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Rigidity Theorems of Complete Ricci Solitons under L^p

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Abstract We prove rigidity results of complete Ricci soliton under some L^p conditions and some curvature assumptions.

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1. Introduction

Let us recall the concept of Ricci solitons, which was introduced by Hamilton [5] in mid 1980's. Let (M, g) be an n-dimensional, complete, connected Riemannian manifold. A Ricci soliton is a Riemannian metric together with a vector field (M, g, X) that satisfies

$$\mathrm{Ricc} + \frac{1}{2} L_X g = \lambda g \tag{1}$$

for some constant λ. It is called shrinking, steady or expanding Ricci soliton depending on whether $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. If there is a smooth function f on M such that $X = \nabla f$, then the equation (1) can be written as

$$\mathrm{Ricc} + \mathrm{Hess}\, f = \lambda g. \tag{2}$$

This case is called a gradient (Ricci) soliton. The Euclidean space is a shrinking, steady or expanding Ricci soliton considering the function $f(x) = \frac{\epsilon}{4}|x|^2$, with $\epsilon \in \{1, 0, -1\}$. Both equations (1) and (2) can be considered as perturbations of the Einstein equation

$$\mathrm{Ricc} = \lambda g$$

and reduce to this latter case if X or ∇f are Killing vector fields. When $X = 0$ or f is constant, we call the underlying Einstein manifold a trivial Ricci soliton.

Ricci solitons are an important object in the study of the Ricci flow, since they are self-similar solutions of the flow. They also serve as model cases of various Harnack inequalities for the Ricci flow, which become equalities when the flow consists of Ricci solitons. From the seminal

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work of Hamilton [5] and Perelman's result [9] that any compact Ricci soliton is necessarily a gradient soliton, it is to see that any compact steady or expanding Ricci soliton must be Einstein [2]. So the classification of complete gradient shrinking solitons plays important roles in the study of the Ricci flow [2, 9]. In recent years, the study of gradient Ricci solitons has become the subject of a rapidly increasing investigation directed mainly towards two goals, classification and triviality [2, 3, 4, 11, 12]. However, there are not much works on the more general case of Ricci solitons, that is, when X is not necessarily the gradient of a potential f, in particular on noncompact Ricci solitons. Recently, Mastrolia and Rigoli [7] established three basic equations for a general soliton structure on the Riemannian manifold and drew some geometric conclusions with the aid of the maximum principle.

Now we improve Theorem 1.1 in [7] to the following

Theorem 1.1 *Let (M, g) be a complete manifold with Ricci tensor satisfying*

$$\mathrm{Ricc} \le \frac{1}{2}a(x)g \tag{3}$$

for some function $a(x)$. Assume that, for some $H > \frac{1}{2}$,

$$\lambda_1^{L_H}(M) \ge 0, \tag{4}$$

where $L_H = \Delta + Ha(x)$. For $2H - \sqrt{4H^2 - 2H} < \beta < 2H + \sqrt{4H^2 - 2H}$, if there exists a soliton structure (M, g, X) on (M, g) with $X \not\equiv 0$ satisfying

$$\lim_{R\to\infty} \frac{1}{R^2} \int_{B(p,2R)\setminus B(p,R)} |X|^\beta = 0, \tag{5}$$

where $B(p, R)$ is the geodesic ball of radius R centered at p in M, then X is a parallel field and (M, g) is Ricci flat Einstein. Furthermore, the simply connected universal cover of M is a warped product $(\mathbb{R} \times_c P, h)$ with $c = |X|, h = dt^2 + cg'$ and (P, g') is Ricci flat Einstein.

Theorem 1.2 *Let (M, g) be a complete manifold with Ricci tensor satisfying* $\mathrm{Ricc} \le 0$. *For some positive number β, if there exists a soliton structure (M, g, X) on (M, g) with $X \not\equiv 0$ satisfying $\int_M |X|^\beta < \infty$, then X is a parallel field and (M, g) is Ricci flat Einstein. Furthermore, the simply connected universal cover of M is a warped product $(\mathbb{R} \times_c P, h)$ with $c = |X|, h = dt^2 + cg'$ and (P, g') is Ricci flat Einstein.*

Theorem 1.3 *Let (M, g) be a complete manifold with Ricci tensor satisfying* $\mathrm{Ricc} < 0$. *Thus, there are no soliton structure (M, g, X) on (M, g) with $X \not\equiv 0$ and $X \in L^p(M)$ for some $p > 0$.*

Corollary 1.4 *Let (M, g) be a complete minimal submanifold in an $(n + p)$-dimensional Euclidean space $\mathbb{R}^{n+p}$. For some $\beta > 0$, if there exists a soliton structures (M, g, X) on (M, g) with $X \not\equiv 0$ satisfying $\int_M |X|^\beta < \infty$, then M must be an affine n-dimensional plane.*

2. Proofs of Theorems

Before we prove Theorem 1.1, we need the following Lemma 2.1. Although Lemma 2.1 was proved in [7, 11], for completeness, we still include it.

Lemma 2.1 *Let (M, g, X) be a Ricci soliton on (M, g). Then*

$$\frac{1}{2}\Delta|X|^2 = |\nabla X|^2 - \mathrm{Ricc}(X, X). \tag{6}$$

Proof It is easy to see that

$$\frac{1}{2}\Delta|X|^2 = |\nabla X|^2 + X_i X_{ikk}. \tag{7}$$

On the other hand

$$\mathrm{div}(L_X g)(X) = X_i X_{ikk} + X_i X_{kik}, \tag{8}$$

and

$$X_{kik} - X_{kki} = X_j R_{ji}. \tag{9}$$

So from (7), (8) and (9), we obtain

$$\frac{1}{2}\Delta|X|^2 = |\nabla X|^2 + \mathrm{div}(L_X g)(X) - X_i X_{kki} - X_i X_j R_{ij}. \tag{10}$$

Contracting the soliton equation (1), we have

$$R + \mathrm{div}\, X = n\lambda. \tag{11}$$

Using (11), we get

$$X_i X_{kki} = \nabla_X(\mathrm{div}\, X) = -\nabla_X(R). \tag{12}$$

Since

$$\mathrm{div}(\lambda g)(Y) = 0, \tag{13}$$

from (1), we obtain

$$\mathrm{div}(L_X g)(X) = -2\mathrm{div}(\mathrm{Ricc})(X). \tag{14}$$

By the second Bianchi's identities, we have

$$\nabla_X(R) = 2\mathrm{div}(\mathrm{Ricc})(X). \tag{15}$$

Combining with (12), we get

$$-X_i X_{kki} = 2\mathrm{div}(\mathrm{Ricc})(X). \tag{16}$$

Substitute (15) and (16) to (10), we complete the proof of the lemma. □

Proof of Theorem 1.1 First of all, using Cauchy-Schwarz inequality, we have that for any vector field Y on M

$$\frac{1}{4}|\nabla|Y|^2|^2 \le |Y|^2|\nabla Y|^2. \tag{17}$$

Setting $u = |X|^2$, we multiply (6) by u and use (17) to obtain

$$\frac{1}{2}u\Delta u + u\mathrm{Ricc}(X, X) \ge \frac{1}{4}|\nabla u|^2. \tag{18}$$

Next we use assumption (3) to deduce

$$u\Delta u + a(x)u^2 \ge \frac{1}{2}|\nabla u|^2. \tag{19}$$

Directly computing (19), we have

$$\begin{aligned} u^\alpha \Delta u^\alpha &= u^\alpha \big(\alpha(\alpha-1)u^{\alpha-2}|\nabla u|^2 + \alpha u^{\alpha-1}\Delta u\big) \\ &= \frac{\alpha-1}{\alpha}|\nabla u^\alpha|^2 + \alpha u^{2\alpha-2} u\Delta u \\ &\geq \frac{\alpha-1}{\alpha}|\nabla u^\alpha|^2 + \alpha u^{2\alpha-2}\big(\frac{1}{2}|\nabla u|^2 - a(x)u^2\big) \\ &\geq \big(1-\frac{1}{2\alpha}\big)|\nabla u^\alpha|^2 - \alpha a(x) u^{2\alpha}, \end{aligned} \tag{20}$$

where α is a positive constant.

Let $q \geq 0$ and $\phi \in C_0^\infty(M)$. Multiplying (20) by $u^{2q\alpha}\phi^2$ and integrating over M, we obtain

$$\begin{aligned} \big(1-\frac{1}{2\alpha}\big)\int_M u^{2q\alpha}|\nabla u^\alpha|^2\phi^2 \leq & \alpha\int_M a(x)u^{2(q+1)\alpha}\phi^2 + \int_M u^{(2q+1)\alpha}\Delta u^\alpha \phi^2 \\ =&\alpha\int_M a(x)u^{2(q+1)\alpha}\phi^2 - (2q+1)\int_M u^{2q\alpha}|\nabla u^\alpha|^2\phi^2 - \\ & 2\int_M u^{(2q+1)\alpha}\phi g(\nabla\phi, \nabla u^\alpha), \end{aligned}$$

which gives

$$\begin{aligned} &\big(2(q+1)-\frac{1}{2\alpha}\big)\int_M u^{2q\alpha}|\nabla u^\alpha|^2\phi^2 \\ &\leq \alpha\int_M a(x)u^{2(q+1)\alpha}\phi^2 - 2\int_M u^{(2q+1)\alpha}\phi g(\nabla\phi,\nabla u^\alpha). \end{aligned} \tag{21}$$

Using the Cauchy-Schwarz inequality, we can rewrite (21) as

$$\begin{aligned} &\big(2(q+1)-\frac{1}{2\alpha}-\epsilon\big)\int_M u^{2q\alpha}\phi^2|\nabla u^\alpha|^2 \\ &\leq \alpha\int_M a(x)u^{2(q+1)\alpha}\phi^2 + \frac{1}{\epsilon}\int_M u^{2(q+1)\alpha}|\nabla\phi|^2. \end{aligned} \tag{22}$$

On the other hand, by using (4), we have

$$\begin{aligned} H\int_M a(x)u^{2(1+q)\alpha}\phi^2 \leq &(1+q)^2\int_M u^{2q\alpha}|\nabla u^\alpha|^2\phi^2 + \int_M u^{2(1+q)\alpha}|\nabla\phi|^2 + \\ & 2(1+q)\int_M u^{(2q+1)\alpha}\phi g(\nabla\phi,\nabla u^\alpha), \end{aligned} \tag{23}$$

which gives

$$\begin{aligned} H\int_M a(x)u^{2(1+q)\alpha}\phi^2 \leq &(1+q)(1+q+\epsilon)\int_M u^{2q\alpha}|\nabla u^\alpha|^2\phi^2 + \\ & \big(1+\frac{1+q}{\epsilon}\big)\int_M u^{2(1+q)\alpha}|\nabla\phi|^2. \end{aligned} \tag{24}$$

If $2(q+1)-\frac{1}{2\alpha}-\epsilon > 0$, then introducing (24) to (22), we obtain

$$\begin{aligned} &[(2(q+1)-\frac{1}{2\alpha}-\epsilon)H - (1+q)(1+q+\epsilon)\alpha]\int_M u^{2q\alpha}|\nabla u^\alpha|^2\phi^2 \\ &\leq \big[\frac{H}{\epsilon} + \alpha\frac{1+q+\epsilon}{\epsilon}\big]\int_M u^{2(1+q)\alpha}|\nabla\phi|^2. \end{aligned} \tag{25}$$

Let $(1+q)\alpha = \frac{\beta}{2}$. Thus for $2H - \sqrt{4H^2 - 2H} < \beta < 2H + \sqrt{4H^2 - 2H}$, it is easy to see that $\left(2(q+1) - \frac{1}{2\alpha}\right) > 0$ and $\left(2(q+1) - \frac{1}{2\alpha}\right) H - (1+q)^2\alpha > 0$. Then we can choose $\epsilon > 0$ sufficiently small so that $\left(2(q+1) - \frac{1}{2\alpha} - \epsilon\right) > 0$ and $\left[\left(2(q+1) - \frac{1}{2\alpha} - \epsilon\right) H - (1+q)(1+q+\epsilon)\alpha\right] > 0$. It follows from (25) that the following inequality holds:

$$\int_M u^{2q\alpha} |\nabla u^\alpha|^2 \phi^2 \leq C \int_M u^\beta |\nabla \phi|^2, \tag{26}$$

where C is a constant that depends on H, α, ϵ and q. Let ϕ be a smooth function on $[0, \infty)$ such that $\phi \geq 0, \phi = 1$ on $[0, R]$ and $\phi = 0$ in $[2R, \infty)$ with $|\phi'| \leq \frac{2}{R}$. Then considering $\phi \circ r$, where r is the function in the definition of $B(R)$, we have from (26)

$$\int_M u^{2q\alpha} |\nabla u^\alpha|^2 \phi^2 \leq \frac{4C_1}{R^2} \int_{B(p,2R)\setminus B(p,R)} u^\beta. \tag{27}$$

Let $R \to +\infty$. By assumption that $\lim_{R\to\infty} \frac{1}{R^2} \int_{B(p,2R)\setminus B(p,R)} u^\beta = 0$, from (27) we conclude $\nabla u^\alpha = 0$, and u^α is constant. Since by assumption $X \neq 0$, from (20) we get $a(x) \geq 0$. It follows by substituting the above u^α into (23) that

$$H \int_{B(p,R)} a(x) u^{2(1+q)\alpha} \leq H \int_M a(x) u^{2(1+q)\alpha} \phi^2 \leq \frac{1}{R^2} \int_{B_{(p,2R)}\setminus B_{(p,R)}} u^\beta.$$

So we conclude by letting $R \to \infty$ that $a(x) \equiv 0$. Thus, by (3) and Lemma 2.1, we have

$$0 \geq \mathrm{Ricc}(X, X) = |\nabla X|^2,$$

i.e., $\nabla X \equiv 0$ and X is a parallel vector field. Thus X is a Killing field and going back to (1)

$$\mathrm{Ricc} = \lambda g \leq 0,$$

that is, (M, g) is Einstein with $\lambda \leq 0$. Thus, if $\lambda < 0$, we obtain the following as the same as (22)

$$\left(2(q+1) - \frac{1}{2\alpha} - \epsilon\right) \int_M u^{2q\alpha} \phi^2 |\nabla u^\alpha|^2 - \alpha\lambda \int_M u^{2(q+1)\alpha} \phi^2 \leq \frac{1}{\epsilon} \int_M u^\beta |\nabla \phi|^2. \tag{28}$$

Then, under assumption that $\lim_{R\to\infty} \frac{1}{R^2} \int_{B(p,2R)\setminus B(p,R)} u^\beta = 0$ and choosing suitably ϕ, from (28) we conclude $X = 0$. Contradiction. Consequently, we have $\lambda = 0$. Hence (M, g) is Ricci flat Einstein.

Now, since X is parallel and a closed conformal field, the final part of the Theorem follows from [8, Proposition 2 (c)] and from [1, Corollary 9.107]. □

Using the same argument as Theorem 1.1, we obtain

Corollary 2.2 *Let (M, g) be a complete manifold with Ricci tensor satisfying* Ricc ≤ 0. *For some $\beta > \frac{1}{2}$, if there exists a soliton structure (M, g, X) on (M, g) with $X \not\equiv 0$ satisfying*

$$\lim_{R\to\infty} \frac{1}{R^2} \int_{B(p,2R)\setminus B(p,R)} |X|^\beta = 0,$$

then X is a parallel field and (M, g) is Ricci flat Einstein. Furthermore, the simply connected universal cover of M is a warped product $(\mathbb{R} \times_c P, h)$ with $c = |X|, h = dt^2 + cg'$ and (P, g') is Ricci flat Einstein.

Proof of Theorem 1.2 We can rewrite (19) as

$$u\Delta u \geq \frac{1}{2}|\nabla u|^2 \geq 0.$$

Thus, by [13, Theorem 3], u is constant under assumption that $\int_M |X|^\beta < \infty$ for all $\beta \neq 1$. The remaining proof now follows exactly as Theorem 1.1. And under assumption that $\int_M |X| < \infty$, by Theorem 1.1, we get this result. □

Using the same argument as Theorems 1.1 and 1.2, we obtain Theorem 1.3.

Proof of Corollary 1.4 By Gauss equation in [6], we have

$$\mathrm{Ricc} \leq 0.$$

By Theorem 1.2, we obtain Ricc $= 0$. Hence, (M, g) must be Einstein. By Gauss equation, we obtain $|B|^2 = n^2|H|^2 - \rho$, where B, H and ρ denote the second fundamental form, the mean curvature and the scalar curvature of M, respectively. Since M is minimal and $\rho = 0$, we obtain $B = 0$. Thus M must be an affine n-dimensional plane. □

Using the same argument as Corollary 1.4, by Corollary 2.2, we obtain

Corollary 2.3 *Let (M, g) be a complete minimal submanifold in an $(n+p)$-dimensional Euclidean space $\mathbb{R}^{n+p}$. For some $\beta > \frac{1}{2}$, if there exists a soliton structures (M, g, X) on (M, g) with $X \not\equiv 0$ satisfying*

$$\lim_{R\to\infty} \frac{1}{R^2} \int_{B(p,2R)\setminus B(p,R)} |X|^\beta = 0,$$

then M must be an affine n-dimensional plane.

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