectrally arbitrary.*

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