An Improved Harnack Inequality for Dirichlet Eigenvalues of Abelian Homogeneous Graphs

Shoudong MAN

Department of Mathematics, School of Information, Renmin University of China,
Beijing 100872, *P. R. China*

Abstract In this paper, we prove an improved Harnack inequality for Dirichlet eigenfunctions of abelian homogeneous graphs and their convex subgraphs. As a consequence, we derive a lower estimate for Dirichlet eigenvalues using the Harnack inequality, extending previous results of Chung and Yau for certain homogeneous graphs.

Keywords Laplace operator; Harnack inequality; eigenvalue estimate.

MR(2010) Subject Classification 05C10; 31C20; 31C05

1. Introduction

Let $G = (V, E)$ denote a graph with vertex set V and edge set E. Suppose a group χ acts on V such that:

(i) For all $a \in \chi$, $\{au, av\} \in E$ if and only if $\{u, v\} \in E$;

(ii) For any two vertices u and v, there is a $a \in \chi$ such that $au = v$.

Then we say G is a homogeneous graph with the associated group χ. The edge set of a homogeneous graph can be described by an (edge) generating set $K \subset \chi$ so that each edge of G is of the form $\{v, av\}$ for some $v \in V$ and $a \in K$, and we let the edge generating set K consist of k generators. In this paper, we also require the generating set K to be symmetric, i.e., $a \in K$ if and only if $a^{-1} \in K$. If for every element $a \in K$, we have $aKa^{-1} = K$, then we say that a homogeneous graph is invariant. If χ is abelian, we say G is an abelian homogeneous graph. For more detailed definition of abelian homogeneous graph, we can refer to [1]. Moreover, we denote $x \sim ax$ if vertex x is adjacent to vertex ax for some $a \in K$. For every vertex x of V, if the number of edges connected to x is finite, we say that G is a locally finite graph. The distance between two vertices is the minimum number of edges to connect them, while the diameter of G is the maximum of all the distances of the graph.

In a graph G_0, for a subset G of the vertex set $V = V(G_0)$, the induced subgraph determined by G has edge set consisting of all edges of G_0 with both endpoints in G. There are two types of boundaries. The (vertex) boundary δG of an induced subgraph G consists of all vertices that are not in G and are adjacent to some vertices in G. The edge boundary, denoted by ∂G, consists of

all edges containing one endpoint in G and one endpoint not in G, but in the host graph. The host graph can be regarded as a special case of a graph with no boundary. We will also use G to denote the induced subgraph on G, if there is no danger of confusion.

For an induced subgraph G with non-empty boundary, there are, in general, two kinds of eigenvalues, the Dirichlet eigenvalues and the Neumann eigenvalues, subject to different boundary conditions. We consider the Dirichlet eigenvalues here, and we consider the family of functions satisfying $f(x) = 0$ for any vertex x in the vertex boundary δG, which is called the Dirichlet boundary condition.

Let $V^R = \{f | f : V \to R\}$, and the Laplace operator L of a graph G be

$$Lf(x) = \frac{1}{k} \sum_{a \in K} [f(x) - f(ax)], \quad \forall f \in V^R.$$

The Dirichlet eigenvalues of an induced subgraph G of G_0 is defined as

$$\lambda = \inf_{f \neq 0} \frac{\sum_{x,y \in G \cup \delta G} (f(x) - f(y))^2}{k \sum_{x \in G} f^2(x)}.$$

An induced subgraph G of a graph G_0 with vertex boundary δG is said to be convex if for any subset $X \subset \delta G$, its neighborhood $N(X) = \{y : y \sim x \in X\}$ satisfies

$$|N(X) \setminus (G \cup \delta G)| = |y \bar{\in} G \cup \delta G : y \sim x \in X| \geq |X|.$$

Suppose a function $f : V \to R$ satisfies $Lf(x) = \lambda f(x)$, then f is called a harmonic eigenfunction of Laplace operator L on graph G with eigenvalue λ, and we can easily note that 0 is a trivial eigenvalue of L associated to the constant eigenfunction.

As we can see from the above definition, at each vertex x, the eigenfunction locally stretches the incident edges in a balanced fashion. Globally, we need to have some tools to capture the notion that how close the adjacent vertices are to each other.

Indeed, a crucial part of spectral graph theory [2] concerns understanding the behavior of eigenfunctions, and Harnack inequalities are one of the main methods for dealing with eigenfunctions. In 1994, Chung and Yau in [3] established the following Harnack inequality for homogeneous graphs and subgraphs G with edge generating set K consisting of k generators,

$$\frac{1}{k} \sum_{a \in K} [f(x) - f(ax)]^2 + \alpha \lambda f^2(x) \leq \frac{\lambda \alpha^2}{\alpha - 2} \sup_{y \in S} f^2(y)$$

for any $\alpha > 2$ and $x \in V$, and using this Harnack inequality, they derived that $\lambda \geq \frac{1}{8kD^2}$ for the Neumann eigenvalues and the Dirichlet eigenvalues in [1] and [3], respectively.

In this paper, we get a similar Harnack inequality, which extends the result of Chung and Yau for certain homogeneous graphs.

According to Bakry and Emery [4], we can define a bilinear operator $\Gamma : V^R \times V^R \to V^R$ by

$$\Gamma(f,g)(x) = \frac{1}{2} \{f(x)Lg(x) + g(x)Lf(x) - L(f(x)g(x))\},$$

and then the Ricci curvature operator on graphs Γ_2 by iterating Γ as

$$\Gamma_2(f,g)(x) = \frac{1}{2}\{\Gamma(f,Lg)(x) + \Gamma(g,Lf)(x) - L\Gamma(f,g)(x)\}.$$

More explicitly, we have

$$\Gamma(f,f)(x) = \frac{1}{2}\rho(x) = \frac{1}{2} \cdot \frac{1}{k}\sum_{a \in K}[f(x) - f(ax)]^2.$$

Definition A *The operator L satisfies the curvature-dimension type inequality $\mathrm{CD}(m,\xi)$ for some $m > 1$ and $\xi \in \mathbb{R}$ if for any $f \in V^R$,*

$$\Gamma_2(f,f)(x) \geq \frac{1}{m}(Lf(x))^2 + \xi\Gamma(f,f)(x).$$

We call m the dimension of the operator L and ξ the lower bound of the Ricci curvature of the operator L. It is easy to see that for $m < \tilde{m}$, the operator L satisfies $\mathrm{CD}(\tilde{m},\xi)$ if it satisfies $\mathrm{CD}(m,\xi)$. If $\Gamma_2(f,f)(x) \geq \xi\Gamma(f,f)(x)$, we say that L satisfies $\mathrm{CD}(\infty,\xi)$.

For Laplace operator L on a complete m dimensional Riemannion manifold, the operator L satisfies $\mathrm{CD}(m,\xi)$ if the Ricci curvature of the Riemanian manifold is bounded below by constant ξ.

In 2010, Lin and Yau proved in [5] that the Ricci flat graphs have the non-negative Ricci curvature in the sense of Bakry and Emery. In fact, in most cases, the Ricci curvature is zero, except for some very special examples like complete graph (every two vertices of graph connected by an edge), the Ricci curvature is positive. The proof is similar to the case of the grid Z^n, see Example 5 in [6]. They also proved in [5] that any locally finite connected graph satisfies either $\mathrm{CD}(2, \frac{2}{d} - 1)$ if d is finite, or $\mathrm{CD}(2,-1)$ if d is infinite, where $d = \sup_{x \in V}\sup_{y \sim x}\frac{d_x}{\mu_{xy}}$.

In this paper, since homogeneous graphs are locally finite and connected Ricci flat graphs, by using the curvature-dimension type inequality $\mathrm{CD}(m,\xi)$, we get different Harnack inequalities and eigenvalue estimate from those as in [1], [3] and [7] for certain homogeneous graphs. Our main theorems are as follows:

Theorem 1.1 *Suppose G is a finite convex subgraph in an abelian homogeneous graph that satisfies the curvature-dimension type inequality $\mathrm{CD}(m,\xi)$, and the edge generating set K consists of k generators. Let $f \in V^R$ be a harmonic eigenfunction of Laplacian L with Dirichlet eigenvalue $\lambda \neq 0$. Then the following inequality holds for all $x \in V$ and $\alpha > 2 - \frac{2\xi+2}{\lambda}$*

$$\frac{1}{k}\sum_{a \in K}[f(x) - f(ax)]^2 + \alpha\lambda f^2(x) \leq \frac{(\alpha^2 - \frac{4}{m} + 2)\lambda + 2\alpha\xi + 2\alpha}{(\alpha - 2)\lambda + 2\xi + 2} \cdot \lambda \cdot \max_{z \in V}f^2(z).$$

By taking $\alpha = 4 - \frac{2\xi+2}{\lambda} \geq 0$, we can easily get

Theorem 1.2 *Suppose G is a finite convex subgraph in an abelian homogeneous graph that satisfies the curvature-dimension type inequality $\mathrm{CD}(m,\xi)$, and the edge generating set K consists of k generators. Let $f \in V^R$ be a harmonic eigenfunction of Laplacian L with Dirichlet*

eigenvalue $\lambda \neq 0$. Then the following inequality holds for all $x \in V$

$$\frac{1}{k} \sum_{a \in K} [f(x) - f(ax)]^2 \leq [(9 - \frac{2}{m})\lambda - 4\xi - 4] \max_{z \in V} f^2(z).$$

We can use the Harnack inequality in Theorem 1.2 to derive a lower eigenvalue estimate as follows.

Theorem 1.3 *Suppose G is a finite convex subgraph in an abelian homogeneous graph that satisfies* $\mathrm{CD}(m, \xi)$, *and λ is a non-zero Dirichlet eigenvalue of Laplace operator L on G. Then*

$$\lambda \geq \frac{m + 4mkD^2(\xi + 1)}{(9m - 2)kD^2},$$

where D is the diameter of G.

Corollary 1.4 *Suppose G is a finite convex subgraph in an abelian homogeneous graph that satisfies* $\mathrm{CD}(m, 0)$, *and λ is a non-zero Dirichlet eigenvalue of Laplace operator L on G. Then*

$$\lambda \geq \frac{1 + 4kD^2}{(9 - \frac{2}{m})kD^2},$$

where D is the diameter of G.

Remark 1.5 The above four results are also applicable for general homogeneous graphs and Neumann eigenvalues defined in [1] and [3]. Moreover, our results extend and strengthen the results in [1] and [3], because in [1] and [3], they proved that $\lambda \geq \frac{1}{8kD^2}$, but from the above Corollary 1.4, since $m > 1$, we can easily check that

$$\lambda \geq \frac{1 + 4kD^2}{(9 - \frac{2}{m})kD^2} > \frac{5}{(9 - \frac{2}{m})kD^2} > \frac{1}{8kD^2}.$$

2. The proof of Theorem 1.1

First, we will establish several basic facts for homogeneous graphs. By using a modification of the proof of Theorem 1.2 in [5], we can get the following Lemma 2.1.

Lemma 2.1 *Let G be a homogeneous graph with edge generating set K consisting of k generators. Then for all $f \in V^R$ and $x \in V$, the following formula holds for the Ricci curvature operator Γ_2 on graph G*

$$\Gamma_2(f, f)(x) = \frac{1}{2}(Lf(x))^2 - \frac{1}{2}\rho(x) + \frac{1}{4k^2} \sum_{b \in K} \sum_{a \in K} [f(x) - f(ax) - f(bx) + f(abx)]^2.$$

Lemma 2.2 *Suppose G is a homogeneous graph with edge generating set K consisting of k generators satisfying* $\mathrm{CD}(m, \xi)$. *Then for all $x \in V$, we have*

$$-\frac{1}{k^2} \sum_{b \in K} \sum_{a \in K} [f(x) - f(ax) - f(bx) + f(abx)]^2 \leq (2 - \frac{4}{m})(Lf(x))^2 - 2(1 + \xi)\rho(x).$$

Proof Since G satisfies $\mathrm{CD}(m, \xi)$, we have

$$\Gamma_2(f, f)(x) \geq \frac{1}{m}(Lf(x))^2 + \xi\Gamma(f, f)(x).$$

By Lemma 2.1 and the above inequality, we get

$$\frac{1}{2}(Lf(x))^2 - \frac{1}{2}\rho(x) + \frac{1}{4k^2}\sum_{b\in K}\sum_{a\in K}[f(x) - f(ax) - f(bx) + f(abx)]^2$$

$$\geq \frac{1}{m}(Lf(x))^2 + \frac{\xi}{2}\rho(x).$$

So we have

$$-\frac{1}{k^2}\sum_{b\in K}\sum_{a\in K}[f(x) - f(ax) - f(bx) + f(abx)]^2 \leq (2 - \frac{4}{m})(Lf(x))^2 - 2(1+\xi)\rho(x). \quad \square$$

Lemma 2.3 ([1]) *For a convex subgraph G of a graph G_0, a function $f : G \cup \delta G \to R$ satisfying*

$$\sum_{y\sim x}(f(x) - f(y)) = \lambda f(x)d_x$$

for $x \in G$ and $f(x) = 0$ for $x \in \delta G$, can be extended to all vertices of G_0 which are adjacent to some vertex in $G \cup \delta G$ such that $f(z)$, for $x \in G \cup \delta G$, satisfies

$$\sum_{y\sim z}(f(z) - f(y)) = \lambda f(z)d_z,$$

where d_x denotes the degree of x in G_0.

Proof of Theorem 1.1 By Lemma 2.3, we can extend f to all vertices adjacent to some vertices in $G \cup \delta G$. Then, we set for any $\alpha > 0$

$$\phi(x) = \frac{1}{k}\sum_{a\in K}[f(x) - f(ax)]^2 + \alpha\lambda f^2(x).$$

We consider

$$L\phi(x) = \frac{1}{k}\sum_{b\in K}[\phi(x) - \phi(bx)]^2 = \frac{1}{k^2}\sum_{b\in K}\sum_{a\in K}\{[f(x) - f(ax)]^2 - [f(bx) - f(abx)]^2\}+$$

$$\frac{\alpha\lambda}{k}\sum_{b\in K}[f^2(x) - f^2(bx)] = Y + Z,$$

where Y denotes the first term and Z denotes the second term in the above second equality.

Since G is an invariant homogeneous graph, we have

$$\sum_{b\in K}[f(abx) - f(bax)] = 0.$$

By Lemma 2.2 and the above fact, we have

$$Y = \frac{1}{k^2}\sum_{b\in K}\sum_{a\in K}\{[f(x) - f(ax)]^2 - [f(bx) - f(abx)]^2\}$$

$$= -\frac{1}{k^2}\sum_{b\in K}\sum_{a\in K}[f(x) - f(ax) - f(bx) + f(abx)]^2+$$

$$\frac{2}{k^2}\sum_{b\in K}\sum_{a\in K}[f(x) - f(ax) - f(bx) + f(abx)][f(x) - f(ax)]$$

$$\leq (2 - \frac{4}{m})(Lf(x))^2 - (2 + 2\xi)\rho(x) + \frac{2}{k^2}\sum_{a\in K}[\sum_{b\in K}(f(abx) - f(bax))][f(x) - f(ax)]+$$

$$\frac{2}{k^2} \sum_{a \in K} \{\sum_{b \in K} [f(x) - f(ax) - f(bx) + f(bax)]\}[f(x) - f(ax)]$$

$$=(2 - \frac{4}{m})(Lf(x))^2 - (2 + 2\xi)\rho(x) + \frac{2}{k^2} \sum_{a \in K} [\sum_{b \in K} (f(abx) - f(bax))][f(x) - f(ax)] +$$

$$\frac{2}{k} \sum_{a \in K} \{\frac{1}{k} \sum_{b \in K} [f(x) - f(bx)] - \frac{1}{k} \sum_{b \in K} [f(ax) - f(bax)]\}[f(x) - f(ax)]$$

$$=(2 - \frac{4}{m})(Lf(x))^2 - (2 + 2\xi)\rho(x) + \frac{2\lambda}{k} \sum_{a \in K} [f(x) - f(ax)]^2$$

$$=(2\lambda^2 - \frac{4\lambda^2}{m})f^2(x) + (2\lambda - 2\xi - 2)\rho(x).$$

On the other hand,

$$Z = \frac{\alpha\lambda}{k} \sum_{b \in K} [f^2(x) - f^2(bx)] = \frac{2\alpha\lambda}{k} \sum_{b \in K} f(x)[f(x) - f(bx)] - \frac{\alpha\lambda}{k} \sum_{b \in K} [f(x) - f(bx)]^2$$

$$= 2\alpha\lambda^2 f^2(x) - \alpha\lambda\rho(x).$$

Combining the above arguments, we have the following inequality for any positive α

$$L\phi(x) \le (2\lambda^2 - \frac{4\lambda^2}{m} + 2\alpha\lambda^2)f^2(x) + (2\lambda - 2\xi - \alpha\lambda - 2)\rho(x).$$

Now we consider a vertex v which achieves the maximum value for $\phi(x)$ over all $x \in V$, and we have

$$0 \le L\phi(v) \le (2\lambda^2 - \frac{4\lambda^2}{m} + 2\alpha\lambda^2)f^2(v) + (2\lambda - 2\xi - \alpha\lambda - 2)\rho(v).$$

This implies

$$\rho(v) \le \frac{(2 - \frac{4}{m} + 2\alpha)\lambda^2}{(\alpha - 2)\lambda + 2\xi + 2} f^2(v)$$

for $\alpha > 2 - \frac{2\xi + 2}{\lambda}$. Therefore for every $x \in V$, we get

$$\rho(x) + \alpha\lambda f^2(x) \le \rho(v) + \alpha\lambda f^2(v)$$

$$\le \frac{(2 - \frac{4}{m} + 2\alpha)\lambda^2}{(\alpha - 2)\lambda + 2\xi + 2} f^2(v) + \alpha\lambda f^2(v)$$

$$\le \frac{(\alpha^2 - \frac{4}{m} + 2)\lambda + 2\alpha\xi + 2\alpha}{(\alpha - 2)\lambda + 2\xi + 2} \cdot \lambda \cdot \max_{z \in V} f^2(z).$$

That is

$$\frac{1}{k} \sum_{a \in K} [f(x) - f(ax)]^2 + \alpha\lambda f^2(x) \le \frac{(\alpha^2 - \frac{4}{m} + 2)\lambda + 2\alpha\xi + 2\alpha}{(\alpha - 2)\lambda + 2\xi + 2} \cdot \lambda \cdot \max_{z \in V} f^2(z). \quad \square$$

3. The proof of Theorem 1.3

Now, we give the proof of Theorem 1.3 by using the Harnack inequality in Theorem 1.2.

Proof Let $f \in V^R$ be a harmonic eigenfunction of Laplacian L with eigenvalue $\lambda \neq 0$. That is

$Lf(x) = \lambda f(x)$ for every $x \in V$. Then $\sum_{x \in V} f(x) = 0$. So we can assume that

$$\sup_{x \in V} f(x) = 1 > \inf_{x \in V} f(x) = k < 0.$$

Take $x_1, x_n \in V$ such that $f(x_1) = \sup_{x \in V} f(x) = 1, f(x_n) = \inf_{x \in V} f(x) = k < 0$ and let $x_1, x_2, \ldots, x_n$ be the shortest path joining x_1 and x_n, where $x_i \sim x_{i+1}$. Then $n \leq D$, where D is the diameter of G. From Theorem 1.2 we have

$$[f(x_i) - f(x_{i+1})]^2 \leq k\rho(x_i) \leq k[(9 - \frac{2}{m})\lambda - 4\xi - 4].$$

Therefore,

$$\sum_{i=1}^{n-1} [f(x_i) - f(x_{i+1})]^2 \leq k[(9 - \frac{2}{m})\lambda - 4\xi - 4]D.$$

On the other hand, by using the Cauchy-Schwartz inequality we have

$$\sum_{i=1}^{n-1} [f(x_i) - f(x_{i+1})]^2 \geq \frac{1}{D}[f(x_n) - f(x_1)]^2 \geq \frac{1}{D}.$$

So we get

$$\lambda \geq \frac{m + 4mkD^2(\xi + 1)}{(9m - 2)kD^2}.$$

This completes the proof of Theorem 1.3. $\square$

References

[1] F. CHUNG, S. T. YAU. *A Harnack inequality for Dirichlet eigenvalues.* J. Graph Theory, 2000, **34**(4): 247–257.
[2] F. CHUNG. *Spectral Graph Theory.* CMBS Lecture Notes, AMS Publication, 1996.
[3] F. CHUNG, S. T. YAU. *A Harnack inequality for homogeneous graphs and subgraphs.* Comm. Anal. Geom., 1994, **2**(4): 627–640.
[4] D. BAKRY, M. EMERY. *Diffusions Hypercontractives.* Springer, Berlin, 1985.
[5] Yong LIN, S. T. YAU. *Ricci curvature and eigenvalue estimate on locally finite graphs.* Math. Res. Lett., 2010, **17**(2): 343–356.
[6] Y. OLLIVIER. *Ricci curvature of Markov chains on metric spaces.* J. Funct. Anal., 2009, **256**(3): 810–864.
[7] F. CHUNG, Yong LIN, S. T. YAU. *Harnack inequalities for graphs with non-negative Ricci curvature.* J. Math. Anal. Appl., 2014, **415**(1): 25–32.