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The Nearest Complex Polynomial with a Prescribed Zero

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Abstract Nearest polynomial with given properties has many applications in control theory and applied mathematics. Given a complex univariate polynomial $f(z)$ and a zero α, in this paper we explore the problem of computing a complex polynomial $\widetilde{f}(z)$ such that $\widetilde{f}(\alpha)=0$ and the distance $\|\widetilde{f}-f\|$ is minimal. Considering most of the existing works focus on either certain polynomial basis or certain vector norm, we propose a common computation framework based on both general polynomial basis and general vector norm, and summarize the computing process into a four-step algorithm. Further, to find the explicit expression of $\widetilde{f}(z)$, we focus on two specific norms which generalize the familiar ℓ_p-norm and mixed norm studied in the existing works, and then compute $\widetilde{f}(z)$ explicitly based on the proposed algorithm. We finally give a numerical example to show the effectiveness of our method.

Keywords nearest polynomial; explicit expression; zero; dual norm

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1. Introduction

Let $\mathbb{C}[z]$ be the polynomial ring in z over $\mathbb{C}$ and $\mathcal{E}:=\{e_j(z)\mid j=1,\ldots,n\}$ an arbitrary polynomial basis in $\mathbb{C}[z]$. Let $f(z)$ be a given polynomial. Set

$$\Xi:=\mathrm{span}\{e_1(z),\ldots,e_n(z)\}=\Big\{\sum_{j=1}^{n}c_je_j(z)\mid c_j\in\mathbb{C},j=1,\ldots,n\Big\}$$

and

$$\Im:=f(z)+\Xi=\Big\{f(z)+\sum_{j=1}^{n}c_je_j(z)\mid c_j\in\mathbb{C},j=1,\ldots,n\Big\}.$$

The aim of this paper is to study the problem of finding a univariate complex polynomial $\widetilde{f}(z)\in\Im$ such that it has a prescribed zero $\alpha\in\mathbb{C}$ and the distance between $\widetilde{f}(z)$ and $f(z)$ is minimal.

Let $\Lambda(\alpha):=\{g\in\Im\mid g(\alpha)=0\}$, i.e., the set of all elements of $\Im$ that have a zero α. The problem can be equivalently formulated as follows:

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Problem 1.1 Find $\widetilde{f} \in \Lambda(\alpha)$ such that

$$\|\widetilde{f} - f\| = \min_{g \in \Lambda(\alpha)} \|g - f\|, \tag{1}$$

where the norm $\|\cdot\|$ of a polynomial is defined as the norm of coefficient vector of the polynomial with respect to $\mathcal{E}$. Assume that $f(\alpha) \notin \Lambda(\alpha)$, otherwise choose $\widetilde{f}(z) = f(z)$.

The sensitivity of the zeros with respect to the uncertainties of the polynomial coefficients has been studied widely; cf. e.g., [1–3]. Ostrowski [1] introduced a continuous sensitivity analysis to study the uncertainties of the coefficients as a continuity problem. Such a direction was developed significantly by Mosier [2], who introduced the notion of a "root neighborhood" or "pseudozero set" of a polynomial, which means the set of all zeros of polynomials that are near to a given polynomial. As noticed in [4], a nearest polynomial with a given zero is needed in computing a pseudozero set. Besides its relationship to the pseudozero set or to the approximate GCD problem, the computation of nearest polynomials with given properties has applications in control theory [5–7]. This motivates our study in the problem of a nearest polynomial.

By choosing $e_j = z^{j-1}$ $(j = 1, 2, \ldots, n-1)$, the monomial basis, Hitz and Kaltofen [8–10] studied Problem 1.1 in case of (weighted) ℓ_2-norm and ℓ_∞-norm. They converted it into a parameterized least squares problem for ℓ_2-norm, and a linear programming problem for ℓ_∞-norm. Stetter [11] extended these results to ℓ_p-norms for all values of p based on dual norm theory. Further, Graillat [12] gave explicit formulas for $\widetilde{f}(z)$ in case of any ℓ_p-norms. Problem 1.1 was also studied in some other polynomial bases. For example, Rezvani and Corless [13,14] chose Lagrangian, Chebyshev, Bernstein and Hermite bases and so forth.

If the point α is replaced with a complex domain, Problem 1.1 becomes a problem of computing the nearest complex polynomial with a zero in a given domain. For such a problem, Qiu and Davison [5] addressed it by computing structured singular values for a special class of rank-one problems [6,7]. Hitz and Kaltofen [8,9] proposed a symbolic-numeric approach for finding a nearest polynomial in weighted ℓ_2-norm. Further, Luo et al. [15] studied the nearest complex polynomial in ℓ_p-norm and mixed norm, while a similar problem for the real case in weighted ℓ_∞-norm was researched in [16,17].

Motivated by the fact that previous studies mainly focus on either ℓ_p-norms and mixed norm or certain specific polynomial basis, this paper tries to unify these results, and proposes a common computation framework for Problem 1.1. We will propose a four-step algorithm for computing $\widetilde{f}(z)$ in both general polynomial basis $\mathcal{E}$ and general vector norm $\|\cdot\|$. Meanwhile, since explicit expressions of $\widetilde{f}(z)$ heavily depend on the dual norm $\|\cdot\|^*$ and not all the norms would have concrete forms for their dual norms, we also study the computation of $\widetilde{f}(z)$ when focusing on two specific norms, i.e., the generalized weighted norm $\|\cdot\|_{p,\mathbf{w}}$ and the generalized mixed norm $\|\cdot\|_{p,p_1,p_2}$.

Apart from the nearest polynomial computation that will be studied in this paper, these two norms have many other applications, such as in (block) sparse signal recovery and in function interpolation [18–20]. Interestingly, we also observe that the second norm can be extended to the corresponding matrix norm, where $\|\cdot\|_{1,2,2}$ was popular in joint feature selection and subspace learning [21–23]. Like [15], the results obtained in this paper can be extended to computing the nearest polynomial with a zero in a given domain, but this will not be simply repeated since the key idea is similar.

The rest of this paper is organized as follows. Section 2 describes some useful notations, and then a common computation framework for Problem 1.1 is proposed in Section 3. Section 4 shows the detailed computing process for explicit solutions. A numerical example is given in Section 5, and finally Section 6 concludes the paper.

2. Notations

As in [15], some useful notations are listed:

(i) Let $\mathbb{C}^n$ denote the n-dimensional complex vector space. Vectors in $\mathbb{C}^n$ will mean column vectors which are typed in bold, e.g.,

$$\mathbf{u} := (u_1, u_2, \ldots, u_n)^T \in \mathbb{C}^n, \quad \mathbf{v} := (v_1, v_2, \ldots, v_n)^T \in \mathbb{C}^n.$$

Let

$$\mathbf{\Phi}(z) := (e_1(z), e_2(z), \ldots, e_n(z))^T \in \mathbb{C}^n \text{ for every } z \in \mathbb{C}. \tag{2}$$

(ii) (Scalar product) The ordinary scalar product on $\mathbb{C}^n$ is denoted by

$$\langle \mathbf{u}, \mathbf{v} \rangle := \overline{\mathbf{v}}^T \mathbf{u} = \sum_{j=1}^{n} u_j \overline{v_j}, \quad \mathbf{u}, \mathbf{v} \in \mathbb{C}^n,$$

where $\overline{v_j}$ means the conjugate number of v_j.

(iii) (ℓ_p-norm $\|\cdot\|_p$) The ℓ_p-norm of the vector $\mathbf{u}$ is defined by:

$$\|\mathbf{u}\|_p := \begin{cases} (\sum_{j=1}^{n} |u_j|^p))^{\frac{1}{p}}, & \text{if } 1 \le p < \infty, \\ \max_{j=1,\ldots,n} |u_j|, & \text{if } \ p = \infty. \end{cases}$$

(iv) (Weighted ℓ_p-norm $\|\cdot\|_{p,\omega}$) Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \in \mathbb{R}_+^n$ be the weight vector, where $\mathbb{R}_+$ denotes the set of all positive real numbers. Then, define

$$\|\mathbf{u}\|_{p,\omega} := \begin{cases} (\sum_{j=1}^{n} |\omega_j u_j|^p))^{\frac{1}{p}}, & \text{if } 1 \le p < \infty, \\ \max_{j=1,\ldots,n} |\omega_j u_j|, & \text{if } \ p = \infty. \end{cases}$$

(v) (Mixed norm $\|\cdot\|_{\text{mix}}$) Let $1 \le r \le n-1$ and $1 \le p_1, p_2 \le \infty$ be given. Then, define

$$\|\mathbf{u}\|_{\text{mix}} := \max\{\|\mathbf{u}^{(1)}\|_{p_1}, \|\mathbf{u}^{(2)}\|_{p_2}\},$$

where $\mathbf{u} = (\mathbf{u}^{(1)^T}, \mathbf{u}^{(2)^T})^T$, $\mathbf{u}^{(1)} = (u_1, \ldots, u_r)^T$ and $\mathbf{u}^{(2)} = (u_{r+1}, \ldots, u_n)^T$.

(vi) (Dual norm $\|\cdot\|^*$) The dual norm $\|\cdot\|^*$ associated to any norm $\|\cdot\|$ is defined by

$$\|\mathbf{u}\|^* := \sup_{\mathbf{v} \ne 0} \frac{|\langle \mathbf{u}, \mathbf{v} \rangle|}{\|\mathbf{v}\|} = \sup_{\|\mathbf{v}\|=1} |\langle \mathbf{u}, \mathbf{v} \rangle|. \tag{3}$$

Conversely, the dual formulation of Eq. (3) is

$$\|\mathbf{v}\| = \sup_{\mathbf{u}\neq 0} \frac{|\langle \mathbf{v}, \mathbf{u}\rangle|}{\|\mathbf{u}\|^*} = \sup_{\|\mathbf{u}\|^*=1} |\langle \mathbf{v}, \mathbf{u}\rangle|. \tag{4}$$

It is well-known that [14,15]

$$\begin{cases} \|\mathbf{u}\|_p^* = \|\mathbf{u}\|_q, & 1 \le p, q \le \infty, \dfrac{1}{p} + \dfrac{1}{q} = 1; \\ \|\mathbf{u}\|_{p,\omega}^* = \|\mathbf{u}\|_{q,\omega^*}, & 1 \le p, q \le \infty, \dfrac{1}{p} + \dfrac{1}{q} = 1, \omega^* = (\omega_1^{-1}, \omega_2^{-1}, \ldots, \omega_n^{-1})^T; \\ \|\mathbf{u}\|_{\mathrm{mix}}^* = \|\mathbf{u}^{(1)}\|_{q_1} + \|\mathbf{u}^{(2)}\|_{q_2}, & 1 \le q_j \le \infty, \dfrac{1}{p_j} + \dfrac{1}{q_j} = 1, j = 1, 2 \end{cases}$$

with the convention that $q = \infty$ if $p = 1$ and $q = 1$ if $p = \infty$ which is used throughout this paper.

3. General solutions for Problem 1.1

In this section, we propose a common framework to compute the nearest complex polynomial in both general norm and general basis, and give the basic forms for nearest polynomial $\widetilde{f}(z)$ and minimal distance $\|\widetilde{f} - f\|$.

We start our discussion with the following lemma:

Lemma 3.1 *Let $\gamma \in \mathbb{C}$, $|\gamma| = 1$ and $\mathbf{u} \in \mathbb{C}^n$, $\|\mathbf{u}\|^* = 1$. Then, there exists at least one vector $\mathbf{v} \in \mathbb{C}^n$, $\|\mathbf{v}\| = 1$, such that $\langle \mathbf{u}, \mathbf{v}\rangle = \overline{\mathbf{v}}^T\mathbf{u} = \gamma$.*

Proof Since $\{\mathbf{v} \in \mathbb{C}^n \mid \|\mathbf{v}\| = 1\}$ is a compact domain, the sup in Eq. (3) can be attained. That is to say, there exists at least one vector $\mathbf{v}_0 \in \mathbb{C}^n$, $\|\mathbf{v}_0\| = 1$, such that $|\langle \mathbf{u}, \mathbf{v}_0\rangle| = \|\mathbf{u}\|^* = 1$.

Suppose $\langle \mathbf{u}, \mathbf{v}_0\rangle = \gamma_0$ where $\gamma_0 \in \mathbb{C}$, $|\gamma_0| = 1$. Taking $\mathbf{v} = \overline{\gamma}\gamma_0\mathbf{v}_0$, we obtain $\|\mathbf{v}\| = 1$, $\langle \mathbf{u}, \mathbf{v}\rangle = \gamma$. □

If we particularly take $\|\cdot\| = \|\cdot\|_p$, namely $\|\cdot\|^* = \|\cdot\|_q$ with $\frac{1}{p} + \frac{1}{q} = 1$, Lemma 3.1 has a more detailed version.

Lemma 3.2 ([13,15]) *Let $\frac{1}{p} + \frac{1}{q} = 1$ with $1 \le p, q \le \infty$. Suppose that $\mathbf{u} \in \mathbb{C}^n$ and $\gamma \in \mathbb{C}$ are given such that $\|\mathbf{u}\|_q = 1$ and $|\gamma| = 1$. Define $\mathbf{v} \in \mathbb{C}^n$ as follows:*

$$v_j = \begin{cases} \overline{\gamma}|u_j|^{q-2}u_j, & \text{if } u_j \neq 0, \\ 0, & \text{if } u_j = 0, \end{cases} \quad \text{for } 1 \le q < \infty, \tag{5}$$

$$v_j = \begin{cases} \overline{\gamma}u_{j_0}, & \text{if } j = j_0, \\ 0, & \text{if } j \neq j_0, \end{cases} \quad \text{for } q = \infty, \tag{6}$$

where j_0 is any, say the least, index with $|u_{j_0}| = 1$. Then, we have

$$\langle \mathbf{u}, \mathbf{v}\rangle = \gamma \quad \textit{with} \quad \|\mathbf{v}\|_p = 1.$$

In Problem 1.1, we observe that

$$g(z) - f(z) = \sum_{j=1}^{n} c_j e_j(z) = \mathbf{c}^T\mathbf{\Phi}(z),$$

where $\mathbf{c} = (c_1, c_2, \ldots, c_n)^T \in \mathbb{C}^n$. Since $\|g - f\|$ is defined as the norm of coefficient vector of $g(z) - f(z)$ with respect to $\{e_1(z), e_2(z), \ldots, e_n(z)\}$, we have $\|g - f\| = \|\mathbf{c}\|$. Thus, to find $\widetilde{f}(z)$, we need to compute the minimum value

$$\min_{f(\alpha)+\mathbf{c}^T\mathbf{\Phi}(\alpha)=0} \|\mathbf{c}\| \tag{7}$$

and a vector $\mathbf{c}$ where it is attained.

Let

$$\mathbf{u} := \tau\mathbf{\Phi}(\alpha) = \mu(e_1(\alpha), e_2(\alpha), \ldots, e_n(\alpha))^T$$

with $\tau = \frac{1}{\|\mathbf{\Phi}(\alpha)\|^*}$, so $\|\mathbf{u}\|^* = 1$. Recalling that $f(\alpha) \notin \Lambda(\alpha)$, we set

$$\gamma = -\frac{f(\alpha)}{|f(\alpha)|}.$$

Then, Lemma 3.1 implies that there exists a vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T \in \mathbb{C}^n$ satisfying

$$\|\mathbf{v}\| = 1 \quad \text{and} \quad \langle \mathbf{u}, \mathbf{v} \rangle = \overline{\mathbf{v}}^T\mathbf{u} = \gamma.$$

Take $\mathbf{c} = \tau|f(\alpha)|\overline{\mathbf{v}}$ and define

$$\widetilde{f}(z) := f(z) + \mathbf{c}^T\mathbf{\Phi}(z) = f(z) + \sum_{j=1}^{n} c_j e_j(z). \tag{8}$$

The next theorem proves that $\widetilde{f}(z)$ is actually a solution to Problem 1.1.

Theorem 3.3 *Let $\|\cdot\|$ and $\{e_1(z), e_2(z), \ldots, e_n(z)\}$ be arbitrary norm of $\mathbb{C}^n$ and polynomial basis in $\mathbb{C}[z]$, respectively. Then, the polynomial $\widetilde{f}(z)$ defined by Eq. (8) solves Problem 1.1 with minimal distance $\frac{|f(\alpha)|}{\|\mathbf{\Phi}(\alpha)\|^*}$.*

Proof From the definitions of $\mathbf{u}$, γ, $\mathbf{v}$ and $\mathbf{c}$, it follows that $\mathbf{c}^T\mathbf{\Phi}(\alpha) = \mu|f(\alpha)|\overline{\mathbf{v}}^T\mathbf{\Phi}(\alpha) = |f(\alpha)|\overline{\mathbf{v}}^T\mathbf{u} = |f(\alpha)|\gamma = -f(\alpha)$, which implies $\widetilde{f}(\alpha) = 0$ and $\|\widetilde{f} - f\| = \|\mathbf{c}\| = \frac{|f(\alpha)|}{\|\mathbf{\Phi}(\alpha)\|^*}$.

It remains to prove that $\|\widetilde{f} - f\| \le \|g - f\|$ for all $g \in \Lambda(\alpha)$. Write an arbitrary $g \in \Lambda(\alpha)$ as follows:

$$g(z) = f(z) + \sum_{j=1}^{n} r_j e_j(z)$$

with $\mathbf{r} = (r_1, r_2, \ldots, r_n)^T$. Then, since $g(\alpha) = 0$, an application of Hölder's inequality yields

$$|f(\alpha)| = |g(\alpha) - f(\alpha)| = \Big|\sum_{j=1}^{n} r_j e_j(\alpha)\Big| = |\mathbf{r}^T\mathbf{\Phi}(\alpha)| \le \|\mathbf{r}\|\,\|\mathbf{\Phi}(\alpha)\|^*,$$

from which it follows that

$$\|\widetilde{f} - f\| = \frac{|f(\alpha)|}{\|\mathbf{\Phi}(\alpha)\|^*} \le \|\mathbf{r}\| = \|g - f\|. \quad \square$$

Denote the minimal distance by

$$\widetilde{d} := \mu|f(\alpha)| = \frac{|f(\alpha)|}{\|\mathbf{\Phi}(\alpha)\|^*}. \tag{9}$$

We summarize the above computation process into the following algorithm.

Algorithm 3.4 (Computation of the nearest polynomial)

Input: $f(z)$, α, $\mathbf{\Phi}(z)$ and a certain norm $\|\cdot\|$.

Output: A nearest polynomial $\widetilde{f}$ and its minimal distance $\widetilde{d}$ from f.

Step 1. Compute $\mathbf{u}=\tau\mathbf{\Phi}(\alpha)$ with $\|\mathbf{u}\|^*=1$, where $\tau=\frac{1}{\|\mathbf{\Phi}(\alpha)\|^*}$.

Step 2. Compute $\mathbf{v}$ with $\|\mathbf{v}\|=1$ and $\langle\mathbf{u},\mathbf{v}\rangle=\gamma$, where $\gamma=-\frac{f(\alpha)}{|f(\alpha)|}$.

Step 3. Compute $\mathbf{c}=\tau|f(\alpha)|\overline{\mathbf{v}}$.

Step 4. Return $\widetilde{f}(z)=f(z)+\mathbf{c}^T\mathbf{\Phi}(z)$ and $\widetilde{d}=\tau|f(\alpha)|$.

Algorithm 3.4 is valid to any norm, where computations of nearest polynomial $\widetilde{f}(z)$ and minimal distance $\widetilde{d}$ come down to computing the desired vector $\mathbf{v}$ and the dual norm of $\mathbf{\Phi}(\alpha)$. However, both of $\mathbf{v}$ and $\|\mathbf{\Phi}(\alpha)\|^*$ may not have explicit forms for certain norms. For this reason, in the next section we shall focus on two specific norms that are not studied in previous works and in which we can obtain explicit $\mathbf{v}$ and $\|\mathbf{\Phi}(\alpha)\|^*$.

4. Explicit solutions for Problem 1.1

We first give the definitions of two norms to be used.

4.1. Norm definitions and dual norms

Motivated by (weighted) ℓ_p-norm and mixed norm introduced in Section 2, we define two more general norms.

Definition 4.1 *Let $\mathbf{W}=(w_{ij})\in\mathbb{R}^{n\times n}$ be a nonsingular and symmetric weight matrix. For $\mathbf{u}=(u_1,u_2,\ldots,u_n)^T\in\mathbb{C}^n$ and $1\le p\le\infty$, we define*

$$\|\mathbf{u}\|_{p,\mathbf{W}}:=\|\mathbf{W}\mathbf{u}\|_p \tag{10}$$

which is called generalized weighted norm.

It is straightforward to prove that $\|\cdot\|_{p,\mathbf{W}}$ is a norm, so we omit the proof.

Definition 4.2 *Suppose $\mathbf{u}=(u_1,u_2,\ldots,u_n)^T\in\mathbb{C}^n$. Let $\mathbf{u}=({\mathbf{u}^{(1)}}^T,{\mathbf{u}^{(2)}}^T)^T$, $\mathbf{u}^{(1)}=(u_1,\ldots,u_r)^T$, $\mathbf{u}^{(2)}=(u_{r+1},\ldots,u_n)^T$ and $1\le r\le n-1$. For $1\le p,p_1,p_2\le\infty$, we define*

$$\|\mathbf{u}\|_{p,\ p_1,\ p_2}:=\|\left(\|\mathbf{u}^{(1)}\|_{p_1},\ \|\mathbf{u}^{(2)}\|_{p_2}\right)\|_p=\left(\|\mathbf{u}^{(1)}\|_{p_1}^p+\|\mathbf{u}^{(2)}\|_{p_2}^p\right)^{\frac{1}{p}} \tag{11}$$

which is called generalized mixed norm.

The positivity and homogeneity of $\|\cdot\|_{p,\ p_1,\ p_2}$ follow directly from its definition. We just prove the correctness of triangle inequality by applying the Minkowski inequality.

For any $\mathbf{u},\mathbf{v}\in\mathbb{C}^n$, $\mathbf{u}=({\mathbf{u}^{(1)}}^T,\ {\mathbf{u}^{(2)}}^T)^T$, $\mathbf{v}=({\mathbf{v}^{(1)}}^T,\ {\mathbf{v}^{(2)}}^T)^T$, we have

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|_{p,\ p_1,\ p_2}&=\left\|\left(\mathbf{u}^{(1)}+\mathbf{v}^{(1)},\ \mathbf{u}^{(2)}+\mathbf{v}^{(2)}\right)\right\|_{p,\ p_1,\ p_2}\\&=\left(\|\mathbf{u}^{(1)}+\mathbf{v}^{(1)}\|_{p_1}^p+\|\mathbf{u}^{(2)}+\mathbf{v}^{(2)}\|_{p_2}^p\right)^{\frac{1}{p}}\\&\le\left((\|\mathbf{u}^{(1)}\|_{p_1}+\|\mathbf{v}^{(1)}\|_{p_1})^p+(\|\mathbf{u}^{(2)}\|_{p_2}+\|\mathbf{v}^{(2)}\|_{p_2})^p\right)^{\frac{1}{p}}\end{aligned}$$

$$
\begin{aligned}
&= \| \left(\|\mathbf{u}^{(1)}\|_{p_1} + \|\mathbf{v}^{(1)}\|_{p_1},\ \|\mathbf{u}^{(2)}\|_{p_2} + \|\mathbf{v}^{(2)}\|_{p_2} \right) \|_p \\
&= \| \left(\|\mathbf{u}^{(1)}\|_{p_1},\ \|\mathbf{u}^{(2)}\|_{p_2} \right) + \left(\|\mathbf{v}^{(1)}\|_{p_1},\ \|\mathbf{v}^{(2)}\|_{p_2} \right) \|_p \\
&\le \| \left(\|\mathbf{u}^{(1)}\|_{p_1},\ \|\mathbf{u}^{(2)}\|_{p_2} \right) \|_p + \| \left(\|\mathbf{v}^{(1)}\|_{p_1},\ \|\mathbf{v}^{(2)}\|_{p_2} \right) \|_p \\
&= \|\mathbf{u}\|_{p,\ p_1,\ p_2} + \|\mathbf{v}\|_{p,\ p_1,\ p_2}\ .
\end{aligned}
$$

According to Algorithm 3.4, dual norm is critical to compute $\widetilde{f}(z)$ and $\widetilde{d}$. Thus, the next theorems reveal the dual norms of $\|\cdot\|_{p,\mathbf{W}}$ and $\|\cdot\|_{p,p_1,p_2}$, respectively.

Theorem 4.3 *Let $\mathbf{u}, p, \mathbf{W}$ be as in Definition 4.1. Then*

$$\|\mathbf{u}\|_{p,\mathbf{W}}^* = \|\mathbf{u}\|_{q,\mathbf{W}^{-1}} \tag{12}$$

where $\mathbf{W}^{-1}$ denotes the inverse of $\mathbf{W}$, q satisfies $1 \le q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof See Appendix A. □

Theorem 4.4 *Let $\mathbf{u}, \mathbf{u}^{(1)}, \mathbf{u}^{(2)}, r, p, p_1, p_2$ be as in Definition 4.2. Then*

$$\|\mathbf{u}\|_{p,\ p_1,\ p_2}^* = \|\mathbf{u}\|_{q,\ q_1,\ q_2}\ , \tag{13}$$

where q, q_1, q_2 satisfy $1 \le q,\ q_1,\ q_2 \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{p_1} + \frac{1}{q_1} = 1$ and $\frac{1}{p_2} + \frac{1}{q_2} = 1$.

Proof See Appendix B. □

By default, in the following discussions we always assume that p's and q's appear in pairs.

Remark 4.5 The introduced generalized weighted norm and generalized mixed norm include many familiar norms.

For example, for any $\mathbf{u} \in \mathbb{C}^n$ and $1 \le p \le \infty$, if $\mathbf{W} = I$ (identity matrix), then $\|\mathbf{u}\|_{p,\mathbf{W}} = \|\mathbf{u}\|_p$; if $\mathbf{W} = \mathrm{diag}(\omega_1, \ldots, \omega_n)$, then $\|\mathbf{u}\|_{p,\mathbf{W}} = \|\mathbf{u}\|_{p,\omega}$; for any $1 \le p_1, p_2 \le \infty$, if $p = \infty$, then $\|\mathbf{u}\|_{\infty,p_1,p_2} = \|\mathbf{u}\|_{\mathrm{mix}}$.

Moreover, Theorems 4.3 and 4.4 imply

$$\|\mathbf{u}\|_p^* = \|\mathbf{u}\|_q, \|\mathbf{u}\|_{p,\mathbf{W}}^* = \|\mathbf{u}\|_{q,\mathbf{W}^{-1}} = \|\mathbf{u}\|_{q,\omega^*}$$

and

$$\|\mathbf{u}\|_{\infty,p_1,p_2}^* = \|\mathbf{u}\|_{1,q_1,q_2} = \|\mathbf{u}^{(1)}\|_{p_1} + \|\mathbf{u}^{(2)}\|_{p_2} = \|\mathbf{u}\|_{\mathrm{mix}}^*.$$

These results are consistent with the ones in Section 2.

4.2. Solutions in generalized weighted norm

Following the flow of Algorithm 3.4, we will find explicit solution formulas in generalized weighted norm for Problem 1.1 in the following steps:

Step 1. Compute $\mathbf{u}$:

For $p \in [1, \infty]$, let

$$\mathbf{u} = \tau \boldsymbol{\Phi}(\alpha)$$

with $\tau = \frac{1}{\|\mathbf{\Phi}(\alpha)\|_{p,\mathbf{W}}^{*}} = \frac{1}{\|\mathbf{\Phi}(\alpha)\|_{q,\mathbf{W}^{-1}}}$ and $\frac{1}{p}+\frac{1}{q}=1$. Then, we have

$$\|\mathbf{u}\|_{p,\mathbf{W}}^{*} = \|\mathbf{u}\|_{q,\mathbf{W}^{-1}} = 1.$$

Step 2. Compute $\mathbf{v}$:

Let $\mathbf{x} = \mathbf{W}^{-1}\mathbf{u} \in \mathbb{C}^n$ so that $\|\mathbf{x}\|_q = \|\mathbf{u}\|_{q,\mathbf{W}^{-1}} = 1$. Set $\gamma = -\frac{f(\alpha)}{|f(\alpha)|}$; then, by using Lemma 3.2, there exists $\mathbf{y} \in \mathbb{C}^n$, $\|\mathbf{y}\|_p = 1$ such that $\langle \mathbf{x}, \mathbf{y}\rangle = \gamma$.

Now, let

$$\mathbf{v} = \mathbf{W}^{-1}\mathbf{y} \tag{14}$$

which satisfies

$$\begin{cases} \|\mathbf{v}\|_{p,\mathbf{W}} = \|\mathbf{W}\mathbf{W}^{-1}\mathbf{y}\|_p = \|\mathbf{y}\|_p = 1, \\ \langle \mathbf{u},\ \mathbf{v}\rangle = \left\langle \mathbf{W}\mathbf{x}, \mathbf{W}^{-1}\mathbf{y}\right\rangle = \langle \mathbf{x}, \mathbf{y}\rangle = \gamma. \end{cases}$$

Step 3. Compute $\mathbf{c} = \tau|f(\alpha)|\overline{\mathbf{v}}$:

Owing to Lemma 3.2, we can deduce each component of $\mathbf{c}$ in details. First, denote the inverse of $\mathbf{W}$ by $\mathbf{W}^{-1} = (\phi_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$. Due to the symmetry of $\mathbf{W}$, $\mathbf{W}^{-1}$ is also symmetrical, namely, $\phi_{ij} = \phi_{ji}$, $i, j = 1, \ldots, n$. Next, we divide $p, q \in [1, \infty]$ that satisfy $\frac{1}{p}+\frac{1}{q}=1$ into two cases to discuss:

Case a $1 < p \le \infty$, $1 \le q < \infty$ and $\frac{1}{p}+\frac{1}{q}=1$.

In this case, τ and the j-th component of $\mathbf{x}$ are

$$\begin{cases} \tau = \dfrac{1}{\|\mathbf{\Phi}(\alpha)\|_{q,\mathbf{W}^{-1}}} = \dfrac{1}{\left(\sum\limits_{i=1}^{n} |\sum\limits_{j=1}^{n} \phi_{ij} e_j(\alpha)|^q\right)^{\frac{1}{q}}}, \\ x_j = \tau \sum\limits_{k=1}^{n} \phi_{jk} e_k(\alpha), \quad j = 1, \ldots, n. \end{cases}$$

By Eq. (5) in Lemma 3.2, if $x_j \neq 0$, i.e., $\sum_{k=1}^{n} \phi_{jk} e_k(\alpha) \neq 0$, one has the j-th component of $\mathbf{y}$ as follows:

$$y_j = \overline{\gamma}|x_j|^{q-2}x_j = -\frac{\overline{f(\alpha)}}{|f(\alpha)|} \frac{|\sum\limits_{k=1}^{n} \phi_{jk} e_k(\alpha)|^{q-1}}{\|\mathbf{\Phi}(\alpha)\|_{q,\mathbf{W}^{-1}}^{q-1}} \frac{\sum\limits_{k=1}^{n} \phi_{jk} e_k(\alpha)}{|\sum\limits_{k=1}^{n} \phi_{jk} e_k(\alpha)|},$$

otherwise, one sets $y_j = 0$.

By using Eq. (14), we compute the i-th component of $\mathbf{v}$ that is

$$v_i = \sum_{j=1}^{n} \phi_{ij} y_j = -\frac{\overline{f(\alpha)}}{|f(\alpha)|} \frac{1}{\|\mathbf{\Phi}(\alpha)\|_{q,\mathbf{W}^{-1}}^{q-1}} \sum_{\substack{j=1 \\ \sum_{k=1}^{n} \phi_{jk} e_k(\alpha) \neq 0}}^{n} \left(\phi_{ij} |\sum_{k=1}^{n} \phi_{jk} e_k(\alpha)|^{q-1} \frac{\sum\limits_{k=1}^{n} \phi_{jk} e_k(\alpha)}{|\sum\limits_{k=1}^{n} \phi_{jk} e_k(\alpha)|} \right),$$

where $i = 1, 2, \ldots, n$. Therefore, the i-th component of $\mathbf{c}$ is

$$c_i = \tau|f(\alpha)|\overline{v_i} = -\frac{f(\alpha)}{\|\mathbf{\Phi}(\alpha)\|_{q,\mathbf{W}^{-1}}^{q}} \sum_{\substack{j=1 \\ \sum_{k=1}^{n} \phi_{jk} e_k(\alpha) \neq 0}}^{n} \left(\phi_{ij} |\sum_{k=1}^{n} \phi_{jk} e_k(\alpha)|^{q-1} \frac{\overline{\sum\limits_{k=1}^{n} \phi_{jk} e_k(\alpha)}}{|\sum\limits_{k=1}^{n} \phi_{jk} e_k(\alpha)|} \right).$$

Case b $p = 1$, $q = \infty$.

In this case, let i_0 be any index with $|\sum_{j=1}^n \phi_{i_0 j} e_j(\alpha)| = \|\mathbf{\Phi}(\alpha)\|_{\infty,\mathbf{W}^{-1}}$. Then, we have

$$\begin{cases} \tau = \dfrac{1}{\|\mathbf{\Phi}(\alpha)\|_{\infty,\mathbf{W}^{-1}}} = \dfrac{1}{|\sum\limits_{j=1}^{n} \phi_{i_0 j} e_j(\alpha)|}, \\ x_j = \tau \sum\limits_{k=1}^{n} \phi_{jk} e_k(\alpha), \quad j = 1, \dots, n. \end{cases}$$

By Eq. (6) in Lemma 3.2, we have

$$y_j = \begin{cases} \overline{\gamma} x_{i_0} = -\dfrac{\overline{f(\alpha)}}{|f(\alpha)|} \dfrac{\sum\limits_{k=1}^{n} \phi_{i_0 k} e_k(\alpha)}{|\sum\limits_{j=1}^{n} \phi_{i_0 j} e_j(\alpha)|}, & j = i_0, \\ 0, & \text{otherwise.} \end{cases}$$

By Eq. (14), one has the i-th component of $\mathbf{v}$ as follows:

$$v_i = \sum_{j=1}^{n} \phi_{ij} y_j = \phi_{i i_0} y_{i_0} = -\frac{\overline{f(\alpha)}}{|f(\alpha)|} \frac{\phi_{i i_0}}{|\sum\limits_{j=1}^{n} \phi_{i_0 j} e_j(\alpha)|} \sum_{k=1}^{n} \phi_{i_0 k} e_k(\alpha), \quad i = 1, \dots, n.$$

Therefore, due to that $\mathbf{c} = \tau |f(\alpha)| \overline{\mathbf{v}}$, the i-th component of $\mathbf{c}$ is

$$c_i = \tau |f(\alpha)| \overline{v_i} \overset{k \to j}{=} -\frac{f(\alpha)}{|\sum\limits_{j=1}^{n} \phi_{i_0 j} e_j(\alpha)|} \frac{\overline{\sum\limits_{j=1}^{n} \phi_{i_0 j} e_j(\alpha)}}{|\sum\limits_{j=1}^{n} \phi_{i_0 j} e_j(\alpha)|} \phi_{i i_0}, \quad i = 1, \dots, n.$$

Step 4. Return $\widetilde{f}(z)$ and $\widetilde{d}$:

After obtaining each component of vector $\mathbf{c}$, we have

$$\begin{cases} \widetilde{f}_{p,\mathbf{W}}(z) := f(z) + \sum\limits_{i=1}^{n} c_i e_i(z), \\ \widetilde{d}_{p,\mathbf{W}} := \dfrac{|f(\alpha)|}{\|\mathbf{\Phi}(\alpha)\|_{q,\mathbf{W}^{-1}}}. \end{cases} \tag{15}$$

Remark 4.6 Note that computing $\mathbf{c}_i$ employs the fact of $\phi_{ij} = \phi_{ji}$. In addition, for different values of p, if we take the weight matrix $\mathbf{W}$ as $\mathbf{I}$ or $\mathrm{diag}(\omega_1, \omega_2, \dots, \omega_n)$, the results implied by Eq. (15) are consistent with the ones in previous works. For example, for $1 < p \le \infty$, take $\mathbf{W} = \mathbf{I}$; then $\phi_{ij} = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta, $i, j = 1, 2, \dots, n$. Consequently, we have

$$\begin{cases} \widetilde{f}_{p,\mathbf{I}}(z) = f(z) - \dfrac{f(\alpha)}{\sum_{k=1}^{n} |e_k(\alpha)|^q} \sum\limits_{\substack{j=1 \\ e_j(\alpha) \neq 0}}^{n} |e_j(\alpha)|^{q-1} \dfrac{\overline{e_j(\alpha)}}{|e_j(\alpha)|} e_j(z), \\ \widetilde{d}_{p,\mathbf{I}} = \dfrac{|f(\alpha)|}{(\sum\limits_{i=1}^{n} |e_i(\alpha)|^q)^{\frac{1}{q}}} \end{cases} \tag{16}$$

which are the same as Eqs. (14) and (15) in [15].

4.3. Solutions in generalized mixed norm

Following the flow of Algorithm 3.4, now we are going to find explicit solution formulas in generalized mixed norm for Problem 1.1 in the following steps:

Step 1. Compute $\mathbf{u}$:

For $p, p_1, p_2 \in [1,\infty]$, denote

$$\begin{cases} \mathbf{\Phi}^{(1)}(\alpha) = (e_1(\alpha),\dots,e_r(\alpha))^T, & \mathbf{u}^{(1)} = \mu^{(1)}\mathbf{\Phi}^{(1)}(\alpha), \quad \mu^{(1)} = \dfrac{1}{\|\mathbf{\Phi}^{(1)}(\alpha)\|_{q_1}}; \\ \mathbf{\Phi}^{(2)}(\alpha) = (e_{r+1}(\alpha),\dots,e_n(\alpha))^T, & \mathbf{u}^{(2)} = \mu^{(2)}\mathbf{\Phi}^{(2)}(\alpha), \quad \mu^{(2)} = \dfrac{1}{\|\mathbf{\Phi}^{(2)}(\alpha)\|_{q_2}}. \end{cases}$$

Then, we have $\|\mathbf{u}^{(j)}\|_{q_j} = 1,\ j = 1,2,\ \mathbf{\Phi}(\alpha) = (\mathbf{\Phi}^{(1)}(\alpha)^T,\ \mathbf{\Phi}^{(2)}(\alpha)^T)^T$ and

$$\|\mathbf{\Phi}(\alpha)\|_{q,q_1,q_2} = \Big(\|\mathbf{\Phi}^{(1)}(\alpha)\|_{q_1}^q + \|\mathbf{\Phi}^{(2)}(\alpha)\|_{q_2}^q\Big)^{\frac{1}{q}} = \frac{\|(\mu^{(1)},\mu^{(2)})^T\|_q}{\mu^{(1)}\mu^{(2)}}.$$

Denote $\mu = \|(\mu^{(1)},\ \mu^{(2)})^T\|_q$, so that $\|\mathbf{\Phi}(\alpha)\|_{q,q_1,q_2} = \frac{1}{\tau} = \frac{\mu}{\mu^{(1)}\mu^{(2)}}$. Let

$$\mathbf{u} = \frac{\mathbf{\Phi}(\alpha)}{\|\mathbf{\Phi}(\alpha)\|_{q,q_1,q_2}} = \big(\frac{\mu^{(2)}}{\mu}\mathbf{u}^{(1)T}, \frac{\mu^{(1)}}{\mu}\mathbf{u}^{(2)T}\big)^T. \tag{17}$$

Then, we have $\|\mathbf{u}\|^*_{p,p_1,p_2} = \|\mathbf{u}\|_{q,q_1,q_2} = 1$.

Step 2. Compute $\mathbf{v}$:

Let $\mathbf{x} = (x_1,x_2) = (\frac{\mu^{(2)}}{\mu}, \frac{\mu^{(1)}}{\mu})^T \in \mathbb{R}^2$. Then from the definition of μ, we have $\|\mathbf{x}\|_q = 1$. So for $\gamma_0 = 1$, by Lemma 3.2 there exists $\mathbf{y} = (y_1,y_2) \in \mathbb{R}^2$, $\|\mathbf{y}\|_p = 1$ such that

$$\langle \mathbf{x},\mathbf{y}\rangle = x_1y_1 + x_2y_2 = \gamma_0 = 1,$$

where $x_1,x_2 \ge 0$ implies $y_1,y_2 \ge 0$ by Eqs. (5) and (6).

Let $\gamma = -\frac{f(\alpha)}{|f(\alpha)|}$. Since $\|\mathbf{u}^{(1)}\|_{q_1} = 1$ and $\|\mathbf{u}^{(2)}\|_{q_2} = 1$, by Lemma 3.2 again there exist $\mathbf{v}^{(1)} = (v_1,\dots,v_r)^T \in \mathbb{C}^r$, $\|\mathbf{v}^{(1)}\|_{p_1} = 1$ and $\mathbf{v}^{(2)} = (v_{r+1},\dots,v_n)^T \in \mathbb{C}^{n-r}$, $\|\mathbf{v}^{(2)}\|_{p_2} = 1$ satisfying

$$\big\langle \mathbf{u}^{(1)},\ \mathbf{v}^{(1)}\big\rangle = \gamma, \quad \big\langle \mathbf{u}^{(2)},\ \mathbf{v}^{(2)}\big\rangle = \gamma.$$

Now, we let

$$\mathbf{v} = \big(y_1\mathbf{v}^{(1)T},\ y_2\mathbf{v}^{(2)T}\big)^T \tag{18}$$

which is the exact vector we are looking for, because $\mathbf{v}$ satisfies

$$\begin{cases} \|\mathbf{v}\|_{p,p_1,p_2} = \|\mathbf{y}\|_p = 1, \\ \langle \mathbf{u},\ \mathbf{v}\rangle = \dfrac{\mu^{(2)}}{\mu}y_1\big\langle \mathbf{u}^{(1)},\mathbf{v}^{(1)}\big\rangle + \dfrac{\mu^{(1)}}{\mu}y_2\big\langle \mathbf{u}^{(2)},\mathbf{v}^{(2)}\big\rangle = \gamma(x_1y_1 + x_2y_2) = \gamma. \end{cases}$$

Step 3. Compute $\mathbf{c} = \tau|f(\alpha)|\overline{\mathbf{v}}$:

The main purpose of this step is to compute each component of $\mathbf{c}$ explicitly by applying Lemma 3.2. For simplicity, we divide $p,q \in [1,\infty]$ into two cases which are further subdivided into several subcases according to the values of p_1,p_2,q_1,q_2 to discuss.

Case a $1 < p \le \infty$, $1 \le q < \infty$.

In this case, we have $\mathbf{y} = (y_1, y_2)^T = ((\frac{\mu^{(2)}}{\mu})^{q-1}, (\frac{\mu^{(1)}}{\mu})^{q-1})^T$ by Eq. (5). Now we shall compute $\mathbf{v}$ by using Eq. (18) and the relationship between $\mathbf{u}^{(j)}$ and $\mathbf{v}^{(j)}$.

Case a$_1$ $1 < p_j \le \infty$, $1 \le q_j < \infty$, $\frac{1}{p_j} + \frac{1}{q_j} = 1$, $j = 1, 2$.

If $e_j(\alpha) \neq 0$, then, by Eqs. (5) and (18) one has the j-th component of $\mathbf{v}$ as follows:

$$v_j = \begin{cases} \overline{\gamma} y_1 |u_j^{(1)}|^{q_1-2} u_j^{(1)} = -\dfrac{\overline{f(\alpha)}}{|f(\alpha)|} \dfrac{{\mu^{(2)}}^{q-1}}{\mu^{q-1}} \dfrac{|e_j(\alpha)|^{q_1-1}}{\|\mathbf{\Phi}^{(1)}(\alpha)\|_{q_1}^{q_1-1}} \dfrac{e_j(\alpha)}{|e_j(\alpha)|}, & j = 1, \dots, r, \\ \overline{\gamma} y_2 |u_j^{(2)}|^{q_2-2} u_j^{(2)} = -\dfrac{\overline{f(\alpha)}}{|f(\alpha)|} \dfrac{{\mu^{(1)}}^{q-1}}{\mu^{q-1}} \dfrac{|e_j(\alpha)|^{q_2-1}}{\|\mathbf{\Phi}^{(2)}(\alpha)\|_{q_2}^{q_2-1}} \dfrac{e_j(\alpha)}{|e_j(\alpha)|}, & j = r+1, \dots, n. \end{cases}$$

Otherwise, one has $v_j = 0$.

Due to $\mathbf{c} = \frac{\mu^{(1)}\mu^{(2)}}{\mu} |f(\alpha)| \overline{\mathbf{v}}$ and the definition of $\mu^{(j)}$, one has the j-th component of $\mathbf{c}$ as follows:

$$c_j = \begin{cases} -f(\alpha)(\dfrac{\mu^{(1)}\mu^{(2)}}{\mu})^q \dfrac{|e_j(\alpha)|^{q_1-1}}{\|\mathbf{\Phi}^{(1)}(\alpha)\|_{q_1}^{q_1-q}} \dfrac{\overline{e_j(\alpha)}}{|e_j(\alpha)|}, & j = 1, \dots, r, \\ -f(\alpha)(\dfrac{\mu^{(1)}\mu^{(2)}}{\mu})^q \dfrac{|e_j(\alpha)|^{q_2-1}}{\|\mathbf{\Phi}^{(2)}(\alpha)\|_{q_2}^{q_2-q}} \dfrac{\overline{e_j(\alpha)}}{|e_j(\alpha)|}, & j = r+1, \dots, n \end{cases} \tag{19}$$

where $\frac{\mu^{(1)}\mu^{(2)}}{\mu} = \frac{1}{\|\mathbf{\Phi}(\alpha)\|_{q,q_1,q_2}} = \frac{|f(\alpha)|}{\left[\left(\sum_{k=1}^{r} |e_k(\alpha)|^{q_1}\right)^{\frac{q}{q_1}} + \left(\sum_{k=r+1}^{n} |e_k(\alpha)|^{q_2}\right)^{\frac{q}{q_2}}\right]^{\frac{1}{q}}}$.

Case a$_2$ $p_1 = 1, q_1 = \infty, 1 < p_2 \le \infty, 1 \le q_2 < \infty, \frac{1}{p_2} + \frac{1}{q_2} = 1$.

Let j_0 $(1 \le j_0 \le r)$ be any index with $|e_{j_0}(\alpha)| = \|\mathbf{\Phi}^{(1)}(\alpha)\|_\infty$. Then we have

$$\mu^{(j)} = \frac{1}{\|\mathbf{\Phi}^{(j)}(\alpha)\|_{q_j}} = \begin{cases} \dfrac{1}{|e_{j_0}(\alpha)|}, & j = 1, \\ \dfrac{1}{\left(\sum_{k=r+1}^{n} |e_k(\alpha)|^{q_2}\right)^{\frac{1}{q_2}}}, & j = 2. \end{cases}$$

By Eqs. (5) and (18), one has the j-th component of $\mathbf{v}$ as follows:

$$v_j = \begin{cases} \overline{\gamma} y_1 u_{j_0}^{(1)} = -\dfrac{\overline{f(\alpha)}}{|f(\alpha)|} \dfrac{{\mu^{(2)}}^{q-1}}{\mu^{q-1}} \dfrac{e_{j_0}(\alpha)}{|e_{j_0}(\alpha)|}, & j = j_0, \\ \overline{\gamma} y_2 |u_j^{(2)}|^{q_2-2} u_j^{(2)} = -\dfrac{\overline{f(\alpha)}}{|f(\alpha)|} \dfrac{{\mu^{(1)}}^{q-1}}{\mu^{q-1}} \dfrac{|e_j(\alpha)|^{q_2-1}}{\|\mathbf{\Phi}^{(2)}(\alpha)\|_{q_2}^{q_2-1}} \dfrac{e_j(\alpha)}{|e_j(\alpha)|}, & j = r+1, \dots, n, \ e_j(\alpha) \neq 0, \\ 0, & \text{otherwise}. \end{cases}$$

Due to $\mathbf{c} = \frac{\mu^{(1)}\mu^{(2)}}{\mu} |f(\alpha)| \overline{\mathbf{v}}$ and the definition of $\mu^{(j)}$ again, one has the j-th component of $\mathbf{c}$ as follows:

$$c_j = \begin{cases} -f(\alpha)(\dfrac{\mu^{(1)}\mu^{(2)}}{\mu})^q \dfrac{1}{|e_{j_0}(\alpha)|^{1-q}} \dfrac{\overline{e_{j_0}(\alpha)}}{|e_{j_0}(\alpha)|}, & j = j_0, \\ -f(\alpha)(\dfrac{\mu^{(1)}\mu^{(2)}}{\mu})^q \dfrac{|e_j(\alpha)|^{q_2-1}}{\|\mathbf{\Phi}^{(2)}(\alpha)\|_{q_2}^{q_2-q}} \dfrac{\overline{e_j(\alpha)}}{|e_j(\alpha)|}, & j = r+1, \dots, n, \ e_j(\alpha) \neq 0, \\ 0, & \text{otherwise} \end{cases} \tag{20}$$

where $\frac{\mu^{(1)}\mu^{(2)}}{\mu} = \frac{1}{\|\mathbf{\Phi}(\alpha)\|_{q,\infty,q_2}} = \frac{1}{\left[|e_{j_0}(\alpha)|^q + \left(\sum_{k=r+1}^{n} |e_k(\alpha)|^{q_2}\right)^{\frac{q}{q_2}}\right]^{\frac{1}{q}}}$.

Case $\mathbf{a}_3$ $p_j=1, q_j=\infty, j=1,2$.

Let j_1 $(1\le j_1\le r)$ and j_2 $(r+2\le j_2\le n)$ be any indices with $|e_{j_1}(\alpha)|=\|\mathbf{\Phi}^{(1)}(\alpha)\|_\infty$ and $|e_{j_2}(\alpha)|=\|\mathbf{\Phi}^{(2)}(\alpha)\|_\infty$. By the discussions in Case a_1 and Case a_2, we can write the j-th component of $\mathbf{c}$ directly:

$$c_j=\begin{cases}-f(\alpha)(\dfrac{\mu^{(1)}\mu^{(2)}}{\mu})^q\dfrac{1}{|e_{j_1}(\alpha)|^{1-q}}\dfrac{\overline{e_{j_1}(\alpha)}}{|e_{j_1}(\alpha)|}, & j=j_1,\\ -f(\alpha)(\dfrac{\mu^{(1)}\mu^{(2)}}{\mu})^q\dfrac{1}{|e_{j_2}(\alpha)|^{1-q}}\dfrac{\overline{e_{j_2}(\alpha)}}{|e_{j_2}(\alpha)|}, & j=j_2,\\ 0, & \text{otherwise}\end{cases}\tag{21}$$

where $\frac{\mu^{(1)}\mu^{(2)}}{\mu}=\frac{1}{\|\mathbf{\Phi}(\alpha)\|_{q,\infty,q_2}}=\frac{1}{\left(|e_{j_1}(\alpha)|^q+|e_{j_2(\alpha)}|^q\right)^{\frac{1}{q}}}$.

Case b $p=1, q=\infty$.

Remember that $\mathbf{x}=(x_1,x_2)^T=\frac{1}{\mu}\left(\mu^{(2)},\mu^{(1)}\right)^T$ and $\mu=\|\left(\mu^{(1)},\mu^{(2)}\right)^T\|_q$. For $q=\infty$, if $\mu=\max\{\mu^{(1)},\mu^{(2)}\}=\mu^{(2)}$, then $x_1=1, x_2\le x_1$ which implies $y_1=1, y_2=0$ by Eq. (6), otherwise, $y_1=0, y_2=1$. Without loss of generality, we only take $y_1=1, y_2=0$ into account. Thus, we have $\mu=\mu^{(2)}, \mu^{(1)}\le\mu^{(2)}$ and $\|\mathbf{\Phi}(\alpha)\|_{\infty,q_1,q_2}=\frac{\mu}{\mu^{(1)}\mu^{(2)}}=\frac{1}{\mu^{(1)}}=\|\mathbf{\Phi}^{(1)}(\alpha)\|_{q_1}$.

Case $\mathbf{b}_1$ $1<p_j\le\infty$, $1\le q_j<\infty$, $\frac{1}{p_j}+\frac{1}{q_j}=1$, $j=1,2$.

Because of Eqs. (5) and (18), one has the j-th component of $\mathbf{v}$ as follows:

$$v_j=\begin{cases}\overline{\gamma}y_1|u_j^{(1)}|^{q_1-2}u_j^{(1)}=-\dfrac{\overline{f(\alpha)}}{|f(\alpha)|}\dfrac{|e_j(\alpha)|^{q_1-1}}{\|\mathbf{\Phi}^{(1)}(\alpha)\|_{q_1}^{q_1-1}}\dfrac{e_j(\alpha)}{|e_j(\alpha)|}, & j=1,\dots,r,\ e_j(\alpha)\ne 0,\\ \overline{\gamma}y_2|u_j^{(2)}|^{q_2-2}u_j^{(2)}=0, & j=r+1,\dots,n.\end{cases}$$

Since $\mathbf{c}=\frac{\mu^{(1)}\mu^{(2)}}{\mu}|f(\alpha)|\overline{\mathbf{v}}=\mu^{(1)}|f(\alpha)|\overline{\mathbf{v}}$, one has the corresponding j-th component of $\mathbf{c}$:

$$c_j=\begin{cases}-f(\alpha)\mu^{(1)}\dfrac{|e_j(\alpha)|^{q_1-1}}{\|\mathbf{\Phi}^{(1)}(\alpha)\|_{q_1}^{q_1-1}}\dfrac{\overline{e_j(\alpha)}}{|e_j(\alpha)|}=-\dfrac{f(\alpha)}{\|\mathbf{\Phi}^{(1)}(\alpha)\|_{q_1}^{q_1}}|e_j(\alpha)|^{q_1-2}\overline{e_j(\alpha)}, & j=1,\dots,r,e_j(\alpha)\ne 0,\\ 0, & j=r+1,\dots,n.\end{cases}\tag{22}$$

Case $\mathbf{b}_2$ $p_1=1, q_1=\infty, 1<p_2\le\infty, 1\le q_2<\infty, \frac{1}{p_2}+\frac{1}{q_2}=1$.

Similarly to Case a_2, we let $j_0(1\le j_0\le r)$ be any index with $|e_{j_0}(\alpha)|=\|\mathbf{\Phi}^{(1)}(\alpha)\|_\infty$. Then, we have the j-th component of $\mathbf{v}$ as follows:

$$v_j=\begin{cases}\overline{\gamma}y_1u_{j_0}^{(1)}=-\dfrac{\overline{f(\alpha)}}{|f(\alpha)|}\dfrac{e_{j_0}(\alpha)}{|e_{j_0}(\alpha)|}, & j=j_0,\\ \overline{\gamma}y_2|u_j^{(2)}|^{q_2-2}u_j^{(2)}=0, & j=r+1,\dots,n,\ e_j(\alpha)\ne 0,\\ 0, & \text{otherwise.}\end{cases}$$

Subsequently, we have the j-th component of $\mathbf{c}$ as follows:

$$c_j=\begin{cases}-f(\alpha)\dfrac{\overline{e_{j_0}(\alpha)}}{|e_{j_0}(\alpha)|^2}, & j=j_0,\\ 0, & \text{otherwise.}\end{cases}\tag{23}$$

Case $\mathbf{b}_3$ $p_j=1, q_j=\infty, j=1,2$.

For this case, because of $y_2 = 0$, v_j $(j = r+1, \ldots, n)$ are still equal to 0. Therefore, vector $\mathbf{c}$ here is the same as that in Case b_2.

Step 4. Return $\widetilde{f}(z)$ and $\widetilde{d}$:

After obtaining each component of vector $\mathbf{c}$, we have

$$\begin{cases} \widetilde{f}_{p,p_1,p_2}(z) := f(z) + \sum_{i=1}^{n} c_i e_i(z), \\ \widetilde{d}_{p,p_1,p_2} \ := \dfrac{|f(\alpha)|}{\|\mathbf{\Phi}(\alpha)\|_{q,q_1,q_2}}. \end{cases} \tag{24}$$

Remark 4.7 The discussions for the case of $1 < p_1 \leq \infty, 1 \leq q_1 < \infty, \frac{1}{p_1}+\frac{1}{q_1} = 1, p_2 = 1, q_2 = \infty$ are similar to the ones in Case a_2 and Case b_2, so we skip the details.

Note that if $p = \infty, q = 1$, the results in Case a degenerate to the ones in [15]. For example, the nearest polynomial and minimal distance of Case a_1 turn to

$$\begin{cases} \widetilde{f}_{\infty,p_1,p_2}(z) = f(z) - \dfrac{f(\alpha)}{\|\mathbf{\Phi}(\alpha)\|_{1,q_1,q_2}} \Big[\sum\limits_{\substack{j=1 \\ e_j(\alpha)\neq 0}}^{r} \dfrac{|e_j(\alpha)|^{q_1-1}}{\|\mathbf{\Phi}^{(1)}(\alpha)\|_{q_1}^{q_1-1}} \dfrac{\overline{e_j(\alpha)}}{|e_j(\alpha)|} e_j(z) + \\ \qquad \sum\limits_{\substack{j=r+1 \\ e_j(\alpha)\neq 0}}^{n} \dfrac{|e_j(\alpha)|^{q_2-1}}{\|\mathbf{\Phi}^{(2)}(\alpha)\|_{q_2}^{q_2-1}} \dfrac{\overline{e_j(\alpha)}}{|e_j(\alpha)|} e_j(z) \Big], \\ \widetilde{d}_{\infty,p_1,p_2} = \dfrac{f(\alpha)}{\|\mathbf{\Phi}(\alpha)\|_{1,q_1,q_2}} = \dfrac{|f(\alpha)|}{\big(\sum_{k=1}^{r} |e_k(\alpha)|^{q_1}\big)^{\frac{1}{q_1}} + \big(\sum_{k=r+1}^{n} |e_k(\alpha)|^{q_2}\big)^{\frac{1}{q_2}}} \end{cases} \tag{25}$$

which are the same as $\widetilde{f}_{mix}(z)$ and $Q_{n,p_1,p_2}(\alpha)$ in Eq. (22) in [15].

5. Examples

For the generalized weighted norm and generalized mixed norm, since there are analytic and explicit expressions for the nearest polynomial $\widetilde{f}(z)$ and minimal distance $\widetilde{d}$, we can calculate them directly and efficiently.

Example 5.1 Suppose that $0 < n < \infty$ is given. Let $f(z) = z^2 + 2$, $e_j(z) = z^{j-1}$ $(j = 1, 2, \ldots, n)$ and $\alpha = i$. Here, i denotes the unit imaginary number with $i^2 = -1$. Then, we have $\Im = \Big\{ f(z) + \sum_{j=1}^{n} c_j e_j(z) \mid c_j \in \mathbb{C}, j = 1, \ldots, n \Big\}$.

First, for $p = \infty$ and $\mathbf{W} = \mathrm{diag}(\omega_1, \omega_2, \ldots, \omega_n) \in \mathbb{R}^{n\times n}$ with $\omega_j \neq 0$ $(j = 1, 2, \ldots, n)$, we consider the problem of computing a polynomial $\widetilde{f}(z) \in \Im$ such that $\widetilde{f}(\alpha) = 0$ and $\|\widetilde{f} - f\|_{p,\mathbf{W}}$ is minimal.

Since $q = 1$ and $\mathbf{W}^{-1} = \mathrm{diag}(\omega_1^{-1}, \omega_2^{-1}, \ldots, \omega_n^{-1})$, Eq. (15) then is equal to

$$\begin{cases} \widetilde{f}_{p,\mathbf{W}}(z) = f(z) - \dfrac{f(\alpha)}{\sum_{k=1}^{n} |\frac{e_k(\alpha)}{\omega_k}|} \sum\limits_{\substack{j=1 \\ e_j(\alpha)\neq 0}}^{n} \dfrac{\overline{e_j(\alpha)}}{|e_j(\alpha)|} \dfrac{e_j(z)}{|\omega_j|}, \\ \widetilde{d}_{p,\mathbf{W}} \ = \dfrac{|f(\alpha)|}{\sum_{k=1}^{n} |\frac{e_k(\alpha)}{\omega_k}|}. \end{cases} \tag{26}$$

Therefore, the desired nearest polynomial is

$$(z^2+2) - \frac{1}{\sum_{k=1}^{n}\frac{1}{|\omega_k|}} \sum_{j=1}^{n} \frac{(-i)^{j-1}}{|\omega_j|} z^{j-1}$$

with minimal distance $\frac{1}{\sum_{k=1}^{n}\frac{1}{|\omega_k|}}$.

For example, considering the special case of $n=2$ and $\mathbf{W}=\mathrm{diag}(1,1)$, we have the nearest polynomial: $\widetilde{f}(z)=z^2+\frac{i}{2}z+\frac{3}{2}$ and the minimal distance: $\widetilde{d}=\frac{1}{2}$; considering another special case of $n=3$ and $\mathbf{W}=\mathrm{diag}(-1,2,-1)$, we have the nearest polynomial: $\widetilde{f}(z)=\frac{7}{5}z^2+\frac{i}{5}z+\frac{8}{5}$ and minimal distance: $\widetilde{d}=\frac{2}{5}$.

Next, for $p=\infty$, consider the problem of computing a polynomial $\widetilde{f}(z)\in\Im$ such that $\widetilde{f}(\alpha)=0$ and $\|\widetilde{f}-f\|_{p,p_1,p_2}$ is minimal.

Recalling Eq. (25), we have the nearest polynomial as follows:

$$\widetilde{f}_{\infty,p_1,p_2}(z) = (z^2+2) - \frac{1}{r^{\frac{1}{q_1}}+(n-r)^{\frac{1}{q_2}}}\Big[\sum_{\substack{j=1\\ e_j(\alpha)\neq 0}}^{r} \frac{(-i)^{j-1}}{r^{\frac{q_1-1}{q_1}}} z^{j-1} + \sum_{\substack{j=r+1\\ e_j(\alpha)\neq 0}}^{n} \frac{(-i)^{j-1}}{(n-r)^{\frac{q_2-1}{q_2}}} z^{j-1}\Big] \quad (27)$$

with minimal distance $\frac{1}{r^{\frac{1}{q_1}}+(n-r)^{\frac{1}{q_2}}}$.

For example, fix $r=1$ and $p_1=p_2=2$. Considering the special case of $n=2$, we have the nearest polynomial: $\widetilde{f}(z)=z^2+\frac{i}{2}z+\frac{3}{2}$ and the minimal distance: $\widetilde{d}=\frac{1}{2}$; considering another special case of $n=3$, we have the nearest polynomial: $\widetilde{f}(z)=(2-\frac{\sqrt{2}}{2})z^2+(1-\frac{\sqrt{2}}{2})iz+(3-\sqrt{2})$ and the minimal distance: $\widetilde{d}=\sqrt{2}-1$.

Remark 5.2 From Example 5.1, the nearest polynomial and minimal distance are related to choices of the polynomial basis and the vector norm. Moreover, there may exist more than one solutions to Problem 1.1, while our aim in this paper is to find any one of them. For example, for the case of $p=\infty$ and $\mathbf{W}=\mathrm{diag}(1,1)$, if given $f(z)=z+1$, $e_1(z)=1, e_2(z)=z$, and $\alpha=0$, by utilizing Algorithm 3.4, one can find that $\widetilde{f}(z)=z$ is a nearest polynomial with $\widetilde{d}=1$. However, in fact, any of the polynomials $f_\beta(z)=\beta z$ with $\beta\in[0,2]$ is also a nearest polynomial.

6. Conclusion

For a given complex polynomial $f(z)$ and a prescribed zero α, we proposed a common framework to compute nearest complex polynomial and minimal distance for Problem 1.1. Besides, we studied the explicit expressions in two generalized norms, which include many previous results. As the explicit expression of nearest real polynomial can be derived only in the case of ℓ_2-norm, in future research we will consider the explicit solutions with real coefficients to Problem 1.1 in the case of other norms.

Appendix A. Proof of Theorem 4.3

(a) Let $\mathbf{v}=(v_1,\ldots,v_n)^T\in\mathbb{C}^n$, $\mathbf{v}\neq 0$. Then we have

$$\langle \mathbf{u},\mathbf{v}\rangle = \overline{\mathbf{v}}^T\mathbf{u} = \overline{\mathbf{v}}^T\mathbf{W}\mathbf{W}^{-1}\mathbf{u} = (\overline{\mathbf{W}\mathbf{v}})^T(\mathbf{W}^{-1}\mathbf{u}) = \left\langle \mathbf{W}^{-1}\mathbf{u},\mathbf{W}\mathbf{v}\right\rangle.$$

By applying the Hölder inequality, we get that

$$|\langle \mathbf{u}, \mathbf{v} \rangle| = |\left\langle \mathbf{W}^{-1}\mathbf{u}, \mathbf{W}\mathbf{v} \right\rangle| \le \|\mathbf{W}^{-1}\mathbf{u}\|_q \|\mathbf{W}\mathbf{v}\|_p = \|\mathbf{u}\|_{q,\mathbf{W}^{-1}} \|\mathbf{v}\|_{p,\mathbf{W}}$$

where $1 \le p, q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. According to the definition of dual norm, this implies that

$$\|\mathbf{u}\|^*_{p,\mathbf{W}} \le \|\mathbf{u}\|_{q,\mathbf{W}^{-1}}.$$

(b) Let $\mathbf{x} = \frac{\mathbf{W}^{-1}\mathbf{u}}{\|\mathbf{u}\|_{q,\mathbf{W}^{-1}}}$, that is $\mathbf{u} = \|\mathbf{u}\|_{q,\mathbf{W}^{-1}} \mathbf{W}\mathbf{x}$ and $\|\mathbf{x}\|_q = 1$. By Lemma 3.2, there exists $\mathbf{y} \in \mathbb{C}^n$, $\|\mathbf{y}\|_p = 1$ such that $\langle \mathbf{x}, \mathbf{y} \rangle = \gamma$ where $\gamma \in \mathbb{C}$, $|\gamma| = 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Let $\mathbf{v} = \mathbf{W}^{-1}\mathbf{y}$. Then we have $\|\mathbf{v}\|_{p,\mathbf{W}} = \|\mathbf{y}\|_p = 1$, and

$$\langle \mathbf{u}, \mathbf{v} \rangle = \left\langle \|\mathbf{u}\|_{q,\mathbf{W}^{-1}} \mathbf{W}\mathbf{x}, \mathbf{W}^{-1}\mathbf{y} \right\rangle = \|\mathbf{u}\|_{q,\mathbf{W}^{-1}} \langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{u}\|_{q,\mathbf{W}^{-1}} \gamma.$$

That is to say, we have found a vector $\mathbf{v} \in \mathbb{C}^n$, $\|\mathbf{v}\|_{p,\mathbf{W}} = 1$ satisfying $|\langle \mathbf{u},\ \mathbf{v} \rangle| = \|\mathbf{u}\|_{q,\mathbf{W}^{-1}}$. Therefore, by Eq. (3), we obtain

$$\|\mathbf{u}\|^*_{p,\mathbf{W}} \ge \|\mathbf{u}\|_{q,\mathbf{W}^{-1}}. \quad \square$$

Appendix B. Proof of Theorem 4.4

The proof is similar to but more complex than the one in Theorem 4.3. Here, some unspecified notations are deemed to be defined as above.

(a) Let $\mathbf{v} = (v_1, \ldots, v_n)^T \in \mathbb{C}^n$, $\mathbf{v} \ne 0$. Then, by the Hölder inequality, we have

$$\begin{aligned}
|\langle \mathbf{u},\ \mathbf{v} \rangle| &= |\left\langle \mathbf{u}^{(1)}, \mathbf{v}^{(1)} \right\rangle + \left\langle \mathbf{u}^{(2)}, \mathbf{v}^{(2)} \right\rangle| \le |\left\langle \mathbf{u}^{(1)}, \mathbf{v}^{(1)} \right\rangle| + |\left\langle \mathbf{u}^{(2)}, \mathbf{v}^{(2)} \right\rangle| \\
&\le \|\mathbf{u}^{(1)}\|_{q_1} \|\mathbf{v}^{(1)}\|_{p_1} + \|\mathbf{u}^{(2)}\|_{q_2} \|\mathbf{v}^{(2)}\|_{p_2} \\
&= \left(\|\mathbf{u}^{(1)}\|_{q_1}, \|\mathbf{u}^{(2)}\|_{q_2}\right) \cdot \left(\|\mathbf{v}^{(2)}\|_{p_1}, \|\mathbf{v}^{(2)}\|_{p_2}\right)^T \\
&\le \|\left(\|\mathbf{u}^{(1)}\|_{q_1}, \|\mathbf{u}^{(2)}\|_{q_2}\right)\|_q \|\left(\|\mathbf{v}^{(2)}\|_{p_1}, \|\mathbf{v}^{(2)}\|_{p_2}\right)\|_p \\
&= \|\mathbf{u}\|_{q,q_1,q_2} \|\mathbf{v}\|_{p,p_1,p_2}.
\end{aligned}$$

Similarly, according to the definition of dual norm, this implies that

$$\|\mathbf{u}\|^*_{p,p_1,p_2} \le \|\mathbf{u}\|_{q,q_1,q_2}.$$

(b) Assume $p = \infty$: Let $\mathbf{x}^{(1)} = \frac{\mathbf{u}^{(1)}}{\|\mathbf{u}^{(1)}\|_{q_1}}$, $\mathbf{x}^{(2)} = \frac{\mathbf{u}^{(2)}}{\|\mathbf{u}^{(2)}\|_{q_2}}$ such that $\|\mathbf{x}^{(1)}\|_{q_1} = 1$ and $\|\mathbf{x}^{(2)}\|_{q_2} = 1$. Suppose $\gamma \in \mathbb{C}$, $|\gamma| = 1$. Then, by Lemma 3.2, there exist $\mathbf{y}^{(1)} \in \mathbb{C}^r$, $\|\mathbf{y}^{(1)}\|_{p_1} = 1$ and $\mathbf{y}^{(2)} \in \mathbb{C}^{n-r}$, $\|\mathbf{y}^{(2)}\|_{p_2} = 1$ satisfying

$$\left\langle \mathbf{x}^{(1)},\ \mathbf{y}^{(1)} \right\rangle = \gamma, \left\langle \mathbf{x}^{(2)},\ \mathbf{y}^{(2)} \right\rangle = \gamma.$$

Let $\mathbf{v} = (\mathbf{y}^{(1)^T}, \mathbf{y}^{(2)^T})^T$, i.e., $\|\mathbf{v}\|_{\infty,\ p_1,p_2} = \max\{\|\mathbf{y}^{(1)}\|_{p_1},\ \|\mathbf{y}^{(2)}\|_{p_2}\} = 1$. Then, we have

$$\begin{aligned}
\langle \mathbf{u},\ \mathbf{v} \rangle &= \left\langle \mathbf{u}^{(1)},\ \mathbf{v}^{(1)} \right\rangle + \left\langle \mathbf{u}^{(2)},\ \mathbf{v}^{(2)} \right\rangle = \|\mathbf{u}^{(1)}\|_{q_1} \left\langle \mathbf{x}^{(1)},\ \mathbf{y}^{(1)} \right\rangle + \|\mathbf{u}^{(2)}\|_{q_2} \left\langle \mathbf{x}^{(2)},\ \mathbf{y}^{(2)} \right\rangle \\
&= \gamma\left(\|\mathbf{u}^{(1)}\|_{q_1} + \|\mathbf{u}^{(2)}\|_{q_2}\right) \\
&= \gamma \|\mathbf{u}\|_{1,\ q_1,\ q_2},
\end{aligned}$$

which implies $|\langle \mathbf{u}, \mathbf{v} \rangle| = \|\mathbf{u}\|_{1,\ q_1,\ q_2}$. Therefore, from the definition of dual norm, we have

$$\|\mathbf{u}\|^*_{\infty,\ p_1,\ p_2} \ge \|\mathbf{u}\|_{1,\ q_1,\ q_2}.$$

Assume $p \in [1,\infty)$: Let $\mathbf{x} = \frac{(\|\mathbf{u}^{(1)}\|_{q_1}, \|\mathbf{u}^{(2)}\|_{q_2})}{\|\mathbf{u}\|_{q,q_1,q_2}}$. Then we have $\mathbf{x} \in \mathbb{R}^2$ and $\|\mathbf{x}\|_q = \|\mathbf{u}\|_{q,q_1,q_2} = 1$. Suppose $\gamma_0 = 1$. Then, using Lemma 3.2, there exists $\mathbf{y} = (y_1, y_2)^T \in \mathbb{R}^2$, $y_1 \geq 0, y_2 \geq 0$ and $\|\mathbf{y}\|_p = 1$ such that $\langle \mathbf{x}, \mathbf{y} \rangle = \gamma_0 = 1$. Equivalently, we have

$$y_1\|\mathbf{u}^{(1)}\|_{q_1} + y_2\|\mathbf{u}^{(2)}\|_{q_2} = \|\mathbf{u}\|_{q,q_1,q_2}\,. \tag{28}$$

Let $\mathbf{x}^{(1)} = \frac{\mathbf{u}^{(1)}}{\|\mathbf{u}^{(1)}\|_{q_1}}$ and $\mathbf{x}^{(2)} = \frac{\mathbf{u}^{(2)}}{\|\mathbf{u}^{(1)}\|_{q_2}}$. Then, we have $\|\mathbf{x}^{(1)}\|_{q_1} = 1$ and $\|\mathbf{x}^{(2)}\|_{q_2} = 1$. For any $\gamma \in \mathbb{C}$, $|\gamma| = 1$, by Lemma 3.2 again, there exist $\mathbf{y}^{(1)} \in \mathbb{C}^r, \mathbf{y}^{(2)} \in \mathbb{C}^{n-r}$, $\|\mathbf{y}^{(1)}\|_{p_1} = 1, \|\mathbf{y}^{(2)}\|_{p_2} = 1$ such that $\langle \mathbf{x}^{(1)}, \mathbf{y}^{(1)} \rangle = \gamma$ and $\langle \mathbf{x}^{(2)},\ \mathbf{y}^{(2)} \rangle = \gamma$, i.e.,

$$\big\langle \mathbf{u}^{(1)}, \mathbf{y}^{(1)} \big\rangle = \gamma\|\mathbf{u}^{(1)}\|_{q_1}, \big\langle \mathbf{u}^{(2)}, \mathbf{y}^{(2)} \big\rangle = \gamma\|\mathbf{u}^{(2)}\|_{q_2}. \tag{29}$$

Set $\mathbf{v} = (y_1{\mathbf{y}^{(1)}}^T, y_2{\mathbf{y}^{(2)}}^T)^T$. Then, combining Eqs. (28) and (29), we have

$$\begin{cases} \|\mathbf{v}\|_{p,p_1,p_2} = \|\big(\|y_1\mathbf{y}^{(1)}\|_{p_1}, \|y_2\mathbf{y}^{(2)}\|_{p_2}\big)\|_p = \|(y_1, y_2)\|_p = \|\mathbf{y}\|_p = 1, \\ \\ \langle \mathbf{u}, \mathbf{v} \rangle = y_1\big\langle \mathbf{u}^{(1)}, \mathbf{y}^{(1)} \big\rangle + y_2\big\langle \mathbf{u}^{(2)}, \mathbf{y}^{(2)} \big\rangle = \gamma\big(y_1\|\mathbf{u}^{(1)}\|_{q_1} + y_2\|\mathbf{u}^{(2)}\|_{q_2}\big) = \gamma\,\|\mathbf{u}\|_{q,q_1,q_2} \end{cases}$$

which implies that there exists $\mathbf{v} \in \mathbb{C}^n$, $\|\mathbf{v}\|_{p,p_1,p_2} = 1$ such that $|\,\langle \mathbf{u}, \mathbf{v} \rangle\,| = \|\mathbf{u}\|_{q,q_1,q_2}$. Therefore, we get that

$$\|\mathbf{u}\|^*_{p,p_1,p_2} \geq \|\mathbf{u}\|_{q,q_1,q_2}\,. \quad \Box$$

References

[1] A. M. OSTROWSKI. *Solution of Equations and Systems of Equations.* Second edition, Pure and Applied Mathematics, Vol. 9, Academic Press, New York-London, 1966.

[2] R. G. MOSIER. *Root neighborhoods of a polynomial.* Math. Comp., 1986, **47**(175): 265–273.

[3] H. J. STETTER. *Polynomials with Coefficients of Limited Accuracy.* Computer Algebra in Scientific Computing–CASC'99 (Munich), 409-430, Springer, Berlin, 1999.

[4] H. J. STETTER. *Numerical Polynomial Algebra.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2004.

[5] Li QIU, E. J. DAVISON. *A simple procedure for the exact stability robustness computation of polynomials with affine coefficient perturbations.* Systems Control Lett., 1989, **13**(5): 413–420.

[6] Jie CHEN, M. K. H. FAN, C. N. NETT. *Structured singular values and stability analysis of uncertain polynomials.* I. The generalized μ. Systems Control Lett., 1994, **23**(1): 53–65.

[7] Jie CHEN, M. K. H. FAN, C. N. NETT. *Structured singular values and stability analysis of uncertain polynomials.* II. a missing link. Systems Control Lett., 1994, **23**(2): 97–109.

[8] M. A. HITZ. *Efficient algorithms for computing the nearest polynomial with constrained roots.* Thesis (Ph.D.)-Rensselaer Polytechnic Institute. 1998.

[9] M. A. HITZ, E. KALTOFEN. *Efficient algorithms for computing the nearest polynomial with constrained roots.* Proceedings of the 1998 International Symposium on Symbolic and Algebraic Computation (Rostock), 236-243, ACM, New York, 1998.

[10] M. A. HITZ, E. KALTOFEN, Y. N. LAKSHMAN. *Efficient algorithms for computing the nearest polynomial with a real root and related problems.* Proceedings of the 1999 International Symposium on Symbolic and Algebraic Computation (Vancouver, BC), 205-212, ACM, New York, 1999.

[11] H. J. STETTER. *The nearest polynomial with a given zero, and similar problems.* SIGSAM Bull., 1999, **33**(4): 2–4.

[12] S. GRAILLAT. *A note on a nearest polynomial with a given root.* SIGSAM Bull., 2005, **39**(2): 53–60.

[13] N. REZVANI, R. M. CORLESS. *The nearest polynomial with a given zero, revisted.* SIGSAM Bull., 2005, **39**(3): 73–79.

[14] N. REZVANI, R. M. CORLESS. *Using weighted norms to find nearest polynomials satisfying linear constraints.* Proceedings of the 2011 International Workshop on Symbolic-Numeric Computation, SNC'11, 2011, 81–87.

[15] Zhongxuan LUO, Wenyu HU, D. W. SHEEN. *The nearest complex polynomial with a zero in a given complex domain*. Theoret. Comput. Sci., 2011, **412**(50): 7029–7043.

[16] H. SEKIGAWA. *The nearest polynomial with a zero in a given domain*. Theoret. Comput. Sci., 2008, **409**(2): 282–291.

[17] H. SEKIGAWA. *Computing the nearest polynomial with a zero in a given domain by using piecewise rational functions*. J. Symbolic Comput., 2011, **46**(12): 1318–1335.

[18] E. J. CANDÈS, M. B. WAKIN, S. P. BOYD. *Enhancing sparsity by reweighted ℓ_1 minimization*. J. Fourier Anal. Appl., 2008, **14**(5-6): 877–905.

[19] Yao WANG, Jianjun WANG, Zongben XU. *On recovery of block-sparse signals via mixed ℓ_2/ℓ_q $(0 < q \leq \infty)$ norm minimization*. EURASIP Journal on Advances in Signal Processing, 2013, **76**(1): 1–17.

[20] P. BOUFOUNOS, G. KUTYNOIK, H. RAUHUT. *Sparse recovery from combined fusion frame measurements*. IEEE Trans. Inform. Theory, 2011, **57**(6): 3864–3876.

[21] Feiping NIE, Heng HUANG, Xiao CAI, et al. *Efficient and robust feature selection via joint $\ell_{2,1}$-norms minimization*. Proceedings of the Neural Information Processing Systems, 2010, 1813–1821.

[22] Quanquan GU, Zhenhui LI, Jiawei HAN. *Joint feature selection and subspace learning*. Proceedings of the Twenty-Second International Joint Conference on Artificial, 2011, 1294–1299.

[23] Xiaoshuang SHI, Yujiu YANG, Zhenhua GUO, et al. *Face recognition by sparse discriminant analysis via joint $L_{2,1}$-norm minimization*. Pattern Recognition, 2014, **47**(7): 2447–2453.