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Bilinear Fourier Multiplier Operator on Morrey Spaces

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Abstract In this paper, by establishing a result concerning the mapping properties for bi(sub)linear operators on Morrey spaces, and the weighted estimates with general weights for the bilinear Fourier multiplier, the author establishes some results concerning the behavior on the product of Morrey spaces for bilinear Fourier multiplier operator with associated multiplier σ satisfying certain Sobolev regularity.

Keywords bilinear Fourier multiplier; Morrey space; weighted norm inequality.

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1. Introduction

The study of mapping properties for bilinear Fourier multiplier operator was originated by Coifman and Meyer in their cerebrated works [6, 7]. Let $\sigma \in L^{\infty}(\mathbb{R}^{2n})$. Define the bilinear Fourier multiplier operator T_σ by

$$T_\sigma(f_1, f_2)(x) = \int_{\mathbb{R}^{2n}} \exp(2\pi i x(\xi_1+\xi_2))\sigma(\xi_1,\xi_2)\mathcal{F}f_1(\xi_1)\mathcal{F}f_2(\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2 \tag{1.1}$$

for $f_1,\ f_2 \in \mathscr{S}(\mathbb{R}^n)$, where and in the following, for $f \in \mathscr{S}(\mathbb{R}^n)$, $\mathcal{F}f$ denotes the Fourier transform of f. As it is well known, the study of bilinear Fourier multiplier operator was originated by Coifman and Meyer, and then by many other authors. Coifman and Meyer [7] proved that if $\sigma \in C^s(\mathbb{R}^{2n}\backslash\{0\})$ satisfies

$$|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\sigma(\xi_1,\, \xi_2)| \le C_{\alpha_1,\,\alpha_2}(|\xi_1|+|\xi_2|)^{-(|\alpha_1|+|\alpha_2|)} \tag{1.2}$$

for all $|\alpha_1|+|\alpha_2| \le s$ with $s \ge 4n+1$, then T_σ is bounded from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p_1,\, p_2,\, p < \infty$ with $1/p = 1/p_1 + 1/p_2$. For the case of $s \ge 2n+1$, Kenig-Stein [14] and Grafakos-Torres [12] improved Coifman and Meyer's multiplier theorem to the indices $1/2 \le p \le 1$ by the multilinear Calderón-Zygmund operator theory. Fairly recently, considerable attention has been paid to the behavior on function spaces for T_σ when the multiplier satisfies certain Sobolev regularity condition. Let $\Phi \in \mathscr{S}(\mathbb{R}^{2n})$ satisfy

$$\begin{cases} \operatorname{supp}\Phi \subset \{(\xi_1,\,\xi_2):\, 1/2 \le |\xi_1|+|\xi_2| \le 2\}; \\ \sum_{\kappa\in\mathbb{Z}} \Phi(2^{-\kappa}\xi_1,\, 2^{-\kappa}\xi_2) = 1 \ \text{ for all } (\xi_1,\,\xi_2) \in \mathbb{R}^{2n}\backslash\{0\}. \end{cases} \tag{1.3}$$

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For $\kappa \in \mathbb{Z}$, set

$$\sigma_\kappa(\xi_1, \xi_2) = \Phi(\xi_1, \xi_2)\sigma(2^\kappa \xi_1, 2^\kappa \xi_2).$$

Tomita [22] proved that if

$$\sup_{\kappa\in\mathbb{Z}} \int_{\mathbb{R}^{2n}} (1+|\xi_1|^2+|\xi_2|^2)^s |\mathcal{F}\sigma_\kappa(\xi_1, \xi_2)|^2 \mathrm{d}\xi_1 \mathrm{d}\xi_2 < \infty \tag{1.4}$$

for some $s > n$, then T_σ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ provided that $p_1, p_2, p \in (1, \infty)$ and $1/p = 1/p_1 + 1/p_2$. Grafakos and Si [11] considered the mapping properties from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for T_σ when σ satisfies (1.4) and $p \le 1$. Miyachi and Tomita [16] considered the problem to find minimal smoothness condition for bilinear Fourier multiplier. Let

$$\|\sigma_\kappa\|_{W^{s_1, s_2}(\mathbb{R}^{2n})} = \Big(\int_{\mathbb{R}^{2n}} \langle\xi_1\rangle^{2s_1}\langle\xi_2\rangle^{2s_2} |\mathcal{F}\sigma_\kappa(\xi_1, \xi_2)|^2 \mathrm{d}\xi_1 \mathrm{d}\xi_2\Big)^{1/2},$$

where $\langle\xi_k\rangle := (1+|\xi_k|^2)^{1/2}$. Miyachi and Tomita [16] proved that if

$$\sup_{\kappa\in\mathbb{Z}} \|\sigma_\kappa\|_{W^{s_1, s_2}(\mathbb{R}^{2n})} < \infty \tag{1.5}$$

for some $s_1, s_2 \in (n/2, n]$, then T_σ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for any $p_1, p_1 \in (1, \infty)$ and $p \ge 2/3$ with $1/p = 1/p_1 + 1/p_2$. Moreover, they also gave minimal smoothness condition for which T_σ is bounded from $H^{p_1}(\mathbb{R}^n) \times H^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. For other works about the behavior of bilinear Fourier multiplier operators on various function spaces, we refer to the papers [3, 8, 10] and the related references therein.

The purpose of this paper is to consider the behavior on the produce of Morrey spaces for T_σ when σ satisfies (1.5). To formulate our resuts, we first recall the definition of Morrey space.

Definition 1.1 *Let $p \in (0, \infty)$, $\lambda \in (0, n)$. The Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ is defined as*

$$L^{p,\lambda}(\mathbb{R}^n) = \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} < \infty\}$$

with

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{y\in\mathbb{R}^n, r>0} \big(\frac{1}{r^\lambda}\int_{B(y, r)} |f(x)|^p \mathrm{d}x\big)^{1/p},$$

where and in the following, $B(y, r)$ denotes the ball in $\mathbb{R}^n$ centered at y and having radius r.

Definition 1.2 *Let $p \in (0, \infty)$, $\lambda \in (0, n)$. The weak type Morrey space $WL^{p,\lambda}(\mathbb{R}^n)$ is defined as*

$$WL^{p,\lambda}(\mathbb{R}^n) = \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{WL^{p,\lambda}(\mathbb{R}^n)} < \infty\}$$

with

$$\|f\|_{WL^{p,\lambda}(\mathbb{R}^n)} = \sup_{y\in\mathbb{R}^n, r>0, t>0} \big(\frac{1}{r^\lambda}|\{x \in B(x, r) : |f(x)| > t\}|\big)^{1/p}.$$

The Morry space was introduced by Morrey [17] in 1938. It is well-known that this space is closely related to some problems in PED (see [19, 20]), and has great interest in harmonic analysis (see [1, 2, 5] and the references therein).

Theorem 1.3 *Let σ be a multiplier satisfying (1.5) for some $s_1, s_2 \in (n/2, n]$ and T_σ be the* *operator defined by (1.1). For $k = 1, 2$, set $t_k = n/s_k$. Then T_σ is bounded from $L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n)$ to $L^{p,\lambda}(\mathbb{R}^n)$ with $1/p = 1/p_1 + 1/p_2$ and $\lambda = \lambda_1 p/p_1 + \lambda_2 p/p_2$, provided that $p_1, p_2, \lambda_1, \lambda_2$ satisfy one of the following three conditions*

(i) $p_1 \in (t_1, \infty)$, $p_2 \in (t_2, \infty)$, $\lambda_1, \lambda_2 \in (0, n)$;

(ii) $p_1 \in (1, \infty)$, $p_2 \in (t_2, \infty)$, $\lambda_1 \in (0, s_1)$ and $\lambda_2 \in (0, n)$;

(iii) $p_1 \in (1, t_1]$, $p_2 \in (1, t_2]$ such that $1/p < 1 + 1/t^*$ with $t^* = \min\{t_1, t_2\}$, $\lambda_1 \in (0, s_1)$ and $\lambda_2 \in (0, s_2)$.

Theorem 1.4 *Let σ be a multiplier satisfying (1.5) for some $s_1, s_2 \in (n/2, n]$ and T_σ be the operator defined by (1.1). Then for $p_2 \in (t_2, \infty)$, $\lambda_1 \in (0, s_1)$ and $\lambda_2 \in (0, n)$, T_σ is bounded from $L^{1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n)$ to $WL^{p,\lambda}(\mathbb{R}^n)$ with $1/p = 1 + 1/p_2$ and $\lambda = \lambda_1 p + \lambda_2 p/p_2$.*

Throughout the article, C always denotes a positive constant that may vary from line to line but remains independent of the main variables. We use the symbol $A \lesssim B$ to denote that there exists a positive constant C such that $A \le CB$. For any set $E \subset \mathbb{R}^n$, χ_E denotes its characteristic function. We use $B(x, R)$ to denote a ball centered at x with radius R. For a ball $B \subset \mathbb{R}^n$ and $\lambda > 0$, we use λB to denote the ball concentric with B whose radius is λ times of B's. For a locally integrable function f, let $M^\sharp f$ be the Fefferman-Stein sharp maximal function of f. For $r \in (0, \infty)$, let $M_r^\sharp$ be the operator defined by

$$M_r^\sharp f(x) = \big(M^\sharp(|f|^r)(x)\big)^{1/r}.$$

For $r \in [0, \infty)$, a suitable function f and a cube Q, let

$$\|f\|_{L(\log L)^r, Q} = \inf\Big\{\lambda > 0 : \frac{1}{|Q|}\int_Q \frac{|f(x)|}{\lambda}\log^r\Big(1 + \frac{|f(x)|}{\lambda}\Big)\mathrm{d}x \le 1\Big\}.$$

The maximal operator $M_{L(\log L)^r}$ is defined by

$$M_{L(\log L)^r} f(x) = \sup_{Q \ni x} \|f\|_{L(\log L)^r, Q}.$$

It is obvious that if $r = 0$, then $M_{L(\log L)^r}$ is just the Hardy-Littlewood maximal operator M.

2. Proofs of Theorems

Let $\sigma \in L^\infty(\mathbb{R}^{2n})$ and $\Phi \in \mathscr{S}(\mathbb{R}^{2n})$ satisfy (1.3). For $\kappa \in \mathbb{Z}$, define

$$\widetilde{\sigma}_\kappa(\xi_1, \xi_2) = \Phi(2^{-\kappa}\xi_1, 2^{-\kappa}\xi_2)\sigma(\xi_1, \xi_2).$$

Then

$$\widetilde{\sigma}_\kappa(\xi_1, \xi_2) = \sigma_\kappa(2^{-\kappa}\xi_1, 2^{-\kappa}\xi_2),\ \mathcal{F}^{-1}\widetilde{\sigma}_\kappa(\xi_1, \xi_2) = 2^{2\kappa n}\mathcal{F}^{-1}\sigma_\kappa(2^\kappa\xi_1, 2^\kappa\xi_2),$$

where $\mathcal{F}^{-1}f$ denotes the inverse Fourier transform of f. For $x, y_1, y_2, y_1' \in \mathbb{R}^n$, let

$$W_{1,\kappa}(x, y_1, y_2; y_1') = \mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x - y_1, x - y_2) - \mathcal{F}^{-1}\widetilde{\sigma}_\kappa(x - y_1', x - y_2).$$

For $\kappa \in \mathbb{Z}$, let $T_{\widetilde{\sigma}_\kappa}$ be the operator defined by

$$T_{\widetilde{\sigma}_\kappa}(f_1, f_2)(x) = \int_{\mathbb{R}^{2n}} \widetilde{\sigma}_\kappa(x - y_1, x - y_2) f_1(y_1) f_2(y_2)\mathrm{d}y_1\mathrm{d}y_2,$$

and set

$$T_{\sigma,N}(f_1,\,f_2)(x)=\sum_{|\kappa|<N}T_{\widetilde{\sigma}_\kappa}(f_1,\,f_2)(x).$$

For each fixed ball B with radius R, let

$$A^{j_2}_{1,\kappa}(B;\,y_1,\,y_1')=\Big(\int_{S_{j_0}(B)}\Big(\int_{S_{j_2}(B(x,\,R))}|W_{1,\kappa}(x,y_1,y_2;y_1')|^{r_2'}\mathrm{d}y_2\Big)^{\frac{r_1'}{r_2'}}\mathrm{d}x\Big)^{\frac{1}{r_1'}}.$$

Lemma 2.1 *Let σ be a multiplier satisfying (1.6) for some $s_1,\,s_2\in(n/2,\,n]$, and $r_1,\,r_2\in(1,\,2]$, B be a ball with radius R.*

(i) If $2^\kappa R<1$, then for $y_1,\,y_1'\in\frac{1}{4}B$, $j_0\in\mathbb{N}$ and nonnegative integers j_2,

$$A^{j_2}_{1,\kappa}(B;\,y_1,\,y_1')\lesssim\frac{R2^{-\kappa(s_1+s_2-n/r_1-n/r_2-1)}}{|2^{j_0}B|^{s_1/n}(2^{j_2}R)^{s_2}};$$

(ii) For each $\kappa\in\mathbb{Z}$, there exists a function H_1^κ, such that if $\operatorname{supp}f_1\subset B$, *$y_1,\,y_1'\in\frac{1}{4}B$, and $x\in\mathbb{R}^n\backslash 4B$,*

$$\begin{aligned}&\int_{\mathbb{R}^{2n}}|W_{1,\kappa}(x,y_1,y_2,\,y_1')|\prod_{l=1}^{2}|f_l(y_l)|\mathrm{d}y_1\mathrm{d}y_2\\&\lesssim\int_{\mathbb{R}^n}|f_1(y_1)|\mathrm{H}_1^\kappa(x,\,y_1,\,y_1')\mathrm{d}y_1M_{r_2}f_2(x)\end{aligned}$$

the function H_1^κ satisfies that

$$\Big(\int_{S_{j_0}(B)}|\mathrm{H}_1^\kappa(x,\,y_k,\,y_k')|^{r_k'}\mathrm{d}x\Big)^{1/r_k'}\lesssim\frac{2^{-\kappa(s_1-n/r_1)}}{(2^{j_0}R)^{s_1}}\quad\text{for integer }\ j_0\geq 3.$$

This lemma is a combination of Lemmas 3.3 and 3.4 in [13].

For a $q\in(0,\,\infty)$, let M_q be the maximal operator defined by $M_qf(x)=(M(|f|^q)(x))^{1/q}$. The following weighted estimate for T_σ will be useful in the proof of Theorem 1.3, and is of independent interest.

Theorem 2.2 *Let σ be a bilinear multiplier satisfying (1.5) for some $s_1,\,s_2\in(n/2,\,n]$, $t_k=n/s_k$ $(k=1,\,2)$. Set $t_k=n/s_k$ for $k=1,\,2$. For weight w, $p_1,\,p_2\in(1,\,\infty)$, $p\in(1/2,\,\infty)$ with $1/p=1/p_1+1/p_2$, the weighted estimate*

$$\|T_\sigma(f_1,\,f_2)\|_{L^p(\mathbb{R}^n,\,w)}\lesssim\prod_{k=1}^{2}\|f_k\|_{L^{p_k}(\mathbb{R}^n,\,M_{q_k}w)}\tag{2.1}$$

holds provided that $p_1,\,p_2,\,q_1,\,q_2$ satisfy one of the following conditions:

(i) $p_k\in(t_k,\,\infty)$ with $k=1,\,2$, and $q_1,\,q_2\in(1,\,\infty)$;

(ii) $p_1\in(1,\,\infty)$, $p_2\in(t_2,\,\infty)$, $q_1\in(t_1,\,\infty)$ and $q_2\in(1,\,\infty)$;

(iii) $p_1,\,p_2\in(1,\,\infty]$ such that $p\in(p_t,\,\infty)$ with $p_t<1+1/t^*$, $q_1\in(t_1,\,\infty)$ and $q_2\in(t_2,\,\infty)$. Moreover, if $p_2\in(t_2,\,\infty)$ and $q_k\in(t_k,\,\infty)$ with $k=1,\,2$, then

$$\|T_\sigma(f_1,\,f_2)\|_{WL^p(\mathbb{R}^n,\,w)}\lesssim\|f_1\|_{L^1(\mathbb{R}^n,\,M_{q_1}w)}\|f_2\|_{L^{p_2}(\mathbb{R}^n,\,M_{q_2}w)},\tag{2.2}$$

where and in the following, $WL^p(\mathbb{R}^n,\, w)$ denotes the weighted weak $L^p(\mathbb{R}^n)$ space with weight w.

Proof Note that for $f_1,\, f_2 \in \mathscr{S}(\mathbb{R}^n)$,

$$\big\|T_\sigma(f_1,\, f_2) - T_{\sigma,\,N}(f_1,\, f_2)\big\|_{L^\infty(\mathbb{R}^n)} \lesssim \big\|(\sigma - \sum_{\kappa:\,|\kappa|\le N} \widetilde{\sigma}_\kappa)\widehat{f_1}\widehat{f_2}\big\|_{L^1(\mathbb{R}^n)} \to 0$$

as $N \to \infty$. By a density argument, it suffices to prove that when $p_1,\, p_2,\, q_1,\, q_2$ satisfy one of the three conditions, the operator $T_{\sigma,\,N}$ satisfies (2.1) with norm independent of N.

As in the proof of Theorem 1.2 in [13], a standard argument shows that for $\gamma_1 > t_1,\, \gamma_2 > t_2$,

$$M^\sharp\big(T_{\sigma,\,N}(f_1,\, f_2)\big)(x) \lesssim M_{\gamma_1} f_1(x) M_{\gamma_2} f_2(x).$$

If $p_k \in (t_k,\, \infty)$ with $k = 1,\, 2$, we can choose $\gamma_k \in (t_k,\, p_k)$, and then by the clever idea of Lerner [15], we have that for any weight w,

$$\begin{aligned}\int_{\mathbb{R}^n} |T_{\sigma,\,N}(f_1,\, f_2)(x)|^p w(x)\,\mathrm{d}x &\lesssim \int_{\mathbb{R}^n} M^\sharp\big(|T_{\sigma,\,N}(f_1,\, f_2)|^p\big)(x) Mw(x)\,\mathrm{d}x \\ &\lesssim \prod_{k=1}^{2} \Big(\int_{\mathbb{R}^n} \big(M_{\gamma_k} f_k(x)\big)^{p_k} Mw(x)\,\mathrm{d}x\Big)^{p/p_k} \\ &\lesssim \prod_{k=1}^{2} \|f_k\|^p_{L^{p_k}(\mathbb{R}^n,\, M_{L\log L} w)},\end{aligned}$$

if $p \in (1/2,\, 1]$. On the other hand, if $p \in (1,\, \infty)$, again by the idea of Lerner [15], we know that for each $h \in L^{p'}(\mathbb{R}^n,\, w^{1-p'})$ with $\|h\|_{L^{p'}(\mathbb{R}^n,\, w^{1-p'})} \le 1$,

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} T_{\sigma,\,N} f(x) h(x)\mathrm{d}x\Big| &\lesssim \int_{\mathbb{R}^n} M^\sharp\big(T_{\sigma,\,N}(f_1, f_2)\big)(x) Mh(x)\mathrm{d}x \\ &\lesssim \prod_{k=1}^{2} \|M_{\gamma_k} f_k\|_{L^{p_k}(\mathbb{R}^n, M_{L(\log L)^{p-1+\delta}} w)} \|Mh\|_{L^{p'}(\mathbb{R}^n, (M_{L(\log L)^{p-1+\delta}} w)^{1-p'})} \\ &\lesssim \prod_{k=1}^{2} \|f_k\|_{L^{p_k}(\mathbb{R}^n,\, M_{L(\log L)^{p+\delta}} w)},\end{aligned}$$

where the last inequality follows from the fact that

$$M\big(M_{L(\log L)^{p-1+\delta}} w\big)(x) \lesssim M_{L(\log L)^{p+\delta}} w(x),$$

see [4], and that for any weight u and $p \in (1,\, \infty)$,

$$\|Mf\|_{L^{p'}(\mathbb{R}^n,\, (M_{L(\log L)^{p-1+\delta}} u)^{1-p'}} \lesssim \|f\|_{L^p(\mathbb{R}^n,\, u^{1-p'})},$$

see [21]. Therefore,

$$\|T_{\sigma,\,N}(f_1,\, f_2)\|_{L^{p_1}(\mathbb{R}^n,\, w)} \lesssim \prod_{k=1}^{2} \|f_k\|_{L^{p_k}(\mathbb{R}^n,\, M_{L(\log L)^{p+\delta}} w)}.$$

Note that for any $r \in (1,\, \infty)$,

$$M_{L\log L} w(x) + M_{L(\log L)^{p+\delta}} w(x) \le M_r w(x).$$

The weighted estimate (2.1) follows when $p_k \in (t_k, \infty)$ with $k = 1, 2$.

We turn our attention to the case $p_1 \in [1, \infty)$ and $p_2 \in (t_2, \infty)$. We first claim that for $p_1 \in [1, \infty)$, $p_2 \in (t_2, \infty)$, $q_1 \in (t_1, \infty)$ and $q_2 \in (1, \infty)$, the inequality

$$\|T_{\sigma, N}(f_1, f_2)\|_{WL^p(\mathbb{R}^n, w)} \lesssim \prod_{k=1}^{2} \|f_k\|_{L^{p_k}(\mathbb{R}^n, M_{q_k} w)} \tag{2.4}$$

holds. To do this, let f_1, f_2 be functions such that

$$\|f_1\|_{L^{p_1}(\mathbb{R}^n, M_{q_1} w)} = \|f_2\|_{L^{p_1}(\mathbb{R}^n, M_{q_2} w)} = 1.$$

Our aim is to prove that for any $\lambda > 0$,

$$w(\{x \in \mathbb{R}^n : |T_{\sigma, N}(f_1, f_2)(x)| > \lambda\}) \lesssim \lambda^{-p}.$$

Take $r_1, r_2 \in (1, 2]$ such that $r_2 \in (t_2, p_2)$, $r_1 > n/r_1$, $n/r_1 + n/r_2 + 1 > s_1 + s_2$. We apply the Calderón-Zygmund decomposition to $|f_1|^{p_1}$ at level λ^p, and then obtain sequence of non-overlapping cubes $\{Q_1^i\}_i$, such that

$$\lambda^p < \frac{1}{|Q_1^i|} \int_{Q_1^i} |f_1(y)|^{p_1} \mathrm{d}y \lesssim \lambda^p,$$

and

$$|f_1(x)| \le \lambda^{p/p_1}, \text{ a. e. } x \in \mathbb{R}^n \backslash \cup_i Q_1^i.$$

Set

$$g_1(y) = f_1(y)\chi_{\mathbb{R}^n \backslash \cup_i Q_1^i}(y) + \sum_i m_{Q_1^i}(f_1)\chi_{Q_1^i}(y),$$

$$b_1(y) = f_1(y) - g_1(y) = \sum_i b_1^i(y) \text{ with } b_1^i(y) = (f_1(y) - m_{Q_1^i}(f_1))\chi_{Q_1^i}(y)$$

with $m_{Q_1^i}(f_1)$ the mean value of f_1 on Q_1^i. For each fixed i, let y_1^i and $\ell(Q_1^i)$ be the center and the side length of Q_1^i, and B_1^i be the ball which is centered at y_1^i and has radius $8\sqrt{n}\ell(Q_1^i)$. Set $\Omega = \cup_i 4B_1^i$. It is obvious that

$$w(\Omega) \le \sum_i w(Q_1^i) \lesssim \lambda^{-p} \sum_i \int_{Q_1^i} |f_1(x)|^{p_1} \mathrm{d}x \inf_{y \in Q_1^i} Mw(y) \lesssim \lambda^{-p}. \tag{2.5}$$

By (2.3) and a standard argument, it is easy to verify that when $r \in (t_2, \infty)$,

$$\|T(h_1, h_2)\|_{L^r(\mathbb{R}^n, M_{q_2} w)} \lesssim \|h_1\|_{L^\infty(\mathbb{R}^n)} \|h_2\|_{L^r(\mathbb{R}^n, M_{q_2} w)}.$$

This via the fact that $\|g_1\|_{L^\infty(\mathbb{R}^n)} \lesssim \lambda^{p/p_1}$, implies that

$$w(\{x \in \mathbb{R}^n : |T(g_1, f_2)(x)| > \lambda/2\}) \lesssim \lambda^{-p_2} \|g_1\|_{L^\infty(\mathbb{R}^n)}^{p_2} \|f_2\|_{L^{p_2}(\mathbb{R}^n, M_{q_2} w)}^{p_2} \lesssim \lambda^{-p}. \tag{2.6}$$

For each fixed B_1^i, $x \in \mathbb{R}^n \backslash \Omega$, $\kappa \in \mathbb{Z}$, let R_1^i be the radius of B_1^i and

$$\mathrm{I}_{1, \kappa, B_1^i}(x, y_1, y_1') = \sum_{j_2=0}^{\infty} (2^{j_2} R_1^i)^{\frac{n}{r_2}} \Big(\int_{S_{j_2}(B(x, R_1^i))} |W_{1, \kappa}(x, y_1, y_2; y_1')|^{r_2'} \mathrm{d}y_2 \Big)^{1/r_2'}.$$

A straightforward computation involving the Hölder inequality leads to that for $x \in \mathbb{R}^n \backslash \Omega$,

$$\int_{\mathbb{R}^{2n}} W_{1, \kappa}(x, y_1, y_2; y_1')|b_1^i(y_1)||f_2^i(y_2)|\mathrm{d}y_1 \mathrm{d}y_2$$

$$\lesssim \int_{Q_1^i} \mathrm{I}_{1,\,\kappa,\,B_1^i}(x,\,y_1,\,y_1')|b_1^i(y_1)|\mathrm{d}y_1 M_{r_2}f_2(x),$$

which in turn implies that for $x \in \mathbb{R}^n\backslash\Omega$,

$$|T_{\sigma,\,N}(b_1^i,\,f_2)(x)| \lesssim G_1^i(x)M_{r_2}f_2(x)$$

with

$$\begin{aligned}\mathrm{G}_1^i(x) = &\sum_{\kappa\in\mathbb{Z}:\,2^\kappa R_1^i\le 1} \int_{Q_1^i} \mathrm{I}_{1,\,\kappa,\,B_1^i}(x,\,y_1,\,y_1')|b_1^i(y_1)|\mathrm{d}y_1+\\ &\sum_{\kappa\in\mathbb{Z}:2^\kappa R_1^i>1} \int_{Q_1^i} H_{1,\,\kappa;\,B_1^i}(x;\,y_1,\,y_1^i)|b_1^i(y_1)|\mathrm{d}y_1.\end{aligned}$$

By the weighted estimate of the Hardy-Littlewood maximal operator, we know that when $p_2 > t_2$,

$$w(\{x \in \mathbb{R}^n:\, M_{r_2}f_2(x) > \lambda^{p/p_2}\}) \lesssim \lambda^{-p}\|f_2\|^{p_2}_{L^{p_2}(\mathbb{R}^n,\,Mw)}.$$

On the other hand, by (i) of Lemma 2.1, for each fixed $j_0 \ge 3$, $y_1 \in B_1^i$,

$$\begin{aligned}\Big(\int_{S_{j_0}(B_1^i)} |\mathrm{I}_{1,\,\kappa,\,B_1^i}(x,\,y_1,\,y_1^i)|^{r_1'}\mathrm{d}x\Big)^{1/r_1'} &\lesssim \sum_{j_2=0}^{\infty} \frac{R_1^i 2^{-\kappa(s_1+s_2-n/r_1-n/r_2-1)}}{(2^{j_0}R_1^i)^{s_1}(2^{j_2}R_1^i)^{s_2-n/r_2}}\\ &\lesssim \frac{R_1^i 2^{-\kappa(s_1+s_2-n/r_1-n/r_2-1)}}{(2^{j_0}R_1^i)^{s_1}(R_1^i)^{s_2-n/r_2}}.\end{aligned}$$

For each fixed i, it follows from (ii) of Lemma 2.1 that

$$\begin{aligned}\int_{\mathbb{R}^n\backslash\Omega} \mathrm{G}_1^i(x)w(x)\mathrm{d}x \lesssim &\sum_{\kappa\in\mathbb{Z}:\,2^\kappa R_1^i\le 1} \int_{Q_1^i}\int_{\mathbb{R}^n\backslash 4B_1^i} \mathrm{I}_{1,\,\kappa,\,B_1^i}(x,\,y_1,y_1^i)w(x)\mathrm{d}x|b_1^i(y_1)|\mathrm{d}y_1+\\ &\sum_{\kappa\in\mathbb{Z}:2^\kappa R_1^i>1} \int_{Q_1^i}\int_{\mathbb{R}^n\backslash 4B_1^i} \mathrm{H}_{1,\,\kappa;\,B_1^i}(x;\,y_1,\,y_1^i)w(x)\mathrm{d}x|b_1^i(y_1)|\,\mathrm{d}y_1\\ \lesssim &\sum_{\kappa\in\mathbb{Z}:\,2^\kappa R_1^i\le 1}\sum_{j_0=3}^{\infty} \int_{Q_1^i}\Big(\int_{S_{j_0}(B_1^i)} |\mathrm{I}_{1,\,\kappa,\,B_1^i}(x,\,y_1,\,y_1^i)|^{r_1'}\mathrm{d}x\Big)^{1/r_1'}\times\\ &|b_1^i(y_1)|\mathrm{d}y_1 \inf_{z\in Q_1^i} M_{r_1}w(z)(2^{j_0}R_1^i)^{n/r_1}+\\ &\sum_{\kappa\in\mathbb{Z}:\,2^\kappa R_1^i>1}\sum_{j_0=3}^{\infty} \int_{Q_1^i}\Big(\int_{S_{j_0}(B_1^i)} |\mathrm{H}_{1,\,\kappa,\,B_1^i}(x,\,y_1,\,y_1^i)|^{r_1'}\mathrm{d}x\Big)^{1/r_1'}\times\\ &|b_1^i(y_1)|\mathrm{d}y_1 \inf_{z\in Q_1^i} M_{r_1}w(z)(2^{j_0}R_1^i)^{n/r_1}\\ \lesssim &\int_{Q_1^i} |b_1^i(y_1)|\mathrm{d}y_1 \inf_{z\in Q_1^i} M_{r_1}w(z)\\ \lesssim &\lambda^{p/p_1}|Q_1^i| \inf_{z\in Q_1^i} M_{r_1}w(z).\end{aligned}$$

Therefore,

$$\begin{aligned}&w(\{x \in \mathbb{R}^n\backslash\Omega:\, |T_{\sigma,\,N}(b_1,\,f_2)(x)| > \lambda/2\})\\ &\quad\lesssim w(\{x \in \mathbb{R}^n:\, M_{r_2}f_2(x) > \lambda^{p/p_1}/4\}) + w(\{x \in \mathbb{R}^n:\, \sum_i G_1^i(x) > \lambda^{p/p_2}/4\})\end{aligned}$$

$$\lesssim \lambda^{-p} + \lambda^{-p/p_1} \sum_i \int_{\mathbb{R}^n \setminus \Omega} \mathrm{G}_1^i(x) w(x) \mathrm{d}x \lesssim \lambda^{-p}. \tag{2.7}$$

Combining the estimates (2.5)–(2.7) yields (2.4).

Now let $p_1 \in (1, \infty)$, $p_2 \in (t_2, \infty)$, $q_1 \in (t_1, \infty)$ and $q_2 \in (1, \infty)$. We can choose $p_{11}, p_{12}, p_{13} \in (1, \infty)$; $p_{21}, p_{22}, p_{23} \in (t_2, \infty)$, such that $(1/p_1, 1/p_2, 1/p)$ is in the open convex hull of the points

$$(1/p_{11},\, 1/p_{21},\, 1/p^1);\ (1/p_{12},\, 1/p_{22},\, 1/p^2);\ (1/p_{13},\, 1/p_{23},\, 1/p^3)$$

where $1/p^j = \sum_{k=1}^2 1/p_{kj}$. We know from (2.4) that for each $j = 1, 2, 3$,

$$\|T_{\sigma, N}(f_1, f_2)\|_{WL^{p_j}(\mathbb{R}^n, w)} \lesssim \|f_1\|_{L^{p_{1j}}(\mathbb{R}^n, M_{q_1} w)} \|f_2\|_{L^{p_{2j}}(\mathbb{R}^n, M_{q_2} w)}.$$

This, via the multilinear Marcinkiewicz interpolation [9, p. 72], shows that for $p_1 \in (1, \infty)$, $p_2 \in (t_2, \infty)$, $q_1 \in (t_1, \infty)$ and $q_2 \in (1, \infty)$,

$$\|T_{\sigma, N}(f_1, f_2)\|_{L^p(\mathbb{R}^n, w)} \lesssim \|f_1\|_{L^{p_1}(\mathbb{R}^n, M_{q_1} w)} \|f_2\|_{L^{p_2}(\mathbb{R}^n, M_{q_2} w)}.$$

Note that (2.4) also implies the inequality (2.2). It remains to consider the case of $p_k \in (1, t_k]$ and $q_k \in (t_k, \infty)$ with $k = 1, 2$. For each fixed $p_1 \in (1, t_1]$, $p_2 \in (1, t_2]$ with $1/p = 1/p_1 + 1/p_2 < 1 + 1/t^*$, we can choose $\theta \in (0, 1)$ such that $1/p < 1 + \theta/t_1 + (1-\theta)/t_2 - \epsilon$ for some $\epsilon > 0$, $p_{11} > 1$, $p_{12} > t_1$, $p_{21} > t_2$ and $p_{22} > 1$ such that

$$\frac{\theta}{p_{11}} + \frac{1-\theta}{p_{21}} = \theta + \frac{1-\theta}{t_2} - \frac{\epsilon}{2},\ \frac{\theta}{p_{12}} + \frac{1-\theta}{p_{22}} = \frac{\theta}{t_1} + 1 - \theta - \frac{\epsilon}{2}.$$

Let $u_1, u_2 \in (1, \infty)$ such that

$$\frac{1}{u_1} = \frac{1}{p_{11}} + \frac{1}{p_{12}},\ \frac{1}{u_2} = \frac{1}{p_{21}} + \frac{1}{p_{22}}.$$

By the inequalities

$$\|T_{\sigma, N}(f_1, f_2)\|_{L^{u_1}(\mathbb{R}^n, w)} \lesssim \prod_{k=1}^2 \|f_k\|_{L^{p_{1k}}(\mathbb{R}^n, M_{q_k} w)}$$

and

$$\|T_{\sigma, N}(f_1, f_2)\|_{L^{u_2}(\mathbb{R}^n, w)} \lesssim \prod_{k=1}^2 \|f_k\|_{L^{p_{2k}}(\mathbb{R}^n, M_{q_k} w)},$$

an application of the bilinear Riesz-Thorin interpolation leads to that

$$\|T_{\sigma, N}(f_1, f_2)\|_{L^p(\mathbb{R}^n, w)} \lesssim \prod_{k=1}^2 \|f_k\|_{L^{p_k}(\mathbb{R}^n, M_{q_k} w)}.$$

This completes the proof of Theorem 2.2. □

By Theorem 2.2, it is easy to see that Theorem 1.3 can be deduced from the following conclusion, which has independent interest.

Theorem 2.3 *Let $p_1, p_2 \in [1, \infty)$, $p \in [1/2, \infty)$ such that $1/p = 1/p_1 + 1/p_2$, $r_1, r_2 \in (1, \infty)$. Let T be a bi(sub)linear operator which satisfies that for any weight w,*

$$\|T(f_1, f_2)\|_{L^p(\mathbb{R}^n, w)} \lesssim \prod_{k=1}^2 \|f_k\|_{L^{p_k}(\mathbb{R}^n, M_{r_k} w)}.$$

Then for any $\lambda_k \in (0,\, n/r_k)$ *with* $k = 1,\, 2$, $\lambda = \lambda_1 p/p_1 + \lambda_2 p/p_2$, T *is bounded from* $L^{p_1,\,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\,\lambda_2}(\mathbb{R}^n)$ *to* $L^{p,\,\lambda}(\mathbb{R}^n)$.

Proof We will employ some ideas from [5]. Let $B = B(x_0,\, R)$ be a ball in $\mathbb{R}^n$. For each fixed $f_1,\, f_2$ with $f_k \in L^{p_k,\,\lambda_k}(\mathbb{R}^n)$ for $k = 1,\, 2$, decompose f_k as

$$f_k(y) = f_k(y)\chi_{2B}(y) + \sum_{j=1}^{\infty} f_k(y)\chi_{2^{j+1}B\setminus 2^j B}(y) =: f_k^{(0)}(y) + \sum_{j=1}^{\infty} f_k^{(j)}(y).$$

The fact that T is bounded from $L^{p_1}(\mathbb{R}^n,\, M_{r_1}w) \times L^{p_2}(\mathbb{R}^n,\, M_{r_2}w)$ to $L^p(\mathbb{R}^n,\, w)$, tells us that

$$\begin{aligned}\Big(\int_B |T(f_1,\, f_2)(x)|^p\,\mathrm{d}x\Big)^{1/p} &= \Big(\int_{\mathbb{R}^n} |T(f_1,\, f_2)(x)|^p \chi_B(x)\,\mathrm{d}x\Big)^{1/p} \\ &\lesssim \prod_{k=1}^{2} \|f_k\|_{L^{p_k}(\mathbb{R}^n,\, M_{r_k}\chi_B)}.\end{aligned}$$

Note that $M\chi_B(x) \lesssim 1$ for any $x \in \mathbb{R}^n$. Thus,

$$\|f_k^{(0)}\|_{L^{p_k}(\mathbb{R}^n,\, M\chi_B)} \lesssim R^{\lambda_k/p_k}\|f_k\|_{L^{p_k,\,\lambda_k}(\mathbb{R}^n)}.$$

On the other hand, a straightforward computation leads to that for $j \in \mathbb{N}$,

$$\|f_k^{(j)}\|_{L^{p_k}(\mathbb{R}^n,\, M_{r_k}\chi_B)} \lesssim 2^{-j(n/r_k-\lambda_k)/p_k} R^{\lambda_k/p_k}\|f\|_{L^{p_k,\,\lambda_k}(\mathbb{R}^n)},$$

since $M\chi_B(x) \sim 2^{-jn}$ for $x \in 2^{j+1}B\setminus 2^j B$, see [18] for details. Therefore,

$$\|f_k\|_{L^{p_k}(\mathbb{R}^n,\, M_{r_k}\chi_B)} \le \sum_{j=0}^{\infty} \|f_k^{(j)}\|_{L^{p_k}(\mathbb{R}^n,\, M_{r_k}\chi_B)} \lesssim R^{\lambda_k/p_k}\|f\|_{L^{p_k,\,\lambda_k}(\mathbb{R}^n)}. \tag{2.8}$$

Our desired conclusion then follows immediately. □

Theorem 1.4 follows from Theorem 2.1 and

Theorem 2.4 *Let* $p_1,\, p_2 \in [1,\, \infty)$, $p \in [1/2,\, \infty)$ *such that* $1/p = 1/p_1 + 1/p_2$, $r_1,\, r_2 \in (1,\, \infty)$. *Let* T *be a bi(sub)linear operator which satisfies that for any weight* w,

$$\|T(f_1,\, f_2)\|_{WL^p(\mathbb{R}^n,\, w)} \lesssim \prod_{k=1}^{2} \|f_k\|_{L^{p_k}(\mathbb{R}^n,\, M_{r_k}w)}. \tag{2.9}$$

Then for any $\lambda_k \in (0,\, n/r_k)$ *with* $k = 1,\, 2$, $\lambda = \lambda_1 p/p_1 + \lambda_2 p/p_2$, T *is bounded from* $L^{p_1,\,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\,\lambda_2}(\mathbb{R}^n)$ *to* $WL^{p,\,\lambda}(\mathbb{R}^n)$.

Proof Let $f_k \in L^{p_k,\,\lambda_k}(\mathbb{R}^n)$ with $k = 1,\, 2$ and B be a ball with radius R. Set $w(x) = \chi_B(x)$. Note that

$$|\{x \in B : |T(f_1,\, f_2)(x)| > t\}| = w(\{x \in \mathbb{R}^n : |T(f_1,\, f_2)(x)| > t\}).$$

We can obtain from the estimate (2.8) and (2.9) that

$$|\{x \in B : |T(f_1,\, f_2)(x)| > t\}| \lesssim R^{\lambda_1/p_1+\lambda_2/p_2} \prod_{k=1}^{2} \|f_k\|_{L^{p_k,\,\lambda_k}(\mathbb{R}^n)},$$

which leads to our desired result and then completes the proof of Theorem 2.4. □

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