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A Note on the Signless Laplacian and Distance Signless Laplacian Eigenvalues of Graphs

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Abstract Let G be a simple graph. We first show that $\delta_i \geq d_i - \sqrt{\lfloor \frac{i}{2} \rfloor \lceil \frac{i}{2} \rceil}$, where δ_i and d_i denote the i-th signless Laplacian eigenvalue and the i-th degree of vertex in G, respectively. Suppose G is a simple and connected graph, then some inequalities on the distance signless Laplacian eigenvalues are obtained by deleting some vertices and some edges from G. In addition, for the distance signless Laplacian spectral radius $\rho_{\mathcal{Q}}(G)$, we determine the extremal graphs with the minimum $\rho_{\mathcal{Q}}(G)$ among the trees with given diameter, the unicyclic and bicyclic graphs with given girth, respectively.

Keywords signless Laplacian; distance signless Laplacian; spectral radius; eigenvalues.

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1. Introduction

Let $G = G(V, E)$ be a graph with vertex set V and edge set E. The order and size of G are defined as $|V|$ and $|E|$, respectively. Denote by $N_G(u)$ the set of vertices adjacent to u, called the neighbor set of u. Then the degree of u is defined as $|N_G(u)|$. The signless Laplacian matrix of a simple graph G is defined to be $Q = A + D$, where A denotes the adjacency matrix and D is the diagonal matrix of vertex degrees of G. We suppose graph G to be connected when distance of vertices is considered in G. The distance between vertex u and v, denoted by $d_G(u, v)$, is the length of a shortest path from u to v. The transmission $Tr(u)$ of vertex u is defined to be the sum of distances from u to all other vertices, i.e., $Tr(u) = \sum_{v \in V(G)} d_G(v, u)$. The distance matrix of G, denoted by $\mathcal{D}(G)$, is a symmetric real matirx with (i, j)-entry being $d_G(v_i, v_j)$. Obviously, $Tr(v_i)$ is the sum of i-th row of $\mathcal{D}(G)$. Denote by $\mathrm{diag}(Tr)$ the diagonal matrix of the vertex transmissions in G. Similar to the signless Laplacian matrix of a graph, the distance signless Laplaian matrix of graph G is introduced by Aouchiche and Hansen [1], defined as $\mathcal{Q}(G) = \mathrm{diag}(Tr) + \mathcal{D}(G)$. The eigenvalues of $\mathcal{Q}(G)$, called distance signless Laplaian eigenvalues of G, are written as $\{q_1(G), q_2(G), \ldots, q_n(G)\}$. Without loss of generality, assume that $q_n(G) \leq \cdots \leq q_2(G) \leq q_1(G)$. Denote by $\rho_{\mathcal{Q}}(G) = q_1(G)$ the distance signless Laplacian spectral radius. Let $P_{\mathcal{Q}}(t)$ denote the distance signless Laplacian characteristic polynomial. As

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usual, we use K_n, C_n, P_n and S_n to denote the complete graph, the cycle, the path and the star with order n, respectively. $K_{a,b}$ means the complete bipartite graph with two colour classes of order a and b. Identity matrix is denoted by I with order following from the context. Let J_n be the matrix of order n with all entries one. The clique, regarded as an induced subgraph of G, is a complete graph. Denote by $G-e$ the graph obtained by removing edge $e\in E(G)$ from G. $G-u$ denotes the graph obtained by deleting the vertex u and all the edges incident to it. Generally, for $S\subset V(G)$, $G-S$ denotes the graph derived from deleting all the vertices of S and edges incident to each vertex of S from graph G. Two graphs G_1 and G_2 are isomorphic if there is a bijection, say φ, from $V(G_1)$ to $V(G_2)$, such that for $x,y\in V(G_1)$, x is adjacent to y if and only if $\varphi(x)$ is adjacent to $\varphi(y)$, in G_2. For the notions not mentioned here, readers can see them among the text and can also refer to [1, 2].

So far, the signless Laplacian eigenvalues and distance eigenvalues have been studied deeply [7–10, 12–14]. But the distance (signless) Laplacian matrix has just been proposed by Aouchiche and Hansen [1], and not many papers are available on it. In [2], Aouchiche and Hansen investigated some particular distance Laplacian eigenvalues and gave some properties of the distance Laplacian spectrum. The unique graphs with minimum and second minimum distance signless Laplacian spectral radius among bicyclic graphs with fixed vertex number were determined in [3]. Xing, Zhou and Li [4] determined the graphs with minimum distance signless Laplacian spectral radius among some classes of graphs with some given conditions.

2. Lower bound for the signless Laplacian eigenvalues of a graph

Before giving the main result, some well-known conclusions are necessary.

Lemma 2.1 (Interlacing theorem)([5, p.30]) *Let A be a symmetric real matrix and B be a principal submatrix of A with order n and s ($s\le n$), respectively. For the eigenvalues of A and B, then*

$$\lambda_{i+n-s}(A)\le\lambda_i(B)\le\lambda_i(A),\quad 1\le i\le s.$$

Lemma 2.2 (Courant-Weyl inequality)([5, p.31]) *Let H_1 and H_2 be symmetric real matrices with order n. For $1\le i\le n$, the eigenvalues of H_1 and H_2 satisfy:*

$$\lambda_n(H_2)+\lambda_i(H_1)\le\lambda_i(H_1+H_2)\le\lambda_i(H_1)+\lambda_1(H_2).$$

Lemma 2.3 ([11, Proposition 2]) *Let G be a simple graph of order n. Then the least eigenvalue $\lambda_n(A)$ of the adjacency matrix A of G satisfies:*

$$\lambda_n(A)\ge-\sqrt{\lfloor\frac{n}{2}\rfloor\lceil\frac{n}{2}\rceil},$$

and the equality holds if and only if $G=K_{\lfloor\frac{n}{2}\rfloor\lceil\frac{n}{2}\rceil}$, where $K_{\lfloor\frac{n}{2}\rfloor\lceil\frac{n}{2}\rceil}$ is the complete bipartite graph with two color classes of order $\lfloor\frac{n}{2}\rfloor$ and $\lceil\frac{n}{2}\rceil$.

The following theorem demonstrates a lower bound for each signless Laplacian eigenvalue.

Theorem 2.4 *Let G be a simple graph of order n, and let $\delta_1\ge\delta_2\ge\cdots\ge\delta_n$ be the signless*

Laplacian eigenvalues of G. For $1 \le i \le n$, then

$$\delta_i \ge d_i - \sqrt{\lfloor \frac{i}{2} \rfloor \lceil \frac{i}{2} \rceil}.$$

Proof Without loss of generality let us take $d_1 \ge d_2 \ge \cdots \ge d_n$, where d_i means the degree of v_i. Let M and A' be the left-top $i \times i$ principal submatrix of the signless Laplacian matrix Q and the adjacency matrix A, respectively. Let $H = d_i I + A'$ where I is the identity matrix of order i. Then let $P = \mathrm{diag}\{d_1 - d_i, d_2 - d_i, \ldots, d_{i-1} - d_i, 0\}$ be a diagonal matrix with the least eigenvalue 0. Obviously, $M = H + P$ where M, H and P are Hermitian matrices of order i.

By Lemmas 2.1 and 2.2, we have $\delta_i \ge \lambda_i(M) \ge \lambda_i(H) + \lambda_i(P) = \lambda_i(H)$. Moreover, the eigenvalues of H are $\lambda_k(H) = d_i + \lambda_k(A')$, $k = 1, 2, \ldots, i$. Actually, the matrix A' is the adjacency matrix of the subgraph indexed by $\{v_1, v_2, \ldots, v_i\}$ of G. Then $\lambda_k(A') \ge -\sqrt{\lfloor \frac{i}{2} \rfloor \lceil \frac{i}{2} \rceil}$ $(k = 1, 2, \ldots, i)$ follows from Lemma 2.3. Finally, we get

$$\delta_i \ge \lambda_i(M) \ge \lambda_i(H) + \lambda_i(P) \ge d_i - \sqrt{\lfloor \frac{i}{2} \rfloor \lceil \frac{i}{2} \rceil}. \quad \square$$

3. Inequalities on the distance signless Laplacian eigenvalues

For a simple and connected graph G, obviously, $\mathcal{Q}(G)$ is a symmetric real matrix. Then by Lemma 2.1, the following corollary is clear.

Corollary 3.1 *Let G be a graph with order n. Let M be the principal submatrix of $\mathcal{Q}(G)$ with order $n-1$. Then,*

$$q_1(G) \ge \lambda_1(M) \ge q_2(G) \ge \cdots \ge \lambda_{n-1}(M) \ge q_n(G).$$

A pendent vertex in a graph is a vertex with degree one. The diameter of graph G, denoted by $d(G)$ (d, for brevity), is defined as the largest value of distances of any two vertices in G. For two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ with order n, if $a_{ij} \le b_{ij}$ $(1 \le i, j \le n)$, we say $A \le B$ and $A < B$, if $a_{ij} < b_{ij}$ $(1 \le i, j \le n)$.

Theorem 3.2 *Let u be a pendent vertex of G and $d(G) = d$ be the diameter of G. For $i = 1, 2, \ldots, n-1$,*

$$q_{i+1}(G) - d \le q_i(G - u) \le q_i(G) - 1.$$

Proof Since u is a pendent vertex, we can easily get $d_{G-u}(x, y) = d_G(x, y)$ for $x, y \in V(G-u)$, and $1 \le d_G(u, w) \le d$ for $w \in V(G-u)$. Therefore, $Tr_G(w) > Tr_{G-u}(w), w \in V(G-u)$. Let M be the principal submatrix of $\mathcal{Q}(G)$ obtained by deleting the row and column corresponding to u. Then $M \ge \mathcal{Q}(G-u)$ and $M \ne \mathcal{Q}(G-u)$. Let $P = M - \mathcal{Q}(G-u)$. Then P is a diagonal matrix with the least diagonal entries not less than one and the largest diagonal entries not more than d obviously, i.e., the eigenvalues of P satisfy

$$1 \le \lambda_i(P) \le d, \quad i = 1, 2, \ldots, n-1. \tag{3.1}$$

Thus by Lemma 2.2 and inequality (3.1), for $M, \mathcal{Q}(G-u)$ and P, we can get

$$q_i(G-u)+1 \le \lambda_i(M) \le q_i(G-u)+d, \quad i=1,2,\ldots,n-1. \tag{3.2}$$

By Corollary 3.1, it is obtained that

$$q_i(G) \ge \lambda_i(M) \ge q_{i+1}(G), \quad i=1,2,\ldots,n-1. \tag{3.3}$$

Combining the left inequalities of (3.2) and (3.3), we have

$$q_i(G-u)+1 \le q_i(G), \quad i=1,2,\ldots,n-1. \tag{3.4}$$

Similarly, combining the right inequalities of (3.2) and (3.3) gives

$$q_{i+1}(G) \le q_i(G-u)+d, \quad i=1,2,\ldots,n-1. \tag{3.5}$$

The proof is completed by (3.4) and (3.5). □

Corollary 3.3 *Let G be a graph on n vertices with diameter $d(G)=2$. Suppose vertex v is adjacent to any other vertex of G and $G-v$ is connected with $d(G-v)=d(G)$, then the eigenvalues of $\mathcal{Q}(G-v)$ interlace those of $\mathcal{Q}(G)-I$, i.e.,*

$$q_{i+1}(G)-1 \le q_i(G-v) \le q_i(G)-1, \quad i=1,2,\ldots,n-1.$$

Proof Since v is adjacent to any other vertex of G and $d(G-v)=d(G)=2$, we obtain $d_{G-v}(x,y)=d_G(x,y)$ for any $x,y \in V(G-v)$. Hence, $Tr_G(x)=Tr_{G-v}(x)+1$ for each $x \in V(G-v)$. Let M be the principal submatrix of $\mathcal{Q}(G)$ derived from deleting the row and column corresponding to v and $P=M-\mathcal{Q}(G-v)$. Then P is equal to the identity matrix I. By Lemma 2.2, it is obtained that

$$q_i(G-v)+1 \le \lambda_i(M) \le q_i(G-v)+1, \quad i=1,2,\ldots,n-1,$$

where $\lambda_i(M)$ denotes the i-th largest eigenvalue of M.

From Corollary 3.1, we see $q_i(G) \ge \lambda_i(M) \ge q_{i+1}(G)$, $i=1,2,\ldots,n-1$, where $\lambda_i(M)$ is defined as above. Thus similar to the method of Theorem 3.2, the conclusion is obtained. □

For graph G, $u,v \in V(G)$ are called multiplicate vertices, if $N_G(u)=N_G(v)$. Suppose u is adjacent to v and $N_{G-v}(u)=N_{G-u}(v)$, then u,v are called quasi-multiplicate vertices. In general, $S \subset V(G)$ is a *multiplicate* vertex set, if $N_G(u)=N_G(v)$ for $u,v \in S$; $C \subset V(G)$ is a quasi-multiplicate vertex set, if the vertices of C induce a clique and $N_G(u)-C=N_G(v)-C$ for $u,v \in C$. Obviously, if we add edges to any two vertices of a multiplicate vertex set, then we obtain a quasi-multiplicate vertex set.

Corollary 3.4 *For graph G of order n and $u,v \in V(G)$, if u,v are multiplicate (or quasi-multiplicate) vertices, then*

$$q_{i+1}(G)-d \le q_i(G-v) \le q_i(G)-1.$$

In fact, in Corollary 3.4, since u,v are multiplicate (or quasi-multiplicate) vertices, then $d_G(u,w)=d_G(v,w)$, for $w \in V(G)$ and $w \ne u,v$. Moreover, for $x,y \in V(G-v)$, $d_{G-v}(x,y)=$

$d_G(x,y)$. Then by Lemma 2.2 and Corollary 3.1, the conclusion can be proved in the similar way as Theorem 3.2.

In [1], the authors demonstrate that the eigenvalues of $\mathcal{Q}(G)$ are non-decreasing when some edges are removed with the resultant graph also connected. The following lemma is on the behavior of distance signless Laplacian eigenvalues when the edge between quasi-multiplicate vertices is removed. And by it, we have a theorem in general.

Lemma 3.5 *Let x and y be quasi-multiplicate vertices of G and $|V(G)| = n \geq 3$. Denote the edge between x and y by e. Let q_i be the eigenvalues of $\mathcal{Q}(G)$ and q_i' be the eigenvalues of $\mathcal{Q}(G-e)$. For $i = 1, 2, \ldots, n$, then $q_i \leq q_i' \leq q_i + 2$.*

Proof As x and y are quasi-multiplicate vertices, apart from the change of distance between x and y from one to two, the distances of other vertices are invariable. So $\mathcal{Q}(G-e) \geq \mathcal{Q}(G)$ and let $P = \mathcal{Q}(G-e) - \mathcal{Q}(G)$. Then P can be partitioned into $\begin{pmatrix} J_2 & 0 \\ 0 & 0 \end{pmatrix}$ and the eigenvalues of P are 2 and 0 with multiplicity 1 and n$-$1, respectively. Thus, the conclusion follows by Lemma 2.2. $\square$

Theorem 3.6 *Let $C \subset V(G)$ be a quasi-multiplicate set of graph G and $2 \leq m = |C| < |V(G)| = n$. Suppose G' is the graph obtained by removing all the edges between vertices of C. Let q_i be the eigenvalues of $\mathcal{Q}(G)$ and q_i' be those of $\mathcal{Q}(G')$, $i = 1, 2, \ldots, n$. Then,*

$$q_i \leq q_i' \leq q_i + 2m - 2, \quad i = 1, 2, \ldots, n.$$

Proof Obviously, C becomes a multiplicate set in G'. Similarly to Lemma 3.5, in the process of deleting edges, only the distances of vertices in C change from one to two. Let $P = \mathcal{Q}(G') - \mathcal{Q}(G)$. Then P can be partitioned into $\begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}$, where $M = (m-2)I + J_m$ with order m. It is easy to know the eigenvalues of M are $2m-2$ and $m-2$ with multiplicity 1 and $m-1$, respectively. Hence, the eigenvalues of P are $2m-2$, $m-2$ and 0 with multiplicity 1, $m-1$ and $n-m$, respectively. Thus the theorem follows from Lemma 2.2. $\square$

4. Extremal graphs with minimum $\rho_{\mathcal{Q}}(G)$

For trees with given diameter d, the following theorem shows that P_{d+1} is the extremal graph with the minimum $\rho_{\mathcal{Q}}(G)$.

Theorem 4.1 *Let $\mathcal{T}_d$ be the set of all trees with given diameter $d \geq 1$. Then for any tree $T \in \mathcal{T}_d$, the distance signless Laplacian spectral radius $\rho_{\mathcal{Q}}(T) \geq \rho_{\mathcal{Q}}(P_{d+1})$ with equality if and only if $T = P_{d+1}$, where P_{d+1} denotes the path of order $d+1$.*

Proof Let tree $T \in \mathcal{T}_d$ with order $n \geq d+1$. From Theorem 3.2, we see that $q_1(G-u) \leq q_1(G)-1$, i.e., $q_1(G-u) < q_1(G)$ where u is a pendent vertex. In other words, the distance signless Laplacian spectral radius $\rho_{\mathcal{Q}}(G)$ strictly decreases when the pendent vertices are removed from

G. Thus, the conclusion follows by continuously deleting the pendent vertices which are not on the diametrical line. $\square$

Lemma 4.2 ([1]) *The distance signless Laplacian characteristic polynomial of cycle C_n is as follows.*

$$P_Q(t)=\begin{cases}(t-\frac{n^2}{4})^{k-1}\cdot(t-\frac{n^2}{2})\cdot\prod_{j=1}^{k}(t-\frac{n^2}{4}+\csc^2(\frac{\pi(2j-1)}{n})), & \text{if } n=2k;\\ (t-\frac{n^2-1}{2})\cdot\prod_{j=1}^{k}(t-\frac{n^2-1}{4}+\frac{1}{4}\sec^2(\frac{\pi j}{n}))(t-\frac{n^2-1}{4}+\frac{1}{4}\csc^2(\frac{\pi(2j-1)}{2n})), & \\ \qquad \text{if } n=2k+1.\end{cases}$$

Then by calculating, for the distance signless Laplacian spectral radius of C_n, we have

$$\rho_Q(C_n)=\begin{cases}\frac{n^2}{2}, & \text{if } n=2k \text{ (i.e., even)};\\ \frac{n^2-1}{2}, & \text{if } n=2k+1 \text{ (i.e., odd)}.\end{cases}$$

A simple connected graph G is called unicyclic if $|V(G)|=|E(G)|$, bicyclic if $|V(G)|+1=|E(G)|$. The girth of graph G is the length of the shortest cycle (if exists).

Theorem 4.3 *Let $\mathcal{U}_g$ be the set of all unicyclic graphs with given girth $g\geq 3$. For any unicyclic graph $G\in\mathcal{U}_g$,*

(i) $\rho_Q(G)\geq\frac{g^2}{2}$, if g is even;

(ii) $\rho_Q(G)\geq\frac{g^2-1}{2}$, if g is odd.

Equalities hold if and only if $G=C_g$.

Proof Let $G\in\mathcal{U}_g$ and $V(G)=V_1\bigcup V_2$. Without loss of generality, let the vertices of the cycle be $V_1=\{v_1,v_2,\ldots,v_g\}$. Then the components of subgraph induced by $V_2=\{v_{g+1},v_{g+2},\ldots,v_n\}$ are isolated vertices or trees. Assume that G has the minimum distance signless Laplacian spectral radius with order $n>g$, then $V_2\neq\emptyset$. By Theorem 3.2, we obtain another graph $G-v_i$ with less distance signless Laplacian spectral radius, where $v_i\in V_2$ is a pendent vertex, a contradiction. Thus $G=C_g$ has the minimum distance signless Laplacian spectral radius and the conclusion follows from Lemma 4.2. $\square$

For graph G, let $e\in E(G)$ and the two incident vertices be u and v. Replace e with a new vertex, say $h\notin V(G)$, and make h adjacent to u and v. This operation of graph is known as *edge subdivision*. Remove e from graph G and identify the two vertices incident to e. We call this operation edge contraction. A cut-edge of connected graph G is an edge $e\in E(G)$ such that $G-e$ is disconnected.

Recall that the spectral radius of a nonnegative irreducible matrix increases if an entry increases [6, p.38]. Then before demonstrating the conclusion on bicyclic graphs, we first give the following important and useful lemmas.

Lemma 4.4 *Let G_s be the graph derived from subdividing an edge, say e, of graph G. Then*

$\rho_{\mathcal{Q}}(G_s) > \rho_{\mathcal{Q}}(G)$.

Proof Let the new vertex be h. Then $V(G_s) = V(G)\bigcup\{h\}$. For $\forall x, y \in V(G)$, by the definition of distance of vertex, we easily obtain $d_G(x,y) \le d_{G_s}(x,y)$ and $Tr_{G_s}(x) > Tr_G(x)$. Suppose M is the principal submatrix of $\mathcal{Q}(G_s)$ derived from deleting the row and column corresponding to h. So $0 < M$ is an irreducible matrix and $M \ge \mathcal{Q}(G)$ ($M \ne \mathcal{Q}(G)$). Thus we get $\rho(M) > \rho_{\mathcal{Q}}(G)$, where $\rho(M)$ denotes the spectral radius of M. Therefore, $\rho_{\mathcal{Q}}(G_s) \ge \rho(M) > \rho_{\mathcal{Q}}(G)$ from Lemma 2.1. □

Lemma 4.5 *Let $e \in E(G)$ be a cut-edge of graph G. Let G_c be the graph obtained by contracting e. Then $\rho_{\mathcal{Q}}(G_c) < \rho_{\mathcal{Q}}(G)$.*

Proof Let the vertices incident to edge e be u and v. By contracting e, without loss of generality, let v be identified with u. Thus $V(G) = V(G_c)\bigcup\{v\}$. Moveover, in fact, $d_G(x,y) \ge d_{G_c}(x,y)$ and $Tr_G(x) > Tr_{G_c}(x)$ for any $x, y \in V(G_c)$. The remaining proof is similar to that of Lemma 4.4, and is omitted. □

For bicyclic graph G, we call it type of ∞, if it has an induced subgraph isomorphic to G_1 (see Figure 1), and type of Θ, if it has an induced subgraph isomorphic to G_2 (see Figure 1).

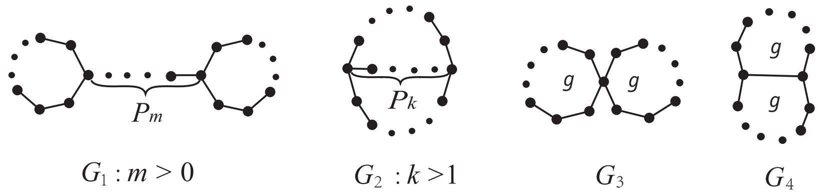

Figure 1 The graphs G_1, G_2 and G_3, G_4 (g denotes the length of cycle)

Theorem 4.6 *Let G be a bicyclic graph with given girth $g \ge 3$. Then,*

(i) If G is type of ∞, $\rho_{\mathcal{Q}}(G) \ge \rho_{\mathcal{Q}}(G_3)$;

(ii) If G is type of Θ, $\rho_{\mathcal{Q}}(G) \ge \rho_{\mathcal{Q}}(G_4)$.

For (i) and (ii), equalities hold if and only if G is isomorphic to G_3 and G_4 (see Figure 1), respectively.

Proof (i) Assume bicyclic graph G with girth g having the minimum distance signless Laplacian spectral radius is not isomorphic to G_3. Then through the following steps we get contradictions.

Step 1. Let $G^{(1)}$ be the induced subgraph of G isomorphic to G_1. If $G^{(1)}$ is the proper induced subgraph of G, i.e., the order of G is more than that of $G^{(1)}$. By the method of deleting pendent vertices and Theorem 3.2, we obtain $\rho_{\mathcal{Q}}(G) > \rho_{\mathcal{Q}}(G^{(1)})$, a contradiction.

Step 2. From Step 1, if G has the minimum $\rho_{\mathcal{Q}}(G)$, G is necessarily isomorphic to G_1. Then we let G be isomorphic to G_1. Furthermore assume the length of the other cycle in G is larger than g. Then by the inverse of Lemma 4.4, we can get a graph, say $G^{(2)}$, possessing less distance signless Laplacian spectral radius, which has two cycles with the same length g, a contradiction.

Step 3. After the above steps, let G be isomorphic to G_1 and have same length g of cycles. Suppose the length of the path P_m (see G_1 in Figure 1) between the two cycles of G is more than zero (i.e., $m \geq 2$). If $m = 2$, by Lemma 4.5, we derive a new graph, say $G^{(3)}$, with less distance signless Laplacian spectral radius than G. If $m > 2$, we also obtain a contradiction from Lemmas 4.4 and 4.5. Thus the length of P_m in G is zero.

By the three steps, if G is type of ∞ and has the minimum $\rho_Q(G)$, G is isomorphic to G_3. Then the proof of (i) is done.

The proof of (ii) can be testified in the similar way, omitted. $\square$

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