Journal of Mathematical Research with Applications
Nov., 2014, Vol. 34, No. 6, pp. 729–735
DOI:10.3770/j.issn:2095-2651.2014.06.011
Http://jmre.dlut.edu.cn

Solutions of Linear Differential Equations in the Unit Disc

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Abstract In this paper, some properties of solutions of linear differential equations $f^{(k)} + A(z)f = 0$ and $f^{(k)} + A(z)f = F(z)$ are discussed. Our results are a generalization of the original results.

Keywords unit disc; linear differential equations; analytic function; Q_p space.

MR(2010) Subject Classification 30D35; 34M10

1. Introduction

In this paper, we will use notation $\triangle = \{z : |z| < 1\}$ to denote the unit disc in the complex plane, D and A denote Dirichlet space and analytic function space, respectively. We still need the following definitions.

Definition 1.1 ([1]) *We say a function $f \in A$ belongs to space $Q_p(p \in (0,\infty))$ if and only if*

$$\sup_{a\in\triangle} \iint_{\triangle} |f'(z)|^2 g^p(z,a)\mathrm{d}\sigma(z) < \infty,$$

where $g(z,a) = \log|\frac{1-\bar{a}z}{z-a}|$ is the Green's function in $\triangle$ and $\mathrm{d}\sigma$ is an area measure on $\triangle$ normalized such that $\sigma(\Delta) = 1$.

Definition 1.2 ([2]) *We say that a function $f \in A$ belongs to the classical Dirichlet space D if and only if,*

$$\iint_{\triangle} |f'(z)|^2 \mathrm{d}\sigma(z) < \infty.$$

Definition 1.3 ([3]) *We say that a function $f \in A$ belongs to the Hardy space H^p $(0 < p < \infty)$ if and only if*

$$\sup_{0\le r<1} \Big(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\varphi})|^p \mathrm{d}\varphi\Big)^{\frac{1}{p}} < \infty.$$

For $p = \infty$, it is natural to say that $f \in H^\infty$ if and only if

$$\sup_{z\in\triangle} |f(z)| < \infty.$$

Received October 1, 2013; Accepted April 16, 2014
Supported by the National Natural Science Foundation of China (Grant Nos. 11301232; 11171119), the Youth Science Foundation of Education Bureau of Jiangxi Province (Grant No. GJJ12207) and the Natural Science Foundation of Jiangxi Province (Grant No. 20132BAB211009).
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For $0 \le q < \infty$, functions $f \in A$ in the corresponding weighted Hardy spaces H_q^p and H_q^∞ satisfy

$$\sup_{0\le r<1}(1-r^2)^q\Big(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\varphi})|^p\mathrm{d}\varphi\Big)^{\frac{1}{p}}<\infty$$

and

$$\sup_{z\in\triangle}(1-|z|^2)^q|f(z)|<\infty.$$

For a basic reference to Hardy spaces, see [3].

Definition 1.4 ([4]) *Let $\beta \in (0,\infty)$ be a constant. Then $f \in A$ is in space ε^β if and only if*

$$|f(z)|\le \exp\big(\frac{\alpha}{(1-|z|)^\beta}\big)$$

for some constant $\alpha \in (0,\infty)$.

In 2000, Heittokangas investigated the properties of solutions of the equation

$$f''+A(z)f=0 \tag{1.1}$$

in $\triangle$ in his Doctoral thesis [2] and obtained the following result.

Theorem 1.5 ([2]) *Let $A(z)=\sum_{n=0}^\infty a_nz^n$, $a_n \in C$ be the analytic coefficient of (1.1) in $\triangle$ with $|a_n|\le 1$ for all n. Then all solutions f of (1.1) belong to $\bigcap_{0<p<\infty} Q_p$.*

In 2011, Li and Wulan improved Theorem 1.5 and obtained the following result.

Theorem 1.6 ([5]) *Let $A(z)=\sum_{n=0}^\infty a_nz^n$, $a_n \in C$ be the analytic coefficient of (1.1) in $\triangle$ with $|a_n|\le 1$ for all n. Then all solutions f of (1.1) belong to Dirichlet space D.*

Remark 1.7 In fact, the inclusion $D\subset \bigcap_{0<p<\infty} Q_p$ is strict, see [1].

Our first result contains Theorem 1.6 as a special case.

Theorem 1.8 *Let $A(z)=\sum_{n=0}^\infty a_nz^n$, $a_n \in C$ be the analytic coefficient of equation*

$$f^{(k)}+A(z)f=0,\quad k\ge 2 \tag{1.2}$$

in $\triangle$ with $|a_n|\le n^{k-2}$ for all n. Then all solutions f of (1.2) belong to D.

In 2000, Heittokangas also investigated the properties of solutions of (1.2) in [2] and obtained the following result.

Theorem 1.9 ([2]) *Let $A(z)$ be analytic coefficient of (1.2) in $\triangle$ satisfying*

$$|A(z)|\le\frac{\alpha}{(1-|z|)^\beta}, \tag{1.3}$$

where $\alpha>0$ and $\beta\ge 0$ are finite constants. Then

(1) $f\in H^\infty$, if $0\le\beta<k$;

(2) $f\in H^\infty_{\frac{\alpha}{(k-1)!}}$, if $\beta=k$;

(3) $f\in \varepsilon^{\beta-k}$, if $k<\beta<\infty$.

In Theorem 1.9, Heittokangas investigated the properties of solutions of the higher-order linear differential equation (1.2). In this paper, we will investigate the properties of solutions of the higher-order non-homogeneous linear differential equation

$$f^{(k)} + A(z)f = F(z), \tag{1.4}$$

and obtain a similar result to Theorem 1.9.

Theorem 1.10 *Let $A(z)$, $F(z)$ be analytic coefficients of (1.3) in $\triangle$ satisfying*

$$|A(z)| \leq \frac{\alpha}{(1-|z|)^{\beta}}, \quad F(z) \in H^p,$$

where $\alpha > 0$, $\beta \geq 0$ are finite constants, $\frac{1}{k} \leq p \leq +\infty$. Then every solution f of (1.4) satisfies:

(1) $f \in H^{\infty}$, if $0 \leq \beta < k$;

(2) $f \in H^{\infty}_{\frac{\alpha}{(k-1)!}}$, if $\beta = k$;

(3) $f \in \varepsilon^{\beta-k}$, if $k < \beta < \infty$.

2. Lemmas for the proof of Theorems

Lemma 2.1 ([6]) *Let $u(x)$,$v(x) \geq 0$, c be a positive constant and $u \leq c + \int_0^t uv\mathrm{d}t_1$. Then $u \leq c\exp(\int_0^t v\mathrm{d}t_1)$.*

Lemma 2.2 ([3]) *If $f' \in H^p(p<1)$, then $f \in H^q$, $q = \frac{p}{1-p}$.*

Lemma 2.3 ([3]) *If $f' \in H^1$, then $f \in H^{\infty}$.*

Lemma 2.4 ([3]) *If $f \in H^p(0 < p < \infty)$, then*

$$|f(z)| \leq 2^{\frac{1}{p}} \|f\|_p (1-r)^{\frac{-1}{p}}, \quad r = |z|,$$

where $\|f\|_p = \sup_{0 \leq r < 1}(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\varphi})|^p \mathrm{d}\varphi)^{\frac{1}{p}}$.

Lemma 2.5 *If $f^{(k)}(z) \in H^p$ ($\frac{1}{k} \leq p \leq \infty, k \geq 2$), then $f(z) \in H^{\infty}$.*

Proof We divide our proof into four cases.

Case 1 $1 < p < \infty$. Since $f^{(k)}(z) \in H^p$, by Definition 1.3 and Lemma 2.4, we have

$$\begin{aligned} |f^{(k)}(z)| \leq & 2^{\frac{1}{p}} \|f^{(k)}(z)\|_p (1-r)^{\frac{-1}{p}} \\ \leq & 2^{\frac{1}{p}} (1-r)^{\frac{-1}{p}} \sup_{0 \leq r < 1} \Big(\frac{1}{2\pi}\int_0^{2\pi} |f^{(k)}(re^{i\varphi})|^p \mathrm{d}\varphi\Big)^{\frac{1}{p}} \\ \leq & M_1 (1-r)^{\frac{-1}{p}}, \end{aligned}$$

where $M_1(>0)$ is a constant. After integrating $f^{(k)}(z)$ from z_0 to z, we get

$$f^{(k-1)}(z) - f^{(k-1)}(z_0).$$

We choose $z_0 = 0$ and the path of integration to be the line segment $[0, z]$. Denoting $z = re^{i\varphi}$ and $\zeta = te^{i\varphi}$, $0 \leq t \leq r < 1$, we obtain

$$\begin{aligned} |f^{(k-1)}(z)| - |f^{(k-1)}(0)| \leq & |f^{(k-1)}(z) - f^{(k-1)}(0)| \\ \leq & \int_0^r |f^{(k)}(te^{i\varphi})|\mathrm{d}t \leq \int_0^r M_1(1-t)^{\frac{-1}{p}}\mathrm{d}t. \end{aligned}$$

Therefore $\sup_{z\in\Delta}|f^{(k-1)}(z)|<\infty$.

Now integrating $f^{(k-1)}(z)$ from z_0 to z, we get

$$f^{(k-2)}(z)-f^{(k-2)}(z_0).$$

We choose $z_0=0$ and the path of integration to be the line segment $[0,z]$. Denoting $z=re^{i\varphi}$ and $\zeta=te^{i\varphi}$, $0\le t\le r<1$, we obtain

$$|f^{(k-2)}(z)|-|f^{(k-2)}(0)|\le|f^{(k-2)}(z)-f^{(k-2)}(0)|\le\int_0^r|f^{(k-1)}(te^{i\varphi})|\mathrm{d}t,$$

which means $\sup_{z\in\Delta}|f^{(k-2)}(z)|<\infty$.

Repeating the above process k times, we can obtain

$$|f(z)|-|f(0)|\le|f(z)-f(0)|\le\int_0^r|f'(te^{i\varphi})|\mathrm{d}t,$$

which means $\sup_{z\in\Delta}|f(z)|<\infty$. Consequently $f(z)\in H^\infty$.

Case 2 $p=1$. By Lemma 2.3, we have $f^{(k-1)}(z)\in H^\infty$ from the fact $f^{(k)}(z)\in H^1$. Using a similar discussion to case 1, we can get $f(z)\in H^\infty$.

Case 3 $\frac{1}{k}\le p<1$. We divide our discussion into two subcases.

Subcase 3.1 When $k=2$, then $\frac{1}{2}\le p<1$. If $p=\frac{1}{2}$, since $f''(z)\in H^{\frac{1}{2}}$, by Lemma 2.2, we have $f'(z)\in H^1$. Then, by Lemma 2.3, we can get $f(z)\in H^\infty$. If $\frac{1}{2}<p<1$, by Lemma 2.2, we have $f'(z)\in H^{\frac{p}{1-p}}$ and $\frac{p}{1-p}>1$. Using a similar discussion to case 1, we conclude that $f(z)\in H^\infty$.

Subcase 3.2 When $k>2$, we have $\frac{1}{k}<\frac{1}{2}$.

If $p=\frac{1}{2}$, it follows from Lemma 2.2 that $f^{(k-1)}(z)\in H^1$. Then, by Lemma 2.3, we get $f^{(k-2)}(z)\in H^\infty$. Using a similar discussion to case 1 again, we obtain $f(z)\in H^\infty$.

If $p=\frac{1}{k}$, it follows from Lemma 2.2 that $f^{(k-1)}(z)\in H^{\frac{1}{k-1}}$ and then $f^{(k-2)}(z)\in H^{\frac{1}{k-2}}$. By the induction, we can get $f'(z)\in H^1$. Thus, by Lemma 2.3, we conclude that $f(z)\in H^\infty$.

If $\frac{1}{2}<p<1$, by Lemma 2.2, we have $f^{(k-1)}(z)\in H^{\frac{p}{1-p}}$ and $\frac{p}{1-p}>1$. Consequently, using a similar discussion to case 1, we can get $f(z)\in H^\infty$.

If $\frac{1}{k}<p<\frac{1}{2}$, it follows from Lemma 2.2 that $f^{(k-1)}(z)\in H^{\frac{p}{1-p}}$ and then $f^{(k-2)}(z)\in H^{\frac{p}{1-2p}}$. By the induction, we know $f^{(k-n)}(z)\in H^{\frac{p}{1-np}}$ $(1\le n<k)$. When $\frac{1}{p}$ is an integer, we know $1<\frac{1}{p}-1<k-1$, and then $f^{(k-(\frac{1}{p}-1))}(z)\in H^{\frac{p}{1-(\frac{1}{p}-1)p}}=H^1$. By Lemma 2.3, we have $f^{(k-\frac{1}{p})}(z)\in H^\infty$, and then using a similar discussion to case 1 again, we can get $f(z)\in H^\infty$. When $\frac{1}{p}$ is not an integer, we know $1<\frac{1}{p}-1<[\frac{1}{p}]<\frac{1}{p}<k$, where $[\frac{1}{p}]$ denotes integer part of $\frac{1}{p}$. Then $f^{(k-[\frac{1}{p}])}(z)\in H^{\frac{p}{1-[\frac{1}{p}]p}}$, where $\frac{p}{1-[\frac{1}{p}]p}>1$. Using a similar discussion to case 1, we can get $f(z)\in H^\infty$.

Case 4 $p=\infty$. Using a similar discussion to case 1, we can conclude that $f(z)\in H^\infty$ from the fact $f^{(k)}(z)\in H^\infty$.

Finally, we complete the proof. $\square$

3. Proof of Theorems 1.8 and 1.10

Theorem 1.8 can be verified by following the proof of Theorem 2.4 in [5] with suitable modifications.

Proof of Theorem 1.8 We can find a positive integer $N_0(>k)$ sufficiently large such that

$$\frac{(n-k-1)^{k-2}}{n^{\frac{1}{2}}(n-1)\cdots(n+2-k)}\sum_{i=1}^{+\infty}\frac{1}{i^{\frac{3}{2}}}+\frac{(n-k)^{k-2}+(n-k+1)^{k-2}}{n^{\frac{1}{2}}(n-1)\cdots(n+2-k)}\leq 1 \tag{3.1}$$

is true for $n>N_0$. Let $f(z)=\sum_{n=0}^{\infty}b_nz^n$ be a formal solution of (1.2). Then

$$f^{(k)}+A(z)f=\sum_{n=0}^{\infty}[(n+k)(n+k-1)\cdots(n+1)b_{n+k}+c_n]z^n=0, \tag{3.2}$$

where $c_n=a_0b_n+a_1b_{n-1}+\cdots+a_nb_0$. Hence (3.2) holds if and only if

$$b_n=-\frac{c_{n-k}}{n(n-1)\cdots(n+1-k)}, \tag{3.3}$$

for all $n=k,k+1,\ldots$. Choose a finite constant $M>0$ such that $|b_0|\leq M$, $|b_1|\leq M$ and $|b_i|\leq\frac{M}{i(i-1)^{\frac{1}{2}}}$ for all $i=2,3,\ldots,n$ $(n>N_0)$. Then it follows from (3.3) that

$$\begin{aligned}
|b_{n+1}| &= \frac{1}{(n+1)n\cdots(n+2-k)}|c_{n-k+1}|\\
&\leq \frac{1}{(n+1)n\cdots(n+2-k)}(|a_0b_{n-k+1}|+\cdots+|a_{n-k+1}b_0|)\\
&\leq \frac{M}{\prod_{i=0}^{k-1}(n+2-k+i)}\Big\{\sum_{i=1}^{n-k-1}\frac{i^{k-2}}{(n-k+1-i)(n-k-i)^{\frac{1}{2}}}+(n-k)^{k-2}+(n-k+1)^{k-2}\Big\}\\
&= \frac{M}{(n+1)n^{\frac{1}{2}}}\Big\{\frac{1}{n^{\frac{1}{2}}\prod_{i=0}^{k-3}(n+2-k+i)}\sum_{i=1}^{n-k-1}\frac{i^{k-2}}{(n-k+1-i)(n-k-i)^{\frac{1}{2}}}+\\
&\quad \frac{(n-k)^{k-2}+(n-k+1)^{k-2}}{n^{\frac{1}{2}}\prod_{i=0}^{k-3}(n+2-k+i)}\Big\}\\
&\leq \frac{M}{(n+1)n^{\frac{1}{2}}}\Big\{\frac{(n-k-1)^{k-2}}{n^{\frac{1}{2}}\prod_{i=0}^{k-3}(n+2-k+i)}\sum_{i=1}^{n-k-1}\frac{1}{(n-k+1-i)(n-k-i)^{\frac{1}{2}}}+\\
&\quad \frac{(n-k)^{k-2}+(n-k+1)^{k-2}}{n^{\frac{1}{2}}\prod_{i=0}^{k-3}(n+2-k+i)}\Big\}\\
&\leq \frac{M}{(n+1)n^{\frac{1}{2}}}\Big\{\frac{(n-k-1)^{k-2}}{n^{\frac{1}{2}}\prod_{i=0}^{k-3}(n+2-k+i)}\sum_{i=1}^{n-k-1}\frac{1}{(n-k-i)^{\frac{3}{2}}}+\frac{(n-k)^{k-2}+(n-k+1)^{k-2}}{n^{\frac{1}{2}}\prod_{i=0}^{k-3}(n+2-k+i)}\Big\}\\
&\leq \frac{M}{(n+1)n^{\frac{1}{2}}}\Big\{\frac{(n-k-1)^{k-2}}{n^{\frac{1}{2}}\prod_{i=0}^{k-3}(n+2-k+i)}\sum_{i=1}^{+\infty}\frac{1}{i^{\frac{3}{2}}}+\frac{(n-k)^{k-2}+(n-k+1)^{k-2}}{n^{\frac{1}{2}}\prod_{i=0}^{k-3}(n+2-k+i)}\Big\}.
\end{aligned}$$

This together with (3.1) and $n>N_0$, gives

$$|b_{n+1}|\leq\frac{M}{n^{\frac{1}{2}}(n+1)}.$$

Therefore, $|b_n|\leq\frac{M}{(n-1)^{\frac{1}{2}}n}$ holds for all $n=2,3,\ldots$ and so $\sum_{n=0}^{\infty}b_nz^n$ is absolutely convergent on Δ. Consequently f is analytic in Δ and by the definition of Dirichlet space, the assertion follows. □

Proof of Theorem 1.10 The method of proof is the same as in the proof of [2, Theorem 4.2] that has been also applied in [7]. After integrating $f^{(k)}(z)$ from z_0 to z k times, we get

$$f(z)-\sum_{n=0}^{k-1}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n. \tag{3.4}$$

On the other hand,

$$A(z)f(z)=\frac{\mathrm{d}^k}{\mathrm{d}z^k}\frac{1}{(k-1)!}\int_{z_0}^{z}(z-\zeta)^{k-1}A(\zeta)f(\zeta)\mathrm{d}\zeta,$$

$$F(z)=\frac{\mathrm{d}^k}{\mathrm{d}z^k}\frac{1}{(k-1)!}\int_{z_0}^{z}(z-\zeta)^{k-1}F(\zeta)\mathrm{d}\zeta.$$

After respectively integrating $A(z)f(z)$, $F(z)$ from z_0 to z k times, we get

$$\frac{1}{(k-1)!}\int_{z_0}^{z}(z-\zeta)^{k-1}A(\zeta)f(\zeta)\mathrm{d}\zeta, \tag{3.5}$$

$$\frac{1}{(k-1)!}\int_{z_0}^{z}(z-\zeta)^{k-1}F(\zeta)\mathrm{d}\zeta. \tag{3.6}$$

So by (1.3), (3.4)–(3.6), we have

$$\begin{aligned}f(z)=&\sum_{n=0}^{k-1}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n+\frac{1}{(k-1)!}\int_{z_0}^{z}(z-\zeta)^{k-1}F(\zeta)\mathrm{d}\zeta-\\&\frac{1}{(k-1)!}\int_{z_0}^{z}(z-\zeta)^{k-1}A(\zeta)f(\zeta)\mathrm{d}\zeta.\end{aligned} \tag{3.7}$$

Since $F(z)\in H^p$ and $\frac{1}{k}\le p\le+\infty$, it follows that $\frac{1}{(k-1)!}\int_{z_0}^{z}(z-\zeta)^{k-1}F(\zeta)d\zeta\in H^\infty$ because of Lemma 2.5. That is to say, there exists a constant $C_1>0$, such that

$$\Big|\frac{1}{(k-1)!}\int_{z_0}^{z}(z-\zeta)^{k-1}F(\zeta)\mathrm{d}\zeta\Big|<C_1,\quad z\in\Delta. \tag{3.8}$$

We choose $z_0=0$ and the path of integration to be the line segment $[0,z]$ in (3.7). Denoting $z=re^{i\varphi}$ and $\zeta=te^{i\varphi}$, $0\le t\le r<1$ and combining with (3.8), we obtain

$$|f(z)|\le\sum_{n=0}^{k-1}\frac{|f^{(n)}(0)|}{n!}r^n+C_1+\frac{1}{(k-1)!}\int_0^r(1-t)^{k-1}|A(te^{i\varphi})||f(te^{i\varphi})|\mathrm{d}t.$$

Hence

$$|f(z)|\le C+\frac{1}{(k-1)!}\int_0^r(1-t)^{k-1}|A(te^{i\varphi})||f(te^{i\varphi})|\mathrm{d}t, \tag{3.9}$$

where $C=\sum_{n=0}^{k-1}\frac{|f^{(n)}(0)|}{n!}r^n+C_1$.

By Lemma 1.1 and (3.9), we have

$$|f(z)|\le C\exp\Big(\frac{1}{(k-1)!}\int_0^r(1-t)^{k-1}|A(te^{i\varphi})|\mathrm{d}t\Big).$$

It follows from (1.3) that

$$|f(z)| \leq C \exp\Big(\frac{1}{(k-1)!}\int_0^r \alpha(1-t)^{k-\beta-1}\mathrm{d}t\Big). \tag{3.10}$$

(1) If $0 \leq \beta < k$, it follows from (3.10) that

$$\sup_{z\in\Delta}|f(z)| \leq \sup_{z\in\Delta}\Big\{C\exp\big(\frac{1}{(k-1)!}\int_0^r \alpha(1-t)^{k-\beta-1}\mathrm{d}t\big)\Big\} < \infty,$$

which means $f \in H^\infty$.

(2) If $\beta = k$, it follows from (3.10) that

$$\begin{aligned}(1-|z|^2)^{\frac{\alpha}{(k-1)!}}|f(re^{i\varphi})| \leq & (1-|z|^2)^{\frac{\alpha}{(k-1)!}} C\exp\Big(\frac{\alpha}{(k-1)!}\int_0^r (1-t)^{-1}\mathrm{d}t\Big)\\ \leq & C(1-|z|^2)^{\frac{\alpha}{(k-1)!}}\big(\frac{1}{1-r}\big)^{\frac{\alpha}{(k-1)!}} \leq C2^{\frac{\alpha}{(k-1)!}}.\end{aligned}$$

Therefore $f \in H^\infty_{\frac{\alpha}{(k-1)!}}$.

(3) If $k < \beta < \infty$, it follows from (3.10) that

$$\begin{aligned}|f(z)| \leq & C\exp\Big(\frac{\alpha}{(k-1)!}\int_0^r (1-t)^{k-\beta-1}\mathrm{d}t\Big)\\ = & C\exp\big\{\frac{\alpha}{(k-1)!}\big(\frac{(1-r)^{k-\beta}}{\beta-k}-\frac{1}{\beta-k}\big)\big\}\\ = & C\exp\big(\frac{-\alpha}{(k-1)!(\beta-k)}\big)\exp\big(\frac{1}{(k-1)!(\beta-k)}\big(\frac{\alpha}{(1-r)^{\beta-k}}\big)\big)\\ < & \exp\big(\frac{\alpha_1}{(1-r)^{\beta-k}}\big),\end{aligned}$$

where α_1 is some positive constant, so $f \in \varepsilon^{\beta-k}$. Now we complete the proof of Theorem 1.10. □

Acknowledgements The authors would like to thank the referees for helpful suggestions to improve this paper.

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