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Homoclinic Orbits for Hamiltonian Systems with Small Forced Terms

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Abstract By Brezis-Nirenberg type Mountain Pass Theorem, the research has focused on the existence of nontrivial homoclinic orbits for a class of second order Hamiltonian systems with non-Ambrosetti-Rabinowitz type superquadratic potentials and small forced terms.

Keywords Homoclinic orbits; Hamiltonian systems; Mountain Pass Theorem; critical points.

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1. Introduction

Since 1990, there have been a vast literatures (cf. [1–10] and references therein) on the subject of homoclinic orbits for Hamiltonian systems by variational methods. Firstly, Rabinowitz [1] discussed the existence of homoclinic orbits for second order periodic Hamiltonian systems

$$\ddot{q}\,(t) - L\,(t)\,q\,(t) + W_q\,(t, q\,(t)) = 0, t \in \mathbb{R} \tag{HS}$$

where $q\,(t)\,, W(t,q) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are $\mathbb{C}^1$-maps, T-periodic in $t, W_q(t,q) = \frac{\partial W}{\partial q}$ denotes the gradient of $W(t,q)$ with respect to q. If $0 \neq q\,(t) \in W^{1,2}\,(\mathbb{R}, \mathbb{R}^n)$ is a solution of (HS) such that $q\,(t) \to 0, \dot{q(t)} \to 0$ as $|t| \to \infty$, then we say it is a nontrivial homoclinic orbit of (HS). Rabinowitz assumed that $L\,(t)$ is a positive symmetrical matrix function, and $W\,(t,q)$ satisfies the so-called Ambrosetti-Rabinowitz type superquadratic condition:

(AR) There exists $\theta > 2$ such that $0 < \theta W\,(t,q) \leq qW_q\,(t,q)\,, \forall\,(t,q) \in \mathbb{R} \times \mathbb{R}^n \setminus \{0\}$.

For $k \geq 1$, he considered the approximate problem

$$\begin{cases} \ddot{q}\,(t) - L\,(t)\,q\,(t) + W_q\,(t, q\,(t)) = 0, t \in (-kT, kT)\,, \\ q\,(-kT) = q\,(kT)\,. \end{cases} \tag{HSk}$$

Solutions of (HSk) are obtained as critical points $q_k\,(t)$ of the functional

$$f_k\,(q) = \frac{1}{2}\int_{-kT}^{kT} \left[|\dot{q}\,(t)|^2 + (L\,(t)\,q\,(t)\,, q\,(t))\right] \mathrm{d}t - \int_{-kT}^{kT} W\,(t, q\,(t))\,\mathrm{d}t$$

via minimax argument, and uniform estimates permit $q_k\,(t)$ to converge weakly to a nontrivial homoclinic orbit of (HS).

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Later, Izydorek and Janczewska [2] used the same idea as in [1] to study homoclinic orbits for more general periodic Hamiltonian systems with a small forced term $f(t)$ as follows

$$\ddot{q}(t) - K_q(t, q(t)) + W_q(t, q(t)) = f(t), \quad t \in \mathbb{R} \tag{HSf}$$

where $K(t,q), W(t,q) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}^n$, $K_q(t,q) = \frac{\partial K}{\partial q}$ denotes the gradient of $K(t,q)$ with respect to q. They proved the following result:

Theorem 1.1([2]) *Under the condition of (AR), suppose that $K(t,q), W(t,q)$ and $f(t)$ satisfy*

(H_1) $K(t,q), W(t,) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are $\mathbb{C}^1$-maps, T-periodic in the variable t;

(H_2) There are constants $b_1 > 0$ and $b_2 > 0$ such that

$$b_1 |q|^2 \le K(t,q) \le b_2 |q|^2, \forall (t,q) \in \mathbb{R} \times \mathbb{R}^n;$$

(H_3) $K(t,q) \le qK_q(t,q) \le 2K(t,q)$, $\forall (t,q) \in \mathbb{R} \times \mathbb{R}^n$;

(H_4) $W_q(t,q) = o(|q|)$ as $q \to 0$ uniformly with respect to t;

(H_5) $f(t) : \mathbb{R} \to \mathbb{R}^n$ is a continuous and bounded function with $f(t) \in L^2(\mathbb{R})$.

Furthermore, if $\|f\|_{L^2(\mathbb{R})}$ is sufficiently small, then (HSf) possesses a nontrivial homoclinic orbit.

For the existence of homoclinic orbits for coercive or subquadratic Hamiltonian systems, we refer the reader to V. Cotizelati, I. Ekeland and E. Sere [3], W.Omana and M.Willem [4], Y.H. Ding and M. Girardi [5], E. Sere [6], P.L.Felmer, and Silva [7], P. Korman, A. C. Lazer [8], Y. Lv, Chun-Lei Tang [9], etc.

Inspired by the above papers, particularly [1] and [2], we consider whether the conclusion of Theorem 1.1 still holds if $W(t,q)$ does not satisfy condition (AR) in Equation (HSf). Exactly, our main result is

Theorem 1.2 *Assume that $K(t,q), W(t,q)$ and $f(t)$ satisfy $(H_1)-(H_4)$. Furthermore, assume that*

($\bar{H}_5$) $f(t) : \mathbb{R} \to \mathbb{R}^n$ is a continuous function with $f(t) \in L^2(\mathbb{R})$;

(H_6) $W(t,q)/|q|^2 \to \infty$ $(|q| \to \infty)$ uniformly with respect to t;

(H_7) there are $d_1, d_2 > 0$ and $\lambda > 1$ such that

$$|W_q(t,q)| \le d_1 |q|^\lambda + d_2, \quad \forall t \in \mathbb{R}, \ q \in \mathbb{R}^n;$$

(H_8) there are $h > 0, d_3 > 0$ and $\mu > \lambda$ such that

$$qW_q(t,q) - 2W(t,q) \ge d_3 |q|^\mu, \quad \forall t \in \mathbb{R}, \ |q| > h;$$

(H_9) $qW_q(t,q) > 2W(t,q)$, $\forall t \in \mathbb{R}$, $q \in \mathbb{R}^n \setminus \{0\}$,

then (HSf) possesses a nontrivial homoclinic orbit provided that $\|f\|_{L^2(\mathbb{R})}$ is sufficiently small.

Remark 1.3 (H_4) and (H_6) show that $W(t,q)$ is superquadratic at the origin and infinity.

Remark 1.4 Combining (H_4) with (H_7) implies that, for any small $\delta > 0$, there exists $\bar{d}_1 = \bar{d}_1(\delta) > 0$ such that

$$|W_q(t,q)| \le \bar{d}_1 |q|^\lambda + \delta |q|^2, \quad \forall t \in \mathbb{R}, \ q \in \mathbb{R}^n.$$

Remark 1.5 From (H_8) and (H_9), we may assume that

$$qW_q(t,q) - 2W(t,q) \geq d_3 |q|^\mu, \quad \forall t \in \mathbb{R},\ |q| > h$$

with the property that $0 < d_3 = d_3(h) \to 0$ as $h \to 0$.

Remark 1.6 If $W(t,q)$ satisfies condition (AR), then there exist $d_4 > 0, h' > 0$ such that $W(t,q) \geqslant d_4 |q|^\theta, \forall t \in \mathbb{R}, |q| \geqslant h'$, thus, whenever $\theta > \gamma$, (AR) implies (H_8). However, for example, if we take $W(t,q) = |q|^2 \ln(1+|q|^2)$, then it satisfies condition (H_4) and (H_6)–(H_9), but does not satisfy (AR). In this sense, our result Theorem 1.2 generalizes the main conclusions in [1] and [2].

Remark 1.7 In [12], the author and Costa studied the existence of homoclinic type solutions of a class of differential equations with periodic potentials which also satisfy conditions (H_6)–(H_9) with $\mu > \lambda - 1$ instead of $\mu > \lambda$ in (H_8), however, all of them do not contain a non-periodic small forced term. By [12], we also guess that our Theorem 1.2 may still hold with $\mu > \lambda - 1$ instead of $\mu > \lambda$ in (H_8).

In the next section, different from the arguments in [1] and [2], we shall employ the following Brezis-Nirenberg type Mountain Pass Theorem [11] to prove our Theorem 1.2 directly.

Theorem 1.8 (Brezis-Nirenberg [11]) *Let X be a Banach space and $\varphi \in C^1(X,\mathbb{R})$ with $\varphi(0) = 0$. Suppose that φ satisfies*

(i) there are constants $\omega > 0$ and $\rho > 0$ such that $\varphi(u) \geq \omega, \forall \|u\| = \rho$;

(ii) there exists $e \in X \setminus B_\rho(0)$ such that $\varphi(e) < 0$.

Define $\beta = \inf_{\gamma\in\Gamma} \varphi(\gamma(s))$ with

$$\Gamma = \{\gamma \in ([0,1], X) : \gamma(0) = 0, \gamma(1) = e\},$$

then $\omega \leq \beta < \infty$ and φ has at least a $(Ce)_\beta$ sequence, namely, there exists a sequence $\{q_m\}$ in X such that

$$\varphi(q_m) \to \beta, (1 + \|q_m\|) \|\varphi'(q_m)\| \to 0.$$

2. Some lemmas

Denote by $E = W^{1,2}(\mathbb{R},\mathbb{R}^n)$ the usual Sobolev space with the norm

$$\|q\| = \Big(\int_{\mathbb{R}} \Big(|\dot{q}(t)|^2 + |q(t)|^2\Big)\,\mathrm{d}t\Big)^{\frac{1}{2}}. \tag{2.1}$$

Let $\eta(q) = (\int_{\mathbb{R}}[|\dot{q}(t)|^2 + 2K(t,q)]\mathrm{d}t)^{\frac{1}{2}}$. Then ($H_2$) implies

$$\bar{b_1} \|q\|^2 \leq \eta^2(q) \leq \bar{b_2} \|q\|^2, \quad \forall q \in E \tag{2.2}$$

with $\bar{b_1} = \min\{1, 2b_1\}$ and $\bar{b_2} = \max\{1, 2b_2\}$. Set

$$\begin{aligned} I(q) &= \frac{1}{2}\int_{\mathbb{R}} \Big[|\dot{q}(t)|^2 + 2K(t,q(t))\Big]\,\mathrm{d}t - \int_{\mathbb{R}} W(t,q(t))\,\mathrm{d}t + \int_{\mathbb{R}} f(t)\,q(t)\,\mathrm{d}t \\ &= \frac{1}{2}\eta^2(q) - \int_{\mathbb{R}} W(t,q(t))\,\mathrm{d}t + \int_{\mathbb{R}} f(t)\,q(t)\,\mathrm{d}t, \quad q \in E. \end{aligned} \tag{2.3}$$

Using (H_1)–(H_4) and (H_5), we know that $I(q) \in \mathbb{C}^1(E)$, critical points of $I(q)$ in E are classical solutions of (HSf).

Lemma 2.1 *Under the assumptions of (H_1)–(H_4) and ($\bar{H}_5$), there exist $\omega > 0$ and $\rho > 0$ such that $I(q) > \omega \forall \|q\| = \rho$.*

Proof By the Sobolev inequalities, we have

$$\|q\|_{L^2(\mathbb{R})} \leq \|q\|, \|q\|_{L^\infty(\mathbb{R})} \leq \|q\|, \quad \forall q \in E.$$

According to (H_4), for $c_1 \triangleq \bar{b_1}/4$, there exists $\rho \in (0,1)$ such that $|W(t,q)| \leq c_1 |q|^2, \forall |q| \leq \rho$ uniformly in $t \in \mathbb{R}$. If $q \in E$ with $\|q\| = \rho$, then $|q(t)| \leq \rho, \forall t \in \mathbb{R}$. Thus we obtain

$$\int_{\mathbb{R}} W(t,q(t))\,\mathrm{d}t \leq c_1 \|q\|^2_{L^2(\mathbb{R})} \leq c_1 \|q\|^2 = c_1\rho^2, \tag{2.4}$$

$$\left|\int_{\mathbb{R}} f(t)\,q(t)\,\mathrm{d}t\right| \leq \|f\|_{L^2(\mathbb{R})} \|q\|_{L^2(\mathbb{R})} \leq \rho \|f\|_{L^2(\mathbb{R})}. \tag{2.5}$$

Hence, we have the estimate

$$I(q) \geq c_1\rho^2 - \rho \|f\|_{L^2(\mathbb{R})}. \tag{2.6}$$

Therefore, for $c_2 \triangleq \frac{1}{2}\rho c_1$, if $\|f\|_{L^2(\mathbb{R})} \leq c_2$, then by (2.6), we have

$$I(q) \geq \frac{1}{2}c_1\rho^2 \triangleq \omega > 0. \tag{2.7}$$

So we complete the proof. □

Lemma 2.2 *Under the assumptions of (H_1)–(H_4), ($\bar{H}_5$) and (H_6), there is $e \in E \setminus B_\rho(0)$ such that $I(e) < 0$.*

Proof Choose $0 \neq g = g(t) \in \mathbb{C}_0^\infty(\mathbb{R},\mathbb{R}^n) \subset E$ and $\sigma > 0$ such that

$$\mathrm{Supp}\,\{g(t)\} \subset (0,T), \quad 0 < \sigma \|g\| \leq \|g\|_{L^2(0,T)}. \tag{2.8}$$

By (H_6), there is $\alpha > 0$ such that

$$W(t,q) \geq c_3 |q|^2 - \alpha, \quad \forall t \in \mathbb{R},\ q \in \mathbb{R}^n \tag{2.9}$$

with $c_3 \triangleq 1 + \frac{\bar{b_2}}{2\sigma^2}$. Consequently, for $\forall \tau > 1$

$$\begin{aligned} I(\tau g) &= \frac{1}{2}\eta^2(\tau g) - \int_0^T W(t,\tau g(t))\,\mathrm{d}t + \tau \int_0^T f(t)\,g(t)\,\mathrm{d}t \\ &\leq \frac{\tau^2 \bar{b_2}}{2\sigma^2} \int_0^T |g(t)|^2\,\mathrm{d}t - \int_0^T W(t,\tau g(t))\,\mathrm{d}t + \tau \int_0^T f(t)\,g(t)\,\mathrm{d}t \\ &\leq \frac{\tau^2 \bar{b_2}}{2\sigma^2} \int_0^T |g(t)|^2\,\mathrm{d}t - c_3\tau^2 \int_0^T |g(t)|^2\,\mathrm{d}t + \tau \int_0^T f(t)\,g(t)\,\mathrm{d}t + \alpha T \\ &= -\frac{1}{2}\tau^2 \int_0^T |g(t)|^2\,\mathrm{d}t + \tau \int_0^T f(t)\,g(t)\,\mathrm{d}t + \alpha T \to -\infty, \quad \text{as } \tau \to \infty. \end{aligned} \tag{2.10}$$

Clearly, we can take $e = e(t) = \tau g(t) \in E$ such that $\|e\| > \rho$ and $\varphi(e) < 0$ for τ large enough. □

Lemma 2.3 *Under the assumptions of Theorem 1.2, there exists $d > 0$ such that $I(q)$ has a bounded $(Ce)_d$ sequence $\{q_m\}$ in E.*

Proof By Lemmas 2.1, 2.2 and Theorem 1.8, for $d > 0$ defined by

$$d = \inf_{\gamma\in\Gamma} \max_{0\leq s\leqslant 1} I(\gamma(s)) \geq \omega > 0 \tag{2.11}$$

with

$$\Gamma = \{\gamma \in \mathbb{C}([0,1], E) : \gamma(0) = 0, \gamma(1) = e\},$$

$I(q)$ has a $(Ce)_d$ sequence $\{q_m\}$ satisfying that

$$I(q_m) \to d > 0,\ (1 + \|q_m\|)\|I'(q_m)\| \to 0. \tag{2.12}$$

Setting $c_4 \triangleq \min\{1, b_1\} > 0$, by Remark 1.4, we can take $\delta = \frac{1}{3}c_4 > 0$ and the corresponding $\bar{d_1} = \bar{d_1}(\delta) > 0$. Clearly, there exists small $h > 0$ such that $0 < \bar{d_1}h^{\lambda-1} < \frac{1}{3}c_4$. So, we have

$$c_5 \triangleq c_4 - \bar{d_1}h^{\lambda-1} - \delta \geq \frac{1}{3}c_4 > 0,$$

thus, from (H_2), (H_3), Remaks 1.4 and 1.5, we infer that

$$\begin{aligned}
I'(q_m)q_m =& \|\dot{q_m}\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}} q_m(t)K_q(t, q_m(t))\,\mathrm{d}t - \int_{\mathbb{R}} W_q(t, q_m(t))q_m(t)\,\mathrm{d}t + \int_{\mathbb{R}} f(t)q_m(t)\,\mathrm{d}t\\
\geq& c_4\|q_m\|^2 - \int_{\mathbb{R}} |W_q(t, q_m(t))||u_m(t)|\,\mathrm{d}t + \int_{\mathbb{R}} f(t)q_m(t)\,\mathrm{d}t\\
\geq& c_4\|q_m\|^2 - \bar{d}1\int_{\mathbb{R}} |q_m(t)|^{\lambda+1}\,\mathrm{d}t - \delta\int_{\mathbb{R}} |q_m(t)|^2\,\mathrm{d}t + \int_{\mathbb{R}} f(t)q_m(t)\,\mathrm{d}t\\
=& c_4\|q_m\|^2 - \bar{d_1}\int_{|q_m(t)|\geq h} |q_m(t)|^{\lambda+1}\,\mathrm{d}t - \bar{d_1}\int_{|q_m(t)|\leq h} |q_m(t)|^{\lambda+1}\,\mathrm{d}t-\\
&\delta\int_{\mathbb{R}} |q_m(t)|^2\,\mathrm{d}t + \int_{\mathbb{R}} f(t)q_m(t)\,\mathrm{d}t\\
\geq& c_4\|q_m\|^2 - \bar{d_1}\int_{|q_m(t)|\geq h} |q_m(t)|^{\lambda+1}\,\mathrm{d}t - \bar{d_1}h^{\lambda-1}\int_{\mathbb{R}} |q_m(t)|^2\,\mathrm{d}t-\\
&\delta\int_{\mathbb{R}} |q_m(t)|^2\,\mathrm{d}t + \int_{\mathbb{R}} f(t)q_m(t)\,\mathrm{d}t\\
\geq& c_4\|q_m\|^2 - \bar{d_1}\int_{|q_m(t)|\geq h} |q_m(t)|^{\lambda+1}\,\mathrm{d}t - \bar{d_1}h^{\lambda-1}\|q_m\|^2 - \delta\|q_m\|^2 - \|f\|_{L^2(\mathbb{R})}\|q_m\|\\
=& c_5\|q_m\|^2 - \bar{d_1}\int_{|q_m(t)|\geq h} |q_m(t)|^{\lambda+1}\,\mathrm{d}t - \|f\|_{L^2(\mathbb{R})}\|q_m\|.
\end{aligned} \tag{2.13}$$

Since $\lambda < \mu$, we can take $r \in (0,1)$ such that $\lambda + 1 - r < \mu$, so, we can make the estimate for the integral term $\int_{|q_m(t)|\geq h} |q_m(t)|^{\lambda+1}\,\mathrm{d}t$ in (2.13) as follows.

$$\begin{aligned}
\int_{|q_m(t)|\geq h} |q_m(t)|^{\lambda+1}\,\mathrm{d}t \leq& \|q_m\|^r_{L^\infty}\int_{|q_m(t)|\geq h} |q_m(t)|^{\lambda+1-r}\,\mathrm{d}t\\
\leq& h^{\lambda+1-r-\mu}\|q_m\|^r_{L^\infty}\int_{|q_m(t)|\geq h} |q_m(t)|^{\mu}\,\mathrm{d}t\\
\leq& h^{\lambda+1-r-\mu}\|q_m\|^r\int_{|q_m(t)|\geq h} |q_m(t)|^{\mu}\,\mathrm{d}t.
\end{aligned} \tag{2.14}$$

Combining (2.13) with (2.14), we deduce that

$$I'(q_m)\,q_m \geq c_5\,\|q_m\|^2 - \bar{d}_1 h^{\lambda+1-r-\mu}\,\|q_m\|^r \int_{|q_m(t)|\geq h} |q_m(t)|^\mu\,\mathrm{d}t - \|f\|_{L^2(\mathbb{R})}\,\|q_m\|. \tag{2.15}$$

Next, with the aid of (H_3), (H_8)–(H_9), we have for the above $h>0$ in (2.13)

$$\begin{aligned} 2I(q_m) - I'(q_m)\,q_m =& \int_{\mathbb{R}} \left[2K(t,q_m(t)) - K_q(t,q_m(t))\,q_m(t)\right]\mathrm{d}t + \\ & \int_{\mathbb{R}} \left[W_q(t,q_m(t))\,q_m(t) - 2W(t,q_m(t))\right]\mathrm{d}t + \int_{\mathbb{R}} f(t)\,q_m(t)\,\mathrm{d}t \\ \geq & \int_{\mathbb{R}} \left[W_q(t,q_m(t))\,q_m(t) - 2W(t,q_m(t))\right]\mathrm{d}t + \int_{\mathbb{R}} f(t)\,q_m(t)\,\mathrm{d}t \\ \geq & d_3(h) \int_{|q_m(t)|\geq h} |q_m(t)|^\mu\,\mathrm{d}t - \|f\|_{L^2(\mathbb{R})}\,\|q_m\|. \end{aligned} \tag{2.16}$$

Thus, by (2.12) and (2.16), there exist $c_6 = c_6(h) > 0, c_7 = c_7(h) > 0$ such that

$$\int_{|q_m(t)|>h} |q_m(t)|^\mu\,\mathrm{d}t \leq c_6 + c_7\,\|f\|_{L^2(\mathbb{R})}\,\|q_m\|. \tag{2.17}$$

We substitute (2.17) into (2.15), and obtain

$$\begin{aligned} I'(q_m)\,q_m \geq & c_5\,\|q_m\|^2 - \bar{d}_1 c_6 h^{\lambda+1-r-\mu}\,\|q_m\|^r - \\ & \bar{d}_1 c_7 h^{\lambda+1-r-\mu}\,\|f\|_{L^2(\mathbb{R})}\,\|q_m\|^{r+1} - \|f\|_{L^2(\mathbb{R})}\,\|q_m\|. \end{aligned} \tag{2.18}$$

By $0<r<r+1<2, c_5>0$, (2.12) and (2.18), we infer that $\{q_m\}$ is bounded in E. □

3. Proof of Theorem 1.2

Proof of Theorem 1.2 By Lemma 2.3, we know that $I(q)$ has a bounded $(Ce)_d$ sequence $\{q_m\}$, thus, without loss of generality, we may assume that there exists $q_0 = q_0(t) \in E$ such that

$$q_m \rightharpoonup q_0 \text{ weakly in } E,\ q_m \to q_0 \text{ in } L^2_{\mathrm{loc}}(\mathbb{R}),\ q_m \to q_0 \text{ in } \mathbb{C}_{\mathrm{loc}}(\mathbb{R}).$$

Therefore, for $\forall \upsilon \in \mathbb{C}_0^\infty(\mathbb{R})$, from the following

$$\begin{aligned} I'(q_m)\,\nu =& \int_{\mathbb{R}} \dot{q_m}(t)\,\nu(t)\,\mathrm{d}t + \int_{\mathbb{R}} K_q(t,q_m(t))\,\nu(t)\,\mathrm{d}t - \\ & \int_{\mathbb{R}} W_q(t,q_m(t))\,\nu(t)\,\mathrm{d}t + \int_{\mathbb{R}} f(t)\,\nu(t)\,\mathrm{d}t \to 0, \end{aligned} \tag{3.1}$$

we can show that

$$0 = \int_{\mathbb{R}} \dot{q_0}(t)\,\nu(t)\,\mathrm{d}t + \int_{\mathbb{R}} K_q(t,q_0(t))\,\nu(t)\,\mathrm{d}t - \int_{\mathbb{R}} W_q(t,q_0(t))\,\nu(t)\,\mathrm{d}t + \int_{\mathbb{R}} f(t)\,\nu(t)\,\mathrm{d}t, \tag{3.2}$$

that is, $q_0 \in E$ is a critical point of the functional I in (2.3), and $q_0 = q_0(t)$ is one solution of (HSf).

Case (I) If $f(t) \neq 0$, then clearly $q_0 \neq 0$.

Case (II) If $f(t) = 0$, then, for $\forall m$, there is $j_m \in \mathbb{Z}$ such that the maximum of $|q_m(t + j_m T)|$ occurs in $[0,T]$. Let $w_m(t) \triangleq q_m(t + j_m T)$. By ($H_1$) and (2.12), we have $\|w_m\| = \|q_m\|$,

$I(w_m) = I(q_m)$, $I'(w_m) = I'(q_m) \to 0$. Thus, we also assume that there exists $w_0 = w_0(t) \in E$ such that

$$w_m \rightharpoonup w_0 \text{ weakly in } E,\ w_m \to w_0 \text{ in } L^2_{\mathrm{loc}}(\mathbb{R}),\ \text{and } w_m \to w_0 \text{ in } \mathbb{C}_{\mathrm{loc}}(\mathbb{R}).$$

Thus for any $\nu \in \mathbb{C}_{\mathrm{loc}}(\mathbb{R})$, we have

$$|I'(w_m)\nu(\cdot)| = |I'(q_m)\nu(\cdot - j_m T)| \le \|I'(q_m)\| \|\nu(\cdot - j_m T)\| = \|I'(q_m)\| \|\nu(\cdot)\| \to 0. \tag{3.3}$$

Just similarly to (3.1), (3.2), we can prove that $w_0 \in E$ is a critical point of the functional I. We claim that $w_0 \neq 0$. If this is not true, then we have

$$\|q_m\|_{L^\infty(\mathbb{R})} = \|w_m\|_{L^\infty(\mathbb{R})} = \|w_m\|_{L^\infty([0,T])} \to 0. \tag{3.4}$$

Thus by (H_4), given $\varepsilon > 0$, we have, for m sufficiently large and $\forall t \in \mathbb{R}$

$$|W(t, q_m(t))| \le \varepsilon |q_m(t)|^2, |q_m(t) W_q(t, q_m)(t)| \le \varepsilon |q_m(t)|^2. \tag{3.5}$$

So, in view of (H_3) and (3.5), noticing $f(t) = 0$, we have

$$\begin{aligned}\|\dot{q}_m\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}} K(t, q_m(t))\,\mathrm{d}t &\le \|\dot{q}_m\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}} K_q(t, q_m(t))\, q_m(t)\,\mathrm{d}t \\ &= I'(q_m)\, q_m + \int_{\mathbb{R}} W_q(t, q_m(t))\, q_m(t)\,\mathrm{d}t \\ &\le I'(q_m)\, q_m + \varepsilon \int_{\mathbb{R}} |q_m(t)|^2\,\mathrm{d}t \\ &\le \|\ I'(q_m)\| \|q_m\| + \varepsilon \|q_m\|^2. \end{aligned} \tag{3.6}$$

And (3.5) and (3.6) imply

$$\begin{aligned} 0 < I(q_m) &= \frac{1}{2}\int_{\mathbb{R}} \left[|\dot{q}_m(t)|^2 + 2K(t, q_m(t))\right]\mathrm{d}t - \int_{\mathbb{R}} W(t, q_m(t))\,\mathrm{d}t \\ &\le \|\dot{q_m}\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}} K(t, q_m(t))\,\mathrm{d}t - \int_{\mathbb{R}} W(t, q_m(t))\,\mathrm{d}t \\ &\le \|I'(q_m)\| \|q_m\| + 2\varepsilon \|q_m\|^2. \end{aligned} \tag{3.7}$$

Since $\|q_m\|$ is bounded, ε is arbitrary, (2.12) and (3.7) show $I(q_m) \to 0$, which contradicts (2.12). So $w_0 \neq 0$.

Under the above two cases, we get a nontrivial solution $q_*(t)$ of (HSf): $q_*(t) = q_0(t)$ or $q_*(t) = w_0(t)$, respectively.

Finally, we claim that $q_*(t) \to 0$ and $\dot{q}_*(t) \to 0$ as $|t| \to \infty$. For the proof we refer to [2]. For the sake of completeness, we sketch it. Indeed, $q_* \in E$ implies $q_*(t) \to 0$ as $|t| \to \infty$. Since $q_*(t)$ is a solution of (HSf), we have

$$\ddot{q_*}(t) = f(t) - V_q(t, q_*(t)) \in L^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n) \tag{3.8}$$

is continuous, where $V(t, q) = -K(t, q) + W(t, q)$. Thus from $\dot{q}_* \in L^2(\mathbb{R})$, $f \in L^2(\mathbb{R})$ and $V_q(t, 0) = 0$, one can show that (see [2] for details)

$$\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\dot{q}_*(s)|^2\,\mathrm{d}s \to 0, \tag{3.9}$$

$$\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\ddot{q}_*(s)|^2 \mathrm{d}s \le \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |f(s)|^2 \mathrm{d}s + \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |V_q(t, q_*(s))|^2 \mathrm{d}s + 2\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |f(s)|\,|V_q(t, q_*(s))| \mathrm{d}s \to 0 \tag{3.10}$$

as $|t| \to \infty$. Therefore, by (3.9), (3.10), we have the estimate as follows

$$|\dot{q}_*(t)|^2 \le 2\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\dot{q}_*(s)|^2 \mathrm{d}s + 2\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\ddot{q}_*(s)|^2 \mathrm{d}s \to 0, \text{ as } |t| \to \infty. \tag{3.11}$$

The proof of Theorem 1.2 is completed. □

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