Journal of Mathematical Research with Applications
Nov., 2014, Vol. 34, No. 6, pp. 713–722
DOI:10.3770/j.issn:2095-2651.2014.06.009
Http://jmre.dlut.edu.cn

Unique Common Fixed Point Theorems for Lipschitz Type Mappings under c-Distance on Cone Metric Spaces

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Abstract In this paper, new unique common fixed point theorems for four mappings satisfying Lipzchitz type conditions in the term of c-distance on normal cone metric spaces were given. The obtained results generalize and improve many known common fixed point theorems.

Keywords normal cone metric space; c-distance; common fixed point; Lipschitz type condition.

MR(2010) Subject Classification 47H05; 47H10; 54E40; 54H25

1. Introduction and preliminaries

Huang and Zhang [1] recently have introduced the concept of cone metric spaces, where the set of real number is replaced by an ordered Banach space, and they have established some fixed point theorems for a contractive type mapping in a normal cone metric space. Subsequently, some other authors [2–13] have generalized the results of Huang and Zhang [1] and have studied the existence of fixed point or common fixed points of mappings satisfying a contractive type condition in the framework of normal or non-normal cone metric spaces.

Fixed point results in metric spaces with the so called w-distance were obtained for the first by Kada et al. in [14] where non-convex minimization problems were treated. Further results were given in [15–17]. The cone metric version of this notion (usually called a c-distance) was used in [18,19].

The author in [20] obtained a fixed point theorem for a mapping in normal cone metric space under some contractive condition expressed in the terms of c-distance, and the author in [21] also obtained fixed point and common fixed point results for mappings in TVS-valued non-normal cone metric spaces under contractive condition expressed in the terms of c-distance. Those results generalize many known ones.

Recently, Wang and Guo [20] obtained a common fixed point theorem for a pare of non-continuous mappings under contractive conditions in the term of c-distance on a normal cone metric space, but they did not discuss the uniqueness of common fixed points of the given mappings.

Here, we will discuss the same problems as that in [20] for four mappings under weaker Lipschitz type conditions and further give the uniqueness of common fixed points.

Received May 5, 2014; Accepted September 4, 2014
Supported by the National Natural Science Foundation of China (Grant No. 11361064).
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Let E be always a real Banach space and P_0 a subset of E. Then P_0 is called a cone whenever

(i) P_0 is closed, nonempty, and $P_0 \neq \{0\}$;

(ii) $ax + by \in P_0$ for all $x, y \in P_0$ and nonnegative real numbers a, b;

(iii) $P_0 \cap (-P_0) = \{0\}$.

In this paper, we shall always assume that the cone P_0 has a nonempty interior, i.e., $\mathrm{int} P_0 \neq \emptyset$ (such cones are called solid).

For a given cone $P_0 \subset E$, we define a partial ordering $\leq$ with respect to P_0 by $x \leq y$ if and only if $y - x \in P_0$. $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \mathrm{int}\, P_0$.

A cone P_0 is called normal if there exists a real number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \Longrightarrow \|x\| \leq K\|y\|.$$

The least positive number K satisfying the above condition is called the normal constant of P_0.

It is known that a metric space is a normal cone metric space with normal constant $K = 1$.

Definition 1.1 *Let X be a nonempty set and E a real Banach space. Suppose that the mapping $d : X \times X \to E$ satisfies*

(d1) $0 \leq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$*;*

(d2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$*;*

(d3) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, z, y \in X$*.*

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Remark 1.2 If E is replaced by a topological vector space which is larger than a Banach space, then Definition 1.1 becomes the concept of a TVS-valued cone metric space [21]. So the cone metric space is a particular form of a TVS-valued cone metric space. Hence the conclusions holding in TVS-valued cone metric space also hold in a cone metric space.

Definition 1.3 *Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}_{n \in \mathbb{N}}$ a sequence in X. Then*

(i) $\{x_n\}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) \ll c$ for all $n, m > N$.

(ii) $\{x_n\}$ converges to x whenever for every $c \in E$ with $0 \ll c$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n > N$. We denote this by $x_n \to x$ or $\lim_{n \to \infty} x_n = x$.

(iii) (X, d) is called complete if every Cauchy sequence in X is convergent.

We shall make use of the following properties:

(p_1) If $u, v, w \in E$, $u \leq v$ and $v \leq w$, then $u \leq w$;

(p_2) If $u \in E$ and $\theta \leq u \ll c$ for each $c \in \mathrm{int}\, P_0$, then $u = 0$;

(p_3) If $u_n, v_n, u, v \in E$, $\theta \leq u_n \leq v_n$ for each $n \in \mathbb{N}$, and $u_n \to u$, $v_n \to v$, then $\theta \leq u \leq v$;

(p_4) If $x_n, x \in X$, $u_n \in E$, $d(x_n, x) \leq u_n$ and $u_n \to \theta$, then $x_n \to x$;

(p_5) If $u \leq \lambda u$, where $u \in P_0$ and $0 \leq \lambda < 1$, then $u = \theta$;

(p_6) If $\theta \ll c$ and $u_n \in E$, $u_n \to \theta$, then there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for all $n \geq n_0$.

Definition 1.4 *Let (X,d) be a cone metric space. A function $q: X \times X \to E$ is called a c-distance in X if:*

(q_1) $\theta \le q(x,y)$ for all $x,y \in X$;

(q_2) $q(x,z) \le q(x,y) + q(y,z)$ for all $x,y,z \in X$;

(q_3) If a sequence $\{y_n\}$ in X converges to a point $y \in X$, and for some $x \in X$ and $u = u_x \in P_0$, $q(x,y_n) \le u$ holds for each $n \in \mathbb{N}$, then $q(x,y) \le u$;

(q_4) For each $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$, such that $q(z,x) \ll e$ and $q(z,y) \ll e$ implies $d(x,y) \ll c$.

The information of examples and notations of c-distance can be found in [20,21].

The following facts can be found in [21].

For c-distance q,

1) $q(x,y) = q(y,x)$ does not necessarily hold for all $x,y \in X$;

2) $q(x,y) = 0$ is not necessarily equivalent to $x = y$.

Definition 1.5 *A sequence $\{u_n\}$ in P_0 is said to be a c-sequence if for each $c \gg 0$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for all $n \ge n_0$.*

It is easy to show that if $\{u_n\}$ and $\{v_n\}$ are c-sequences in E and $\alpha, \beta > 0$, then $\{\alpha u_n + \beta v_n\}$ is a c-sequence.

The following conclusion is a cone metric version of Lemma 1 in [21].

Lemma 1.6 *Let (X,d) be a cone metric space and q a c-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x,y,z \in X$. Suppose that $\{u_n\}$ and $\{v_n\}$ are c-sequences in P_0. Then the following hold*

(1) If $q(x_n,y) \le u_n$ and $q(x_n,z) \le v_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $q(x,y) = 0$ and $q(x,z) = 0$, then $y = z$.

(2) If $q(x_n,y_n) \le u_n$ and $q(x_n,z) \le v_n$ for all $n \in \mathbb{N}$, then $\{y_n\}$ converges to z.

(3) If $q(x_n,x_m) \le u_n$ for all $m > n > n_0$, then $\{x_n\}$ is a Cauchy sequence in X.

(4) If $q(y,x_n) \le u_n$ for all $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence in X.

2. Common fixed points under c-distance

The following is the main result in this paper.

Theorem 2.1 *Let (X,d) be a cone metric space and P_0 a normal cone with normal constant K. Let $a_i, a_i' \in [0,+\infty), i = 1,2,3,4$, be real numbers satisfying $a_3 + a_4 < 1$, $a_3' + a_4' < 1$ and $\frac{a_1+a_2+a_4}{1-a_3-a_4}\frac{a_1'+a_2'+a_4'}{1-a_3'-a_4'} < 1$. Suppose that four mappings $S,T,I,J: X \to X$ satisfy that $S(X) \subset I(X)$ and $T(X) \subset J(X)$ and for each $x,y \in X$,*

$$q(Sx,Ty) \le a_1 q(Jx,Iy) + a_2 q(Jx,Sx) + a_3 q(Iy,Ty) + a_4 q(Jx,Ty), \tag{2.1}$$

$$q(Tx,Sy) \le a_1' q(Ix,Jy) + a_2' q(Ix,Tx) + a_3' q(Jy,Sy) + a_4' q(Ix,Sy). \tag{2.2}$$

If any one of $S(X)$, $T(X)$, $I(X)$ and $J(X)$ is complete, and for any $u \in \{y \in X : \exists F \in$

$\{S,T,I,J\}, Fy \neq y\}$, *one of the following conditions holds:*

$$\inf\{\|q(Sx,u)\|+\|q(Jx,u)\|+\|q(Jx,Sx)\|: x\in X\}>0;$$

$$\inf\{\|q(Tx,u)\|+\|q(Ix,u)\|+\|q(Ix,Tx)\|: x\in X\}>0.$$

Then S,T,I,J *have a unique common fixed point* $u\in X$ *and* $q(u,u)=0$.

Proof Let $x_0\in X$ be arbitrary. Since $S(X)\subset I(X)$, there exists $x_1\in X$ such that $Sx_0=Ix_1$; by $T(X)\subset J(X)$, there exists $x_2\in X$ such that $Tx_1=Jx_2$. By induction, two sequences $\{x_n\}$ and $\{y_n\}$ can be chosen such that

$$y_{2n}=Sx_{2n}=Ix_{2n+1},\ y_{2n+1}=Tx_{2n+1}=Jx_{2n+2},\quad n=0,1,\ldots.$$

For any $n\in\mathbb{N}$, by (2.1) and (q_2),

$$\begin{aligned}
q(y_{2n},y_{2n+1})&=q(Sx_{2n},Tx_{2n+1})\\
&\le a_1q(Jx_{2n},Ix_{2n+1})+a_2q(Jx_{2n},Sx_{2n})+a_3q(Ix_{2n+1},Tx_{2n+1})+a_4q(Jx_{2n},Tx_{2n+1})\\
&=a_1q(y_{2n-1},y_{2n})+a_2q(y_{2n-1},y_{2n})+a_3q(y_{2n},y_{2n+1})+a_4q(y_{2n-1},y_{2n+1})\\
&\le a_1q(y_{2n-1},y_{2n})+a_2q(y_{2n-1},y_{2n})+a_3q(y_{2n},y_{2n+1})+a_4[q(y_{2n-1},y_{2n})+q(y_{2n},y_{2n+1})].
\end{aligned}$$

Hence

$$q(y_{2n},y_{2n+1})\le L_1\,q(y_{2n-1},y_{2n}),\tag{2.3}$$

where $L_1=\frac{a_1+a_2+a_4}{1-a_3-a_4}$. Similarly, by (2.2) and (q_2),

$$\begin{aligned}
q(y_{2n+1},y_{2n+2})&=q(Tx_{2n+1},Sx_{2n+2})\\
&\le a_1'q(Ix_{2n+1},Jx_{2n+2})+a_2'q(Ix_{2n+1},Tx_{2n+1})+a_3'q(Jx_{2n+2},Sx_{2n+2})+a_4'q(Ix_{2n+1},Sx_{2n+2})\\
&=a_1'q(y_{2n},y_{2n+1})+a_2'q(y_{2n},y_{2n+1})+a_3'q(y_{2n+1},y_{2n+2})+a_4'q(y_{2n},y_{2n+2}))\\
&\quad a_1'q(y_{2n},y_{2n+1})+a_2'q(y_{2n},y_{2n+1})+a_3'q(y_{2n+1},y_{2n+2})+a_4'[q(y_{2n},y_{2n+1})+q(y_{2n+1},y_{2n+2})].
\end{aligned}$$

Hence

$$q(y_{2n+1},y_{2n+2})\le L_2\,q(y_{2n},y_{2n+1}),\tag{2.4}$$

where $L_2=\frac{a_1'+a_2'+a_4'}{1-a_3'-a_4'}$.

Now, using (2.3) and (2.4), we obtain

$$\begin{aligned}
q(y_{2n+1},y_{2n+2})&\le L_2q(y_{2n},y_{2n+1})\le L_1L_2q(y_{2n-1},y_{2n})\le\cdots\\
&\le (L_1L_2)^nq(y_1,y_2)\le L^nL_2q(y_0,y_1),
\end{aligned}$$

where $L=L_1L_2<1$, and

$$q(y_{2n},y_{2n+1})\le L_1q(y_{2n-1},y_{2n})\le L_1L^{n-1}L_2q(y_0,y_1)\le L^nq(y_0,y_1).$$

Hence for $n,m\in\mathbb{N}$ with $m>n$,

$$\begin{aligned}
q(y_{2n+1},y_{2m+1})\le&q(y_{2n+1},y_{2n+2})+q(y_{2n+2},y_{2n+3})+\cdots+q(y_{2m-1},y_{2m})+q(y_{2m},y_{2m+1})\\
\le&L_2L^nq(y_0,y_1)+L^{n+1}q(y_0,y_1)+\cdots+L_2L^{m-1}q(y_0,y_1)+L^mq(y_0,y_1)
\end{aligned}$$

$$\begin{aligned}
&\le (\frac{L_2 L^n}{1-L} + \frac{L^{n+1}}{1-L}) q(y_0, y_1) \le \frac{L^n}{1-L}(L_2+1) q(y_0, y_1)\\
&\le M L^n q(y_0, y_1),
\end{aligned}$$

where $M = \frac{2}{1-L}\max\{1, L_2\}$. Similarly,

$$\begin{aligned}
q(y_{2n}, y_{2m+1}) &\le (\frac{L^n}{1-L} + \frac{L^n L_2}{1-L}) q(y_0, y_1) \le M L^n q(y_0, y_1);\\
q(y_{2n}, y_{2m}) &\le (\frac{L^n}{1-L} + \frac{L^n L_2}{1-L}) q(y_0, y_1) \le M L^n q(y_0, y_1);\\
q(y_{2n+1}, y_{2m}) &\le (\frac{L_2 L^n}{1-L} + \frac{L^{p+1}}{1-L}) q(y_0, y_1) \le M L^n q(y_0, y_1).
\end{aligned}$$

So for any $m > n > 0$, there exists $c(n) \in \mathbb{N}$ such that $\frac{n-1}{2} \le c(n) \le \frac{n}{2}$ and

$$q(y_n, y_m) \le M L^{c(n)} q(y_0, y_1).$$

Let $c_n = M L^{c(n)} q(y_0, y_1)$. Then

$$q(y_n, y_m) \le c_n, \quad \forall\, m, n \in \mathbb{N}, m > n. \tag{2.5}$$

Since $L < 1$, c_n is a c-sequence and $\|c_n\| \to 0$ as $n \to +\infty$. Hence $\{y_n\}$ is a Cauchy sequence by Lemma 1.6(3).

Suppose that $I(X)$ is complete. Then since $\{y_{2n}\}$ is also Cauchy sequence and $y_{2n} \in I(X)$, there exists $u \in I(X)$ such that $y_{2n} = Sx_{2n} = Ix_{2n+1} \to u$. (If $S(X)$ is complete, then there exists $u \in S(X) \subset I(X)$ such that $y_{2n} = Sx_{2n} = Ix_{2n+1} \to u$, so the conclusion remains the same.)

By (2.5), $q(y_{2n}, y_{2m}) \le c_{2n}, \forall m > n > 0$. Fix n and let $m \to \infty$, then by (q_3),

$$q(y_{2n}, u) \le c_{2n}, \quad \forall\, n > 0. \tag{2.6}$$

By (q_2), (2.5) and (2.6),

$$q(y_{2n+1}, u) \le q(y_{2n+1}, y_{2n+2}) + q(y_{2n+2}, u) \le c_{2n+1} + c_{2n+2}, \quad \forall\, n > 0. \tag{2.7}$$

Since P_0 is a normal cone with normal constant K, by (2.5), (2.6) and (2.7), for $m > n > 0$,

$$\|q(y_n, y_m)\| \le K\|c_n\|; \ \|q(y_{2n}, u)\| \le K\|c_{2n}\|, \ \|q(y_{2n+1}, u)\| \le K[\|c_{2n+1}\| + \|c_{2n+2}\|].$$

If u is not a common fixed point of S,T,I and J, then

$$\begin{aligned}
0 &< \inf\{\|q(Sx, u)\| + \|q(Jx, u)\| + \|q(Jx, Sx)\| : x \in X\}\\
&\le \inf\{\|q(Sx_{2n+2}, u)\| + \|q(Jx_{2n+2}, u)\| + \|q(Jx_{2n+2}, Sx_{2n+2})\| : n \in \mathbb{N}\}\\
&= \inf\{\|q(y_{2n+2}, u)\| + \|q(y_{2n+1}, u)\| + \|q(y_{2n+1}, y_{2n+2})\| : n \in \mathbb{N}\}\\
&\le \inf\{2K[\|c_{2n+1}\| + \|c_{2n+2}\|] : n \in \mathbb{N}\} = 0
\end{aligned}$$

or

$$\begin{aligned}
0 &< \inf\{\|q(Tx, u)\| + \|q(Ix, u)\| + \|q(Ix, Tx)\| : x \in X\}\\
&\le \inf\{\|q(Tx_{2n+1}, u)\| + \|q(Ix_{2n+1}, u)\| + \|q(Ix_{2n+1}, Tx_{2n+1})\| : n \in \mathbb{N}\}
\end{aligned}$$

$$= \inf\{\|q(y_{2n+1},u)\| + \|q(y_{2n},u)\| + \|q(y_{2n},y_{2n+1})\| : n \in \mathbb{N}\}$$
$$\le \inf\{K[2\|c_{2n}\| + \|c_{2n+1}\| + \|c_{2n+2}\| : n \in \mathbb{N}\} = 0.$$

These are all contradiction, hence u is a common fixed point of S,T,I and J.

Suppose that $J(X)$ is complete. Then since $\{y_{2n+1}\}$ is also Cauchy sequence and $y_{2n+1} \in J(X)$, there exists $u \in J(X)$ such that $y_{2n+1} = Tx_{2n+1} = Jx_{2n+2} \to u$. (If $T(X)$ is complete, then there exists $u \in T(X) \subset J(X)$ such that $y_{2n+1} = Tx_{2n+1} = Jx_{2n+2} \to u$, so the conclusion remains the same.)

From (2.5), we have that $q(y_{2n+1}, y_{2m+1}) \le c_{2n+1}, \forall m > n > 0$. Fix n and let $m \to \infty$, then by (q_3),

$$q(y_{2n+1},u) \le c_{2n+1}, \quad \forall n > 0. \tag{2.8}$$

By (q_2), (2.5) and (2.8),

$$q(y_{2n+2},u) \le q(y_{2n+2},y_{2n+3}) + q(y_{2n+3},u) \le c_{2n+2} + c_{2n+3}, \quad \forall n > 0. \tag{2.9}$$

Since P_0 is a normal cone with normal constant K, by (2.5), (2.8) and (2.9), for $m > n > 0$,

$$\|q(y_n,y_m)\| \le K\|c_n\|;\ \|q(y_{2n+1},u)\| \le K\|c_{2n+1}\|,\ \|q(y_{2n+2},u)\| \le K[\|c_{2n+2}\| + \|c_{2n+3}\|].$$

If u is not a common fixed point of S, T, I and J, then

$$0 < \inf\{\|q(Sx,u)\| + \|q(Jx,u)\| + \|q(Jx,Sx)\| : x \in X\}$$
$$\le \inf\{\|q(Sx_{2n+2},u)\| + \|q(Jx_{2n+2},u)\| + \|q(Jx_{2n+2},Sx_{2n+2})\| : n \in \mathbb{N}\}$$
$$= \inf\{\|q(y_{2n+2},u)\| + \|q(y_{2n+1},u)\| + \|q(y_{2n+1},y_{2n+2})\| : n \in \mathbb{N}\}$$
$$\le \inf\{K[2\|c_{2n+1}\| + \|c_{2n+2}\| + \|c_{2n+3}\|] : n \in \mathbb{N}\} = 0,$$

or

$$0 < \inf\{\|q(Tx,u)\| + \|q(Ix,u)\|| + \|q(Ix,Tx)\| : x \in X\}$$
$$\le \inf\{\|q(Tx_{2n+1},u)\| + \|q(Ix_{2n+1},u)\| + \|q(Ix_{2n+1},Tx_{2n+1})\| : n \in \mathbb{N}\}$$
$$= \inf\{\|q(y_{2n+1},u)\| + \|q(y_{2n},u)\| + \|q(y_{2n},y_{2n+1})\| : n \in \mathbb{N}\}$$
$$\le \inf\{2K[\|c_{2n}\| + \|c_{2n+1}\|] : n \in \mathbb{N}\} = 0.$$

These are all contradiction. Hence u is a common fixed point of S, T, I and J.

By (2.1),

$$q(u,u) = q(Su,Tu) \le a_1q(Ju,Iu) + a_2q(Ju,Su) + a_3q(Iu,Tu) + a_4q(Ju,Tu)$$
$$\le [a_1 + a_2 + a_3 + 2a_4]q(u,u),$$

hence $q(u,u) \le L_1q(u,u)$. Similarly, by (2.2), we obtain $q(u,u) \le L_2q(u,u)$. So $q(u,u) \le L_1L_2q(u,u)$. But $L = L_1L_2 < 1$, hence $q(u,u) = 0$ by (p_5).

Suppose that v is also a common fixed point of S, T, I and J. Then similarly, we obtain $q(v,v) = 0$.

By (2.1),

$$q(u,v) = q(Su,Tv) \le a_1q(Ju,Iv) + a_2q(Ju,Su) + a_3q(Iv,Tv) + a_4q(Ju,Tv)$$

$$= [a_1 + a_4]q(u, v) \leq [a_1 + a_2 + a_3 + 2a_4]q(u, v),$$

hence $q(u, v) \leq L_1 q(u, v)$. Similarly, by (2.2), we obtain $q(u, v) \leq L_2 q(u, v)$. So $q(u, v) \leq L_1 L_2 q(u, v)$. But $L = L_1 L_2 < 1$, hence $q(u, v) = 0$ by (p_5). Therefore, $u = v$ by Lemma 1.6(1). This completes that u is the unique common fixed point of S,T,I and J, also $q(u, u) = 0$.

We say that $\phi \in \Phi$ if $\phi : [0, +\infty)^4 \to [0, \infty)$ satisfies (i) $\phi(\cdot, \cdot, \cdot, x)$ is non-decreasing about x, (ii) there exists $L_\phi \in [0, +\infty)$ such that $u \leq \phi(v, v, u, v + u)$ implies $u \leq L_\phi v$. In this case, L_ϕ is said to be a companion constant of ϕ.

Theorem 2.2 *Let (X, d) be a metric space, $S, T, I, J : X \to X$ four mappings satisfying that $S(X) \subset I(X)$ and $T(X) \subset J(X)$ and for each $x, y \in X$,*

$$q(Sx, Ty) \leq \phi(q(Jx, Iy), q(Jx, Sx), q(Iy, Ty), q(Jx, Ty)), \tag{2.10}$$

$$q(Tx, Sy) \leq \phi'(q(Ix, Jy), q(Ix, Tx), q(Jy, Sy), q(Ix, Sy)), \tag{2.11}$$

where $\phi, \phi' \in \Phi$. If a) $L_\phi L_{\phi'} < 1$, b) any one of $S(X)$, $T(X)$, $I(X)$ and $J(X)$ is complete, c) for each $u \in \{y \in X : \exists F \in \{S, T, I, J\}, Fy \neq y\}$, one of the following conditions holds:

$$\inf\{\|q(Sx, u)\| + \|q(Jx, u)\| + \|q(Jx, Sx)\| : x \in X\} > 0;$$

$$\inf\{\|q(Tx, u)\| + \|q(Ix, u)\| + \|q(Ix, Tx)\| : x \in X\} > 0,$$

then S, T, I, J have a common fixed point $u \in X$ and $q(u, u) = 0$. Furthermore, ϕ and ϕ' satisfy that for all $r > 0$, $r > \phi(r, 0, 0, r)$ or $r > \phi'(r, 0, 0, r)$, then u is the unique common fixed point of S, T, I, J.

Proof Let $x_0 \in X$ be arbitrary. Since $S(X) \subset I(X)$, there exists $x_1 \in X$ such that $Sx_0 = Ix_1$; by $T(X) \subset J(X)$, there exists $x_2 \in X$ such that $Tx_1 = Jx_2$. By induction, two sequences $\{x_n\}$ and $\{y_n\}$ can be chosen such that

$$y_{2n} = Sx_{2n} = Ix_{2n+1}, \; y_{2n+1} = Tx_{2n+1} = Jx_{2n+2}, \quad n = 0, 1, \ldots.$$

For any n, by (2.10) and (q_2) and $\phi \in \Phi$,

$$\begin{aligned}
q(y_{2n}, y_{2n+1}) &= q(Sx_{2n}, Tx_{2n+1}) \\
&\leq \phi(q(Jx_{2n}, Ix_{2n+1}), q(Jx_{2n}, Sx_{2n}), q(Ix_{2n+1}, Tx_{2n+1}), q(Jx_{2n}, Tx_{2n+1})) \\
&= \phi(q(y_{2n-1}, y_{2n}), q(y_{2n-1}, y_{2n}), q(y_{2n}, y_{2n+1}), q(y_{2n-1}, y_{2n+1})) \\
&\leq \phi(q(y_{2n-1}, y_{2n}), q(y_{2n-1}, y_{2n}), q(y_{2n}, y_{2n+1}), q(y_{2n-1}, y_{2n}) + q(y_{2n}, y_{2n+1})).
\end{aligned}$$

Hence

$$q(y_{2n}, y_{2n+1}) \leq L_\phi \, q(y_{2n-1}, y_{2n}). \tag{2.12}$$

Similarly, by (2.11) and (q_2) and $\phi' \in \Phi$,

$$\begin{aligned}
q(y_{2n+1}, y_{2n+2}) &= q(Tx_{2n+1}, Sx_{2n+2}) \\
&\leq \phi('q(Ix_{2n+1}, Jx_{2n+2}), q(Ix_{2n+1}, Tx_{2n+1}), q(Jx_{2n+2}, Sx_{2n+2}), q(Ix_{2n+1}, Sx_{2n+2})) \\
&= \phi'(q(y_{2n}, y_{2n+1}), q(y_{2n}, y_{2n+1}), q(y_{2n+1}, y_{2n+2}), q(y_{2n}, y_{2n+2}))
\end{aligned}$$

$$\leq \phi'(q(y_{2n}, y_{2n+1}), q(y_{2n}, y_{2n+1}), q(y_{2n+1}, y_{2n+2}), q(y_{2n}, y_{2n+1}) + q(y_{2n+1}, y_{2n+2})).$$

Hence

$$q(y_{2n+1}, y_{2n+2}) \leq L_{\phi'}\, q(y_{2n}, y_{2n+1}). \tag{2.13}$$

Now, from (2.12) and (2.13), we obtain

$$\begin{aligned} q(y_{2n+1}, y_{2n+2}) \leq & L_{\phi'} q(y_{2n}, y_{2n+1}) \leq L_\phi L_{\phi'} q(y_{2n-1}, y_{2n}) \leq \cdots \\ \leq & (L_\phi L_{\phi'})^n q(y_1, y_2) \leq L^n L_{\phi'} q(y_0, y_1), \end{aligned}$$

where $L = L_\phi L_{\phi'} < 1$, and

$$q(y_{2n}, y_{2n+1}) \leq L_\phi q(y_{2n-1}, y_{2n}) \leq L_\phi L^{n-1} L_{\phi'} q(y_0, y_1) \leq L^n q(y_0, y_1).$$

Following the process of the proof of Theorem 2.1, we obtain that S, T, I, J have a common fixed point $u \in X$.

On the other hand, by (2.10),

$$\begin{aligned} q(u, u) = & q(Su, Tu) \leq \phi(q(Ju, Iu), q(Ju, Su), q(Iu, Tu), q(Ju, Tu)) \\ \leq & \phi(q(u, u), q(u, u), q(u, u), q(u, u) + q(u, u)), \end{aligned}$$

hence $q(u, u) \leq L_\phi\, q(u, u)$. Similarly, $q(u, u) \leq L_{\phi'}\, q(u, u)$ by using (2.11), hence $q(u, u) \leq L_\phi L_{\phi'}\, q(u, u)$. So $q(u, u) = 0$.

If v is also a common fixed point of S, T, I, J, then similarly, $q(v, v) = 0$ also holds.

Using (2.10) and (2.11), we obtain

$$\begin{aligned} q(u, v) = & q(Su, Tv) \leq \phi(q(Ju, Iv), q(Ju, Su), q(Iv, Tv), q(Ju, Tv)) \\ \leq & \phi(q(u, v), 0, 0, q(u, v)) \end{aligned}$$

and

$$\begin{aligned} q(u, v) = & q(Tu, Sv) \leq \phi'(q(Iu, Jv), q(Iu, Tu), q(Jv, Sv), q(Iu, Sv)) \\ \leq & \phi'(q(u, v), 0, 0, q(u, v)). \end{aligned}$$

Hence $q(u, v) = 0$. Therefore $u = v$ by Lemma 1.6(1). So u is the unique common fixed point of S, T, I, J.

Remark 2.3 If (X, d) is a metric space, Theorem 2.1 is a particular form of Theorem 2.2. In fact, define two functions $\phi, \phi' : [0, +\infty)^4 \to [0, +\infty)$ by

$$\begin{aligned} \phi(u_1, u_2, u_3, u_4) &= a_1 u_1 + a_2 u_2 + a_3 u_3 + a_4 u_4; \\ \phi'(u_1, u_2, u_3, u_4) &= a'_1 u_1 + a'_2 u_2 + a'_3 u_3 + a'_4 u_4, \end{aligned}$$

where $a_i, a'_i \in [0, +\infty), i = 1, 2, 3, 4$, satisfy $a_3 + a_4 < 1$, $a'_3 + a'_4 < 1$, $\frac{a_1+a_2+a_4}{1-a_3-a_4} \frac{a'_1+a'_2+a'_4}{1-a'_3-a'_4} < 1$. Let $L_\phi = \frac{a_1+a_2+a_4}{1-a_3-a_4}$ and $L_{\phi'} = \frac{a'_1+a'_2+a'_4}{1-a'_3-a'_4}$. Obviously, ϕ is non-decreasing about u_4 and $u \leq \phi(v, v, u, u+v)$ implies that $u \leq (a_1 + a_2 + a_4)v + (a_3 + a_4)u$, hence $u \leq L_\phi v$, so $\phi \in \Phi$. Similarly, $\phi' \in \Phi$. On the other hand, $L_\phi L_{\phi'} < 1$ implies that $L_\phi < 1$ or $L_{\phi'} < 1$. If $L_\phi < 1$, then $a_1 + a_4 < 1$. So $\phi(u, 0, 0, u) = (a_1 + a_4)u < u$ for all $u > 0$. Similarly, $L_{\phi'} < 1$ implies that

$\phi'(u,0,0,u) = (a'_1 + a'_4)u < u$ for all $u > 0$. Hence $\phi, \phi', L_\phi, L_{\phi'}$ satisfy all conditions of Theorem 2.2. Therefore, the conclusion of Theorem 2.1 follows from Theorem 2.2.

The next two results are all particular forms of Theorems 2.1 and 2.2, respectively.

Theorem 2.4 *Let (X,d) be a cone metric space and P_0 a normal cone with normal constant K. Let $a_i \in [0,+\infty), i = 1,2,3,4$, be real numbers satisfying $a_1 + a_2 + a_3 + 2a_4 < 1$. Suppose that two mappings $S, I : X \to X$ satisfy that $S(X) \subset I(X)$ and for each $x, y \in X$,*

$$q(Sx,Sy) \le a_1 q(Ix,Iy) + a_2 q(Ix,Sx) + a_3 q(Iy,Sy) + a_4 q(Ix,Sy). \tag{2.14}$$

If $S(X)$ or $I(X)$ is complete, and for any $u \in \{y \in X : \exists F \in \{S,I,\}, Fy \ne y\}$,

$$\inf\{\|q(Sx,u)\| + \|q(Ix,u)\| + \|q(Ix,Sx)\| : x \in X\} > 0,$$

then S and I have a unique common fixed point u in X and $q(u,u) = 0$.

Proof Let $S = T$, $I = J$ and $a'_i = a_i$ $(i = 1,2,3,4)$. Then the conclusion follows from Theorem 2.1. $\square$

Theorem 2.5 *Let (X,d) be a metric space, $S, I : X \to X$ two mappings satisfying that $S(X) \subset I(X)$ and for each $x, y \in X$,*

$$q(Sx,Sy) \le \phi(q(Ix,Iy), q(Ix,Sx), q(Iy,Sy), q(Ix,Sy)), \tag{2.15}$$

where $\phi \in \Phi$ with $L_\phi < 1$. If $S(X)$ or $I(X)$ is complete, and for each $u \in \{y \in X : \exists F \in \{S,I\}, Fy \ne y\}$,

$$\inf\{\|q(Sx,u)\| + \|q(Jx,u)\| + \|q(Jx,Sx)\| : x \in X\} > 0,$$

then S and I have a common fixed point u in X and $q(u,u) = 0$.

Proof Let $S = T$, $I = J$ and $\phi = \phi'$. Then the conclusion follows from Theorem 2.2. $\square$

Remark 2.6 Theorem 2.4 is the main result in [20]. Hence Theorem 2.1 is a generalization of the main result in [20].

Acknowledgements We thank the referees for their time and comments.

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