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Rings with Symmetric Endomorphisms and Their Extensions

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Abstract Let R be a ring with an endomorphism α and an α-derivation δ. We introduce the notions of symmetric α-rings and weak symmetric α-rings which are generalizations of symmetric rings and weak symmetric rings, respectively, discuss the relations between symmetric α-rings and related rings and investigate their extensions. We prove that if R is a reduced ring and $\alpha(1)=1$, then R is a symmetric α-ring if and only if $R[x]/(x^n)$ is a symmetric $\bar{\alpha}$-ring for any positive integer n. Moreover, it is proven that if R is a right Ore ring, α an automorphism of R and $Q(R)$ the classical right quotient ring of R, then R is a symmetric α-ring if and only if $Q(R)$ is a symmetric $\bar{\alpha}$-ring. Among others we also show that if a ring R is weakly 2-primal and (α,δ)-compatible, then R is a weak symmetric α-ring if and only if the Ore extension $R[x;\alpha,\delta]$ of R is a weak symmetric $\bar{\alpha}$-ring.

Keywords symmetric α-ring; weak symmetric α-ring; polynomial extension; classical quotient ring extension; Ore extension

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1. Introduction

Throughout this paper R denotes an associative ring with identity, and α is a nonzero endomorphism of R. Recall that a ring R is called reduced if it has no nonzero nilpotent elements; R is reversible if $ab=0$ implies $ba=0$ for all $a,b\in R$; R is semicommutative if $ab=0$ implies $aRb=0$ for all $a,b\in R$; an endomorphism α of a ring R is called rigid if $a\alpha(a)=0$ implies $a=0$ for $a\in R$, and R is called α-rigid if there exists a rigid endomorphism α of R. Baser et al. [3] introduced the concept of α-shifting rings and investigated characterizations of the generalized reversible rings. A ring R is said to be right (left) α-shifting if whenever $a\alpha(b)=0$ ($\alpha(a)b=0$) for $a,b\in R$, $b\alpha(a)=0$ ($\alpha(b)a=0$). Baser et al. [2] extended the concept of semicommutative rings and called a ring R α-semicommutative if $ab=0$ implies $aR\alpha(b)=0$ for all $a,b\in R$. Recently, we introduced the concept of semicommutative α-rings in [17]. A ring R is called a

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right (left) semicommutative α-ring if $a\alpha(b) = 0$ ($\alpha(a)b = 0$) implies $\alpha(a)Rb = 0$ ($aR\alpha(b) = 0$) for all $a, b \in R$. According to Lambek [12], a ring R is called symmetric if $abc = 0$ implies $acb = 0$ for all $a, b, c \in R$. Anderson and Camillo [1] showed that a ring R is symmetric if and only if $r_1 r_2 \cdots r_n = 0$ implies $r_{\sigma(1)} r_{\sigma(2)} \cdots r_{\sigma(n)} = 0$ for any permutation σ of the set $\{1, 2, \ldots, n\}$ and $r_i \in R$. There are many papers to study symmetric rings and their generalization [5,8,11,14,18]. In Kwak [10], an endomorphism α of a ring R is called right (left) symmetric if whenever $abc = 0$ for $a, b, c \in R$, $ac\alpha(b) = 0$ ($\alpha(b)ac = 0$). A ring R is called right (left) α-symmetric if there exists a right (left) symmetric endomorphism α of R. The notion of an α-symmetric ring is a generalization of α-rigid rings as well as an extension of symmetric rings. Following [15], a ring R is called weak symmetric if $abc \in \mathrm{nil}(R)$ implies $acb \in \mathrm{nil}(R)$ for all $a, b, c \in R$, where $\mathrm{nil}(R)$ is the set of all nilpotent elements of R. Let α be an endomorphism and δ an α-derivation of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for $a, b \in R$. When $\alpha = id_R$, an α-derivation δ is called a derivation of R. A ring R is said to be weak α-symmetric if $abc \in \mathrm{nil}(R)$ implies $ac\alpha(b) \in \mathrm{nil}(R)$ for $a, b, c \in R$. Moreover, R is said to be weak δ-symmetric if $abc \in \mathrm{nil}(R)$ implies $ac\delta(b) \in \mathrm{nil}(R)$ for $a, b, c \in R$. If R is both weak α-symmetric and weak δ-symmetric, then R is called weak (α, δ)-symmetric. Ouyang and Chen [15] studied the related properties of weak symmetric rings and weak (α, δ)-symmetric rings.

Motivated by the above, for an endomorphism α of a ring R, we introduce the notions of symmetric α-rings and weak symmetric α-rings to extend symmetric rings and weak symmetric rings, respectively, discuss the relations between symmetric α-rings and related rings and investigate their extensions.

Let α be an endomorphism and δ an α-derivation of R. We denote by $R[x; \alpha, \delta]$ the Ore extension whose elements are the polynomials over R, the addition is defined as usual, and the multiplication subject to the relation $xr = \alpha(r)x + \delta(r)$ for any $r \in R$. In particular, if $\delta = 0_R$, we denote by $R[x; \alpha]$ the skew polynomial ring; if $\alpha = 1_R$, we denote by $R[x; \delta]$ the differential polynomial ring. For an endomorphism α and an α-derivation δ, a ring R is said to be α-compatible if for each $a, b \in R$, $ab = 0$ if and only if $a\alpha(b) = 0$. Moreover, R is called δ-compatible if $ab = 0$ implies $a\delta(b) = 0$ for each $a, b \in R$. If R is both α-compatible and δ-compatible, then R is called (α, δ)-compatible. In the following, for integers i, j with $0 \leqslant i \leqslant j$, $f_i^j \in \mathrm{End}(R, +)$ will denote the map which is the sum of all possible words in α, δ built with i letters α and $j - i$ letters δ. For instance, $f_2^4 = \alpha^2\delta^2 + \delta^2\alpha^2 + \delta\alpha^2\delta + \alpha\delta^2\alpha + \alpha\delta\alpha\delta + \delta\alpha\delta\alpha$. In particular, $f_0^0 = 1$, $f_i^i = \alpha^i$, $f_0^i = \delta^i$, $f_{j-1}^j = \alpha^{j-1}\delta + \alpha^{j-2}\delta\alpha + \cdots + \delta\alpha^{j-1}$. For every $f_i^j \in \mathrm{End}(R, +)$ with $0 \leqslant i \leqslant j$, it has C_j^i monomials in α, δ built with i letters α and $j - i$ letters δ. It is well known that for any integer n and $r \in R$, we have $x^n r = \sum_{i=0}^{n} f_i^n(r)x^i$ in the ring $R[x; \alpha, \delta]$.

2. Symmetric α-rings and related rings

As an extension of symmetric rings, now we give the following

Definition 2.1 *Let R be a ring and α a nonzero endomorphism of R. We say that R is a (right)* *symmetric α-ring if $ab\alpha(c)=0$ implies $ac\alpha(b)=0$ for $a,b,c\in R$.*

Similarly, a ring R is said to be a left symmetric α-ring whenever $\alpha(a)bc=0$ for $a,b,c\in R$, $\alpha(b)ac=0$.

Obviously, if $\alpha=\mathrm{id}_R$, the identity endomorphism of R, a (left) symmetric α-ring is a symmetric ring. In general, a right α-symmetric ring need not be a symmetric α-ring.

Example 2.2 Let $R=F[x]$ be the polynomial ring over a field F and $\alpha: R\to R$, $\alpha(f(x))=f(0)$ for $f(x)\in R$. The α is an endomorphism of R but not a monomorphism, and R is an α-symmetric ring by Kwak [10, Example 2.7 (2)]. But for any $0\neq f(x)\in R$ and $g(x)=x+a, h(x)=x\in R$ where $a\neq 0$, we have $f(x)g(x)\alpha(h(x))=0$, $f(x)h(x)\alpha(g(x))\neq 0$. Hence R is not a symmetric α-ring.

The next example shows that if $\alpha\neq \mathrm{id}_R$, a symmetric α-ring need not be symmetric and a left symmetric α-ring also need not be a left symmetric α-ring. Therefore, the classes of symmetric α-rings and left symmetric α-rings are both non-trivial extension of symmetric rings, the symmetric α-property for a ring is not left-right symmetric and the concepts of symmetric α-rings and left symmetric α-rings are independent of each other.

Example 2.3 Consider the ring $R=\left\{\begin{pmatrix} a & b\\ 0 & c\end{pmatrix} | a,b,c\in\mathbb{Z}\right\}$, where $\mathbb{Z}$ is the ring of integers and the endomorphism $\alpha: R\to R$, $\alpha\left(\begin{pmatrix} a & b\\ 0 & c\end{pmatrix}\right)=\begin{pmatrix} a & 0\\ 0 & 0\end{pmatrix}$. It is easy to verify that R is not symmetric. Let $\mathbf{A}=\begin{pmatrix} a_1 & b_1\\ 0 & c_1\end{pmatrix}, \mathbf{B}=\begin{pmatrix} a_2 & b_2\\ 0 & c_2\end{pmatrix}, \mathbf{C}=\begin{pmatrix} a_3 & b_3\\ 0 & c_3\end{pmatrix}\in R$ with $\mathbf{AB}\alpha(\mathbf{C})=0$. Then $a_1a_2a_3=0$, so we have $a_1a_3a_2=0$ and $\mathbf{AC}\alpha(\mathbf{B})=0$, concluding that R is a symmetric α-ring. For $\mathbf{A}=\begin{pmatrix} 0 & 1\\ 0 & 1\end{pmatrix}, \mathbf{B}=\begin{pmatrix} 1 & 1\\ 0 & 0\end{pmatrix}, \mathbf{C}=\begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}\in R$, we have $\alpha(\mathbf{A})\mathbf{BC}=0$, but $\alpha(\mathbf{B})\mathbf{AC}=\begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}\neq 0$. So R is not a left symmetric α-ring.

In the following, we focus our attention on symmetric α-rings.

Proposition 2.4 *For a nonzero endomorphism α of a ring R, the following statements are equivalent:*

(1) R is a symmtric α-ring;

(2) $l_R(b\alpha(c))\subseteq l_R(c\alpha(b))$, for any $a,b,c\in R$;

(3) $AB\alpha(C)=0$ if and only if $AC\alpha(B)=0$, for any $A,B,C\subseteq R$;

(4) $l_R(B\alpha(C))\subseteq l_R(C\alpha(B))$, for any $A,B,C\subseteq R$.

Proof (1) $\Longleftrightarrow$ (3). Suppose that $AC\alpha(B)=0$ for $A,B,C\subseteq R$. Then $ab\alpha(c)=0$ for any $a\in A, b\in B, c\in C$, and hence $ac\alpha(b)=0$. Therefore, $AC\alpha(B)=\{\sum a_ic_i\alpha(b_i)|a_i\in A, b_i\in B, c_i\in C\}=0$. The converse is obvious.

(1) $\Longleftrightarrow$ (2) and (3) $\Longleftrightarrow$ (4) is obvious. $\square$

Proposition 2.5 *For a nonzero endomorphism α of a ring R, the following statements are equivalent:*

(1) R is an α-rigid ring;

(2) R is a symmetric α-ring and $aR\alpha(a) = 0$ implies $a = 0$ for any $a \in R$;

(3) R is a left symmetric α-ring and $\alpha(a)Ra = 0$ implies $a = 0$ for any $a \in R$.

Proof (1) $\Rightarrow$ (2). Assume that R is α-rigid. Then R is reduced and α is a monomorphism by [6]. For $a, b, c \in R$ with $ab\alpha(c) = 0$, we have $0 = ab\alpha(c)c\alpha(a)\alpha(b) = abc\alpha(a)\alpha(b)\alpha(c) = abc\alpha(abc)$ and hence $abc = 0$, $bac = 0$ since R is an α-rigid ring. It gives that $0 = ac\alpha(bac)\alpha^2(b) = ac\alpha(b)\alpha(a)\alpha(c)\alpha^2(b) = ac\alpha(b)\alpha(ac\alpha(b))$, and hence $ac\alpha(b) = 0$. So R is a symmetric α-ring. For $a \in R$ with $aR\alpha(a) = 0$, we have $a\alpha(a) = 0$. This implies that $a = 0$.

(2) $\Rightarrow$ (1). Suppose that $a\alpha(a) = 0$ for $a \in R$. Then we have $1 \cdot a\alpha(a)\alpha(r) = 1 \cdot a\alpha(ar) = 0$ for all $r \in R$. Since R is a symmetric α-ring, $1 \cdot ar\alpha(a) = ar\alpha(a) = 0$. Thus, we get $a = 0$ by the assumption, concluding that R is an α-rigid ring.

Similarly, we can prove (3) $\Longleftrightarrow$ (1). $\square$

Proposition 2.6 *Let α be a nonzero endomorphism of a ring R. Then we have the following:*

(1) The class of symmtric α-rings is closed under α-subrings (not necessarily with identity);

(2) If R is a (left) symmetric α-ring, then R is a right (left) α-shifting ring.

Proof (1) By Definition 2.1.

(2) Suppose that $a\alpha(b) = 0$ for $a, b \in R$. Then $0 = 1 \cdot a\alpha(b)$ implies that $1 \cdot b\alpha(a) = b\alpha(a) = 0$. $\square$

In general, the converses of Proposition 2.6(2) does not hold, and a right α-shifting ring need not be a right semicommutative α-ring.

Example 2.7 Let $\mathbb{Z}_2$ be the ring of integers module 2, $R = \mathbb{Z}_2 \bigoplus \mathbb{Z}_2$ and $\alpha : R \to R$ be an endomorphism of R defined by $\alpha((a, b)) = (b, a)$ for any $(a, b) \in R$. Suppose $(a, b)\alpha((c, d)) = (ad, bc) = 0$ for $(a, b), (c, d) \in R$. Then we have $(c, d)\alpha((a, b)) = (cb, da) = 0$, concluding that R is a right α-shifting ring. However, R is not a symmetric α-ring. In fact, for $A = (1, 0), B = (0, 1), C = (1, 1)$, we have $AB\alpha(C) = (1, 0)(0, 1)(1, 1) = 0$, but $AC\alpha(B) = (1, 0)(1, 1)(1, 0) = (1, 0) \neq 0$.

Example 2.8 Let R and α be as in Example 2.3. It is easy to verify that R is a right α-shifting ring. Taking $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \in R$, we have $\mathbf{A}\alpha(\mathbf{B}) = 0$, but $\alpha(\mathbf{A})\mathbf{CB} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \neq 0$. So R is not a right semicommutative α-ring.

[9, Example 1.5] provides an example which is a right semicommutative α-ring but not a right α-shifting ring.

Let R be a ring and α an endomorphism of R. According to Pourtaherian-Rakhimov [16], a ring R is called satisfying the condition (C_α) if whenever $a\alpha(b) = 0$ with $a, b \in R$, then $ab = 0$.

α-rigid rings and α-compatible rings are such rings.

Proposition 2.9 *Let α be an endomorphism of a ring R. If R satisfies the condition (C_α), then the following statements are equivalent:*

(1) R is a symmetric ring;

(2) R is a right α-symmetric ring;

(3) R is a (right) symmetric α-ring.

Proof (1) $\Leftrightarrow$ (2). It is a straight corollary of [18, Lemma 3.1 (2)].

(2) $\Rightarrow$ (3). Suppose that R is a right α-symmetric ring and $a, b, c \in R$ with $ab\alpha(c) = 0$. Then we have that $abc = 0$ by the condition (C_α) and $ac\alpha(b) = 0$ since R is right α-symmetric. This shows that R is a symmetric α-ring.

(3) $\Rightarrow$ (2). Assume that R is a symmetric α-ring. Then R is a right α-shifting ring by Proposition 2.6, and hence R is reversible by [18, Lemma 3.1(1)]. Now let $abc = 0$ for $a, b, c \in R$. Then $\alpha(ab)\alpha(c) = 0$, and hence $\alpha(c)\alpha(ab) = 0$ by the reversibility. So $\alpha(c)ab = 0 = ab\alpha(c)$ by the condition (C_α). It follows that $ac\alpha(b) = 0$ since R is a symmetric α-ring. This shows that R is a right α-symmetric ring. $\Box$

Corollary 2.10 *Let α be a monomorphism of a ring R. If R is an α-compatible ring, then the following are equivalent:*

(1) R is a symmetric ring;

(2) R is a right α-symmetric ring;

(3) R is a symmetric α-ring.

Proposition 2.11 *Let R be a ring with an endomorphism α. If R is a symmetric α-ring, then the following are equivalent:*

(1) α is a monomorphism;

(2) $\alpha(1) = 1$, where 1 is the identity of R.

Proof (1) $\Rightarrow$ (2). Assume that α is a monomorphism. Then $(1-\alpha(1))\alpha(1) = 1\cdot(1-\alpha(1))\alpha(1) = 0$ implies $1 \cdot 1 \cdot \alpha(1-\alpha(1)) = \alpha(1-\alpha(1)) = 0$. So we have $1-\alpha(1) = 0, \alpha(1) = 1$.

(2) $\Rightarrow$ (1). Suppose that $\alpha(1) = 1$. Let $\alpha(a) = \alpha(b)$ for $a, b \in R$. Then we have $\alpha(a-b) = 1 \cdot 1 \cdot \alpha(a-b) = 0$ and $(a-b)\alpha(1) = a-b = 0$ since R is a symmetric α-ring. Hence α is a monomorphism. $\Box$

A ring R is called Armendariz if whenever polynomials $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for all i, j. For an endomorphism α of a ring R, R is called α-Armendariz if for $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ and $g(x) = b_0 + b_1 x + \cdots + b_m x^m \in R[x;\alpha]$, $fg = 0$ implies $a_i b_j = 0$ for all $0 \le i \le m$ and $0 \le j \le n$.

Proposition 2.12 *Let R be an α-Armendariz ring with an endomorphism α. Then following statements are equivalent:*

(1) $R[x;\alpha]$ is symmetric;

(2) R is α-symmetric;

(3) R is right α-symmetric;

(4) R is symmetric;

(5) R is a (left) symmetric α-ring.

Proof By Kwak [10, Theorem 2.10], we can see that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

Now, we show (2) $\Rightarrow$ (5). Assume that R is α-symmetric. Then R is symmetric by (4). Let $ab\alpha(c) = 0$ for $a, b, c \in R$. Since R is an α-Armendariz ring, we get $abc = 0$ by [7, Proposition 1.3 (2)]. This implies that $ac\alpha(b) = 0$ since R is right α-symmetric, and hence R is a symmetric α-ring. On the other hand, suppose that $a, b, c \in R$ with $\alpha(a)bc = 0$. We have $bc\alpha(a) = 0$ by the symmetry. Since R is an α-Armendariz ring, this implies $bca = 0$, and hence $abc = 0$. So $\alpha(b)ac = 0$ since R is a left α-symmetric ring. Therefore, R is a left symmetric α-ring.

Next, we show (5) $\Rightarrow$ (2). Assume that R is a left symmetric α-ring. If $abc = 0$ for $a, b, c \in R$, then $\alpha(a)bc = 0$ by [7, Proposition 1.3 (1)]. It follows that $\alpha(b)ac = 0$ since R is a left symmetric α-ring. Hence R is left α-symmetric. On the other hand, assume that R is a symmetric α-ring. If $abc = 0$ for $a, b, c \in R$, then $cab = 0$ since R is symmetric, and hence $\alpha(c)ab = 0$ by [7, Proposition 1.3 (1)]. Thus, we get $ab\alpha(c) = 0$. It implies $ac\alpha(b) = 0$ since R is a symmetric α-ring, concluding that R is right α-symmetric. $\square$

3. Extensions of symmetric α-rings

For an endomorphism α of a ring R, an ideal I of R is called α-ideal if $\alpha(I) \subseteq I$. For an α-ideal I of R, the map $\bar{\alpha} : R/I \longrightarrow R/I$ defined by $\bar{\alpha}(\bar{a}) = \overline{\alpha(a)}$ is an endomorphism of the factor ring R/I. Recall that if α is an endomorphism of a ring R, then the map $\bar{\alpha}$: $R[x] \longrightarrow R[x]$ defined by $\bar{\alpha}(\sum_{i=0}^{m} a_i x^i) = \sum_{i=0}^{m} \alpha(a_i)x^i$ is an endomorphism of the polynomial ring $R[x]$ and clearly this map extends α.

Theorem 3.1 *Let R be a reduced ring and $\alpha(1) = 1$. Then R is a symmetric α-ring if and only if $R[x]/(x^n)$ is a symmetric $\bar{\alpha}$-ring, where (x^n) is the ideal generated by x^n, for any positive integer n.*

Proof (1) Suppose that R is a symmetric α-ring and set $S = R[x]/(x^n)$. If $n = 1$, then $S \cong R$. Now we assume $n \geq 2$. Let $A = \sum_{i=0}^{n-1} a_i \mu^i, B = \sum_{j=0}^{n-1} b_j \mu^j, C = \sum_{k=0}^{n-1} c_k \mu^k \in S$ with $AB\bar{\alpha}(C) = 0$, where $\mu = x + (x^n)$. Note that if $i + j + k \geq n$, then $a_i b_j \alpha(c_k)\mu^{i+j+k} = 0$. Hence it suffices to show the cases $i + j + k \leq n - 1$. We proceed by induction on $i + j + k$. From $AB\bar{\alpha}(C) = 0$, we have the following equations:

$$a_0 b_0 \alpha(c_0) = 0, \tag{1}$$

$$a_0 b_0 \alpha(c_1) + a_0 b_1 \alpha(c_0) + a_1 b_0 \alpha(c_0) = 0, \tag{2}$$

$$a_0 b_0 \alpha(c_2) + a_0 b_1 \alpha(c_1) + a_0 b_2 \alpha(c_0) + a_1 b_1 \alpha(c_0) + a_2 b_0 \alpha(c_0) + a_1 b_0 \alpha(c_1) = 0, \tag{3}$$

$$\cdots$$

$$a_0b_0\alpha(c_{n-2}) + \cdots + a_0b_{n-2}\alpha(c_0) + \cdots + a_{n-2}b_0\alpha(c_0) + \cdots + a_1b_0\alpha(c_{n-3}) = 0, \tag{n-1}$$

$$a_0b_0\alpha(c_{n-1}) + \cdots + a_0b_{n-1}\alpha(c_0) + \cdots + a_{n-1}b_0\alpha(c_0) + \cdots + a_1b_0\alpha(c_{n-2}) = 0. \tag{n}$$

Note that reduced ring R is semicommutative, and hence if $ab = 0$ for $a, b \in R$, then $arb = 0$ for any $r \in R$. In the following computations, we use freely this fact.

Multiplying Eq. (2) by a_0b_0 on the left side gives $a_0b_0a_0b_0\alpha(c_1)+a_0b_0a_0b_1\alpha(c_0)+a_0b_0a_1b_0\alpha(c_0) = 0$, then $0 = a_0b_0a_0b_0\alpha(c_1) = (a_0b_0\alpha(c_1))^2$, so $a_0b_0\alpha(c_1) = 0$. Thus we have

$$a_0b_1\alpha(c_0) + a_1b_0\alpha(c_0) = 0. \tag{2$'$}$$

Multiplying Eq. (2$'$) by a_0 on the left side gives $0 = a_0a_0b_1\alpha(c_0) = (a_0b_1\alpha(c_0))^2$, then we obtain $a_1b_0\alpha(c_0) = 0$ and $a_0b_1\alpha(c_0) = 0$.

Thus we obtain $a_1b_0\alpha(c_1) = 0$, $a_0b_1\alpha(c_1) = 0$ and $a_1b_1\alpha(c_0) = 0$ in turn, and hence $a_ic_k\alpha(b_j) = 0$ for $i + j + k = 1$ since R is a symmetric α-ring, so $AC\bar{\alpha}(B) = 0$.

Inductively we assume that $a_ib_j\alpha(c_k) = 0$ for $i + j + k \leq n - 2$. Now for $i + j + k = n - 1$, multiplying Eq. (n) by a_0b_0 on the right side gives $a_0b_0\alpha(c_{n-1})a_0b_0 = 0$ and $a_0b_0\alpha(c_{n-1}) = 0$, so we get

$$a_0b_1\alpha(c_{n-2}) + \cdots + a_0b_{n-1}\alpha(c_0) + \cdots + a_{n-1}b_0\alpha(c_0) + \cdots + a_1b_0\alpha(c_{n-2}) = 0. \tag{n$'$}$$

If we multiply Eq. (n)$'$ by a_0 on the left side and by $\alpha(c_0)$ on the right side, then we get $a_0a_0b_{n-1}\alpha(c_0)\alpha(c_0) = 0$, $a_0b_{n-1}\alpha(c_0) = 0$. Thus we have

$$a_0b_1\alpha(c_{n-2}) + \cdots + a_0b_{n-2}\alpha(c_1) + \cdots + a_{n-1}b_0\alpha(c_0) + \cdots + a_1b_0\alpha(c_{n-2}) = 0. \tag{n$''$}$$

Multiplying Eq. (n)$''$ by $b_0\alpha(c_0)$ on the right side, we get $a_{n-1}b_0\alpha(c_0)b_0\alpha(c_0) = 0$, then $a_{n-1}b_0\alpha(c_0) = 0$. So we have

$$a_0b_1\alpha(c_{n-2}) + \cdots + a_0b_{n-2}\alpha(c_1) + \cdots + a_{n-2}b_1\alpha(c_0) + \cdots + a_1b_0\alpha(c_{n-2}) = 0. \tag{n$'''$}$$

If we multiply Eq. (n)$'''$ on the right side by $b_1\alpha(c_0), b_0\alpha(c_1), \ldots$, and $b_0\alpha(c_{n-2})$, respectively, then we obtain $a_{n-2}b_1\alpha(c_0) = 0, a_{n-2}b_0\alpha(c_1) = 0, \ldots$, and $a_1b_0\alpha(c_{n-2}) = 0$ in turn. This shows that $a_ib_j\alpha(c_k) = 0$, and then $a_ic_k\alpha(b_j) = 0$ for all i, j and k with $i + j + k = n - 1$. It follows that $AC\bar{\alpha}(B) = 0$. Therefore, S is a symmetric $\bar{\alpha}$-ring.

Conversely, since R is a $\bar{\alpha}$-subring of $R[x]/(x^n)$, it is obvious by Proposition 2.6. □

Corollary 3.2 ([8, Theorem 2.3]) *Let R be a reduced ring. Then $R[x]/(x^n)$ is a symmetric ring, where (x^n) is the ideal generated by x^n and n is any positive integer.*

Recall that an element μ of a ring R is right regular if $\mu r = 0$ implies $r = 0$ for $r \in R$. Similarly, left regular is defined, and regular means if it is both left and right regular. Let Δ be a multiplicatively closed subset of R consisting of central regular elements. For an automorphism α of R with $\alpha(\Delta) \subseteq \Delta$, the induced map $\bar{\alpha} : \Delta^{-1} \to \Delta^{-1}$ defined by $\bar{\alpha}(\mu^{-1}a) = \alpha(\mu)^{-1}\alpha(a)$ is also an automorphism.

Proposition 3.3 *Let α be an automorphism of R and Δ be a multiplicatively closed subset of R consisting of central regular elements with $\alpha(\Delta) \subseteq \Delta$. Then R is a symmetric α-ring if and* *only if $\Delta^{-1}R$ is a symmetric $\bar{\alpha}$-ring.*

Proof It is enough to show the necessity by Proposition 2.6 (1).

Assume that R is a symmetric α-ring. Let $AB\bar{\alpha}(C)=0$ for $A=\mu^{-1}a, B=\nu^{-1}b, C=\omega^{-1}c\in\Delta^{-1}R$, where $a,b,c,\mu,\nu,\omega\in R$ with μ,ν,ω regular. Since Δ is contained in the central of R, we have $AB\bar{\alpha}(C)=\mu^{-1}a\nu^{-1}b\alpha(\omega)^{-1}\alpha(c)=(\mu^{-1}\nu^{-1}\alpha(\omega)^{-1})(ab\alpha(c))=(\mu\nu\alpha(\omega))^{-1}(ab\alpha(c))=0$. This implies $ab\alpha(c)=0$, and hence $ac\alpha(b)=0$ since R is a symmetric α-ring. Thus, we have $AC\bar{\alpha}(B)=\mu^{-1}a\omega^{-1}c\alpha(\nu)^{-1}\alpha(b)=(\mu^{-1}\omega^{-1}\alpha(\nu)^{-1})(ac\alpha(b))=(\mu\omega\alpha(\nu))^{-1}(ac\alpha(b))=0$, proving that $\Delta^{-1}R$ is a symmetric $\bar{\alpha}$-ring. □

Corollary 3.4 ([8, Lemma 3.2 (1)]) *Let R be a ring and Δ a multiplicatively closed subset of R consisting of central regular elements. Then R is symmetric if and only if so is $\Delta^{-1}R$.*

The ring of Laurent polynomials in x, coefficients in a ring R, consists of all formal sums $\sum_{i=k}^{n} m_i x^i$ with obvious addition and multiplication, where $m_i\in R$ and k,n are (possibly negative) integers, denote it by $R[x;x^{-1}]$. If α is an endomorphism R, then the map $\bar{\alpha}: R[x;x^{-1}]\to R[x;x^{-1}]$ defined by $\bar{\alpha}(\sum_{i=k}^{n} a_i x^i)=\sum_{i=k}^{n}\alpha(a_i)x^i$ extends α and also is an endomorphism of $R[x;x^{-1}]$.

Proposition 3.5 *Let R be a ring with an endomorphism α. Then $R[x]$ is a symmetric $\bar{\alpha}$-ring if and only if $R[x;x^{-1}]$ is a symmetric $\bar{\alpha}$-ring.*

Proof Let $\Delta=\{1,x,x^2,\ldots\}$. Clearly, Δ is a multiplicatively closed subset of $R[x]$ consisting of central regular elements and $R[x;x^{-1}]=\Delta^{-1}R[x]$. It follows that $R[x;x^{-1}]$ is a symmetric $\bar{\alpha}$-ring by Proposition 3.3. □

Corollary 3.6 ([8, Lemma 3.2 (2)]) *Let R be a ring and Δ a multiplicatively closed subset of R consisting of central regular elements. Then R is symmetric if and only if so is $\Delta^{-1}R$.*

Proposition 3.7 *Let R be a ring with an endomorphism α. If R is an Armendariz ring, then the following are equivalent:*

(1) R is a symmetric α-ring;

(2) $R[x]$ is a symmetric $\bar{\alpha}$-ring;

(3) $R[x;x^{-1}]$ is a symmetric $\bar{\alpha}$-ring.

Proof (1) $\Rightarrow$ (2). Assume that R is a symmetric α-ring and $f(x)=\sum_{i=0}^{l} a_i x^i, g(x)=\sum_{j=0}^{m} b_j x^j, h(x)=\sum_{k=0}^{n} c_k x^k\in R[x]$ with $f(x)g(x)\bar{\alpha}(h(x))=0$. By [7, Lemma 3.5], we have $a_i b_j\alpha(c_k)=0$ for all i,j,k since R is Armendariz. So $a_i c_k\alpha(b_j)=0$ by (1), and hence $f(x)h(x)\bar{\alpha}(g(x))=0$. Therefore, $R[x]$ is a symmetric $\bar{\alpha}$-ring.

(2) $\Rightarrow$ (3). Assume that $R[x]$ is a symmetric $\bar{\alpha}$-ring. For $f(x),g(x),h(x)\in R[x;x^{-1}]$ with $f(x)g(x)\bar{\alpha}(h(x))=0$, there exists a positive integer n such that $f_1(x)=f(x)x^n, g_1(x)=g(x)x^n, h_1(x)=h(x)x^n\in R[x]$ and $f_1(x)g_1(x)\bar{\alpha}(h_1(x))=0$. Since $R[x]$ is a symmetric $\bar{\alpha}$-ring, $f_1(x)h_1(x)\bar{\alpha}(g_1(x))=0$, then $f(x)h(x))\bar{\alpha}(g(x))=x^{-3n}f_1(x)h_1(x)\bar{\alpha}(g_1(x))=0$. This proves that $R[x;x^{-1}]$ is a symmetric $\bar{\alpha}$-ring.

(3) $\Rightarrow$ (1). It follows from the fact that R is a $\bar{\alpha}$-subring of $R[x;x^{-1}]$. $\square$

Corollary 3.8 ([8, Proposition 3.4]) *Let R be an Armendariz ring. Then the following are equivalent:*

(1) *R is a symmetric ring;*

(2) *$R[x]$ is a symmetric ring;*

(3) *$R[x;x^{-1}]$ is a symmetric ring.*

A ring R is called right *Ore* if given $a,b \in R$ with b regular, then there exist $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$. It is a well-known fact that R is a right *Ore* ring if and only if the classical right quotient ring of R exists.

Suppose that the classical right quotient ring $Q(R)$ of R exists. Then for an automorphism α of R and any $ab^{-1} \in Q(R)$ where $a,b \in R$ with b regular, the induced map $\bar{\alpha}: Q(R) \to Q(R)$ defined by $\bar{\alpha}(ab^{-1}) = \alpha(a)\alpha(b)^{-1}$ is an automorphism of $Q(R)$.

Theorem 3.9 *Let R be a right Ore ring, α an automorphism of R and $Q(R)$ the classical right quotient ring of R. Then R is a symmetric α-ring if and only if $Q(R)$ is a symmetric $\bar{\alpha}$-ring.*

Proof It suffices to establish the necessity by Proposition 2.6 (1).

Assume that R is a symmetric α-ring. Let $A = a\mu^{-1}, B = b\nu^{-1}, C = c\omega^{-1} \in Q(R)$ with $AB\bar{\alpha}(C) = a\mu^{-1}b\nu^{-1}\alpha(c)\alpha(\omega)^{-1} = 0$, where $a,b,c,\mu,\nu,\omega \in R$ with μ,ν,ω regular. Now, there exist $b_1, \mu_1 \in R$ with μ_1 regular such that $b\mu_1 = \mu b_1, \mu^{-1}b = b_1\mu_1^{-1}$. Hence, $AB\bar{\alpha}(C) = ab_1\mu_1^{-1}\nu^{-1}\alpha(c)\alpha(\omega)^{-1} = 0$. Next, for $\alpha(c), v \in R$ there exist $c_1, \nu_1 \in R$ with ν_1 regular such that $\alpha(c)\nu_1 = \nu c_1, \nu^{-1}\alpha(c) = c_1\nu_1^{-1}$, so $AB\bar{\alpha}(C) = ab_1\mu_1^{-1}c_1\nu_1^{-1}\alpha(\omega)^{-1} = 0$. Similarly, also there exist $c_2, \mu_2 \in R$ with μ_2 regular such that $c_1\mu_2 = \mu_1c_2, \mu_1^{-1}c_1 = c_2\mu_2^{-1}$. Thus, we obtain that $AB\bar{\alpha}(C) = ab_1c_2\mu_2^{-1}\nu_1^{-1}\alpha(\omega)^{-1} = 0$ and hence $ab_1c_2 = 0$. This implies $0 = ab_1c_2\mu = a\mu b_1c_2 = ab\mu_1c_2 = abc_2\mu_1$, and $0 = abc_2 = abc_2\mu_1 = ab\mu_1c_2 = abc_1\mu_2$. So we have $0 = abc_1 = abc_1\nu = ab\nu c_1 = ab\alpha(c)\nu_1$. It follows that $ab\alpha(c) = 0$, and hence $ac\alpha(b) = 0$ since R is a symmetric α-ring.

Similarly, there exist $c_3, \mu_3, b_2, \omega_2, b_4, \mu_4 \in R$ with μ_3, ω_2, μ_4 regular such that $c\mu_3 = \mu c_3, \alpha(b)\omega_2 = \omega b_2, b_2\mu_4 = \mu_3b_4$, and

$$AC\bar{\alpha}(B) = ac_3\mu_3^{-1}\omega^{-1}\alpha(b)\alpha(\nu)^{-1} = ac_3\mu_3^{-1}b_2\omega_2^{-1}\alpha(\nu)^{-1} = ac_3b_4\mu_4^{-1}\omega_2^{-1}\alpha(\nu)^{-1}.$$

From $ac\alpha(b) = 0$, we have $0 = ac\alpha(b)\omega_2 = ac\omega b_2 = acb_2\omega$, and hence $0 = acb_2 = acb_2\mu_4 = ac\mu_3b_4 = acb_4\mu_3$. It follows that $0 = acb_4 = acb_4\mu_3 = ac\mu_3b_4 = a\mu c_3b_4 = ac_3b_4\mu$, and hence $ac_3b_4 = 0$. Now we have $AC\bar{\alpha}(B) = 0$, proving that $Q(R)$ is a symmetric $\bar{\alpha}$-ring. $\square$

Corollary 3.10 ([8, Theorem 4.1]) *Let R be a ring and Δ a multiplicatively closed subset of R consisting of central regular elements. Then R is symmetric if and only if so is $\Delta^{-1}R$.*

Theorem 3.11 *Let R be a right Ore ring, α an automorphism of R and $Q(R)$ the classical right quotient ring of R. Assume that $aR\alpha(a) = 0$ implies $a = 0$ for any $a \in R$. Then the following statements are equivalent:*

(1) *R is a symmetric ring;*

(2) R is a right α-symmetric ring;
(3) R is a symmetric α-ring;
(4) Q is a symmetric ring;
(5) Q is an $\bar{\alpha}$-symmetric ring;
(6) Q is a symmetric $\bar{\alpha}$-ring.

Proof We claim that if R satisfies that $aR\alpha(a) = 0$ implies $a = 0$ for any $a \in R$, then $Q(R)$ satisfies that $AQ(R)\bar{\alpha}(A) = 0$ implies $A = 0$ for any $A \in Q(R)$. Let $A = a\mu^{-1} \in Q(R)$ with $AQ(R)\bar{\alpha}(A) = 0$. Then $0 = a\mu^{-1}Q(R)\bar{\alpha}(a\mu^{-1}) = aQ(R)\alpha(a)\alpha(\mu^{-1}) = 0$, since $\mu^{-1}Q(R) = Q(R)$. This implies $aQ(R)\alpha(a) = 0$, and so $aR\alpha(a) = 0$. By assumption, we get $a = 0$ and hence $A = 0$, completing the claim. Thus the proof is done by Proposition 2.5, Proposition 2.9, Theorem 3.9 and the claim. □

For an algebra R over a nonzero commutative ring S, the Dorroh extension of R by S is the ring $D = R \times S$ with operations $(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$ and $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$, where $r_i \in R, s_i \in S$. For an endomorphism α of R and the Dorroh extension of R by S, the nonzero map $\bar{\alpha} : D \to D$ defined by $\bar{\alpha}(r, s) = (\alpha(r), s)$ is an endomorphism of D.

Theorem 3.12 *Let S be a commutative domain, α be a monomorphism of a ring R and D be the Dorroh extension of R by S. If R is a symmetric α-ring, then D is a symmetric $\bar{\alpha}$-ring.*

Proof Assume that R is a symmetric α-ring. Let $D_1 = (r_1, s_1), D_2 = (r_2, s_2), D_3 = (r_3, s_3) \in D$ with $D_1D_2\bar{\alpha}(D_3) = 0$. Then we have $s_1s_2s_3 = 0$ and $r_1r_2\alpha(r_3) + s_1r_2\alpha(r_3) + s_2r_1\alpha(r_3) + s_1s_2\alpha(r_3) + s_3r_1r_2 + s_3s_1r_2 + s_3s_2r_1 = 0$. Since S is a domain, $s_1 = 0$ or $s_2 = 0$ or $s_3 = 0$. If $s_1 = 0$, then $r_1r_2\alpha(r_3) + s_2r_1\alpha(r_3) + s_3r_1r_2 + s_3s_2r_1 = 0$. Since R is a symmetric α-ring with a monomorphism α, we have $\alpha(1) = 1$ by Proposition 2.10. It follows that $r_1(r_2\alpha(r_3) + s_2\alpha(r_3) + s_3r_2 + s_3s_2) = r_1(r_2 + s_2 \cdot 1)(\alpha(r_3) + s_3\alpha(1)) = r_1(r_2 + s_2)\alpha(r_3 + s_3 \cdot 1) = 0$. Then $r_1(r_3+s_3)\alpha(r_2+s_2) = 0$, and hence $r_1r_3\alpha(r_2)+r_1r_3s_2+r_1s_3\alpha(r_2)+r_1s_3s_2 = 0$. So $D_1D_3\bar{\alpha}(D_2) = (r_1r_3\alpha(r_2)+s_1r_3\alpha(r_2)+s_3r_1\alpha(r_2)+s_1s_3\alpha(r_2)+s_2r_1r_3+s_2s_1r_3+s_2s_3r_1, s_1s_3s_2) = (r_1r_3\alpha(r_2)+s_3r_1\alpha(r_2) + s_2r_1r_3 + s_2s_3r_1, 0) = (r_1r_3\alpha(r_2) + r_1r_3s_2 + r_1s_3\alpha(r_2) + r_1s_3s_2, 0) = 0$. If $s_2 = 0$ or $s_3 = 0$, we also have $D_1D_3\bar{\alpha}(D_2) = 0$, including that the Dorroh extension D of R by S is a symmetric $\bar{\alpha}$-ring. □

Corollary 3.13 ([8, Proposition 4.2 (1)]) *Let R be an algebra over a commutative ring S and D be the Dorroh extension of R by S. If R is symmetric and S is a domain, then D is also symmetric.*

4. Weak symmetric α-rings

For a ring R, we denote by $\mathrm{Nil}_*(R)$ its lower nil-radical, $\mathrm{Nil}^*(R)$ its upper nil-radical and L-rad(R) its Levitzki radical. For a nonempty subset M of a ring R, the symbol $\langle M\rangle$ denotes the subring (may not with 1) generated by M. A ring R is called NI if $\mathrm{nil}(R) = \mathrm{Nil}^*(R)$, and a ring R is called 2-primal if $\mathrm{nil}(R) = \mathrm{Nil}_*(R)$. According to Chen et al.[4], a ring R is called weakly 2-primal if $\mathrm{nil}(R) = L\text{-rad}(R)$, and following Hong et al. [5], a ring R is called locally 2-primal if each finite subset generates a 2-primal ring. The following implications hold: reduced $\Rightarrow$ symmetric $\Rightarrow$ semicommutative $\Rightarrow$ 2-primal $\Rightarrow$ locally 2-primal $\Rightarrow$ weakly 2-primal $\Rightarrow$ NI-ring.

As an extension of weak symmetric rings, we now introduce the notion of a weak symmetric α-ring.

Definition 4.1 *Let α be an endomorphism of a ring R. A ring R is called (right) weak symmetric α-ring if $ab\alpha(c) \in \mathrm{nil}(R)$ implies $ac\alpha(b) \in \mathrm{nil}(R)$ for $a, b, c \in R$.*

Similarly, a ring R is said to be a left weak symmetric α-ring if $\alpha(a)bc \in \mathrm{nil}(R)$, then $\alpha(b)ac \in \mathrm{nil}(R)$ for $a, b, c \in R$.

It is easy to see that every subring S with $\alpha(S) \subseteq S$ of a (left) weak symmetric α-ring is also a (left) weak symmetric α-ring.

Obviously, if $\alpha = id_R$, then a (left) weak symmetric α-ring is a weak symmetric ring. Example 2.3 provides that if $\alpha \neq id_R$, there exists a weak symmetric ring which is not a weak symmetric α-ring.

Lemma 4.2 *Let R be an (α, δ)-compatible ring. Then we have the following:*

(1) If $ab = 0$, then $af_i^j(b) = 0$ for all $0 \leqslant i \leqslant j$ and $a, b \in R$;

(2) If $abc = 0$, then $a\delta(b)c = 0$, $a\delta^n\alpha^m(b)c = 0$, $af_i^j(b)c = 0$ for all $0 \leqslant i \leqslant j$, any non-negative integer m, n and $a, b, c \in R$;

(3) For $a, b \in R$ and any positive integer m, $ab \in \mathrm{nil}(R)$ if and only if $a\alpha^m(b) \in \mathrm{nil}(R)$;

(4) If $ab \in \mathrm{nil}(R)$, then $a\delta^m(b) \in \mathrm{nil}(R)$ for any $a, b \in R$ and any positive integer m;

(5) If R is an NI ring, then $ab \in \mathrm{nil}(R)$ implies $af_i^j(b) \in \mathrm{nil}(R)$ for all $0 \leqslant i \leqslant j$ and $a, b \in R$;

(6) If R is a weak symmetric α-ring, then $ab\alpha(c) \in \mathrm{nil}(R)$ implies $ac\alpha^n(b) \in \mathrm{nil}(R)$ for any positive integer n and $a, b, c \in R$.

Proof (1) Since R is (α, δ)-compatible, $ab = 0 \Longrightarrow a\alpha(b) = 0, a\delta(b) = 0 \Longrightarrow a\alpha^i(b) = 0, a\delta^j(b) = 0$ for all positive integer i, j. This implies that $af_i^j(b) = 0$ for all $0 \leqslant i \leqslant j$ and $a, b \in R$.

(2) First, we have $abc = 0 \Longrightarrow \alpha(ab)c = 0 \Longrightarrow \alpha(ab)\delta(c) = 0 \Longrightarrow \alpha(a)\alpha(b)\delta(c) = 0 \Longrightarrow 1.\alpha(a)\alpha(\alpha(b)\delta(c)) = 0 \Longrightarrow a\alpha(b)\delta(c) = 0$ and $abc = 0 \Longrightarrow a\delta(bc) = 0 \Longrightarrow a\delta(b)c = 0$. On the other hand, $abc = 0 \Longrightarrow a\alpha(bc) = 0 \Longrightarrow a\alpha(b)c = 0 \Longrightarrow a\alpha^m(b)c = 0 \Longrightarrow a\delta^n\alpha^m(b)c = 0$ for any positive integer m, n. Thus we obtain that $a\alpha^i\delta^j(b)c = 0$, and hence $af_i^j(b)c = 0$ for all $0 \leqslant i \leqslant j$.

(3) It is an immediate consequence of [13, Lemma 3.1] and [15, Lemma 2.8].

(4) Since $ab \in \mathrm{nil}(R)$, there exists some positive integer k such that $(ab)^k = 0$. In the following computations, we use freely (2):

$$\begin{aligned}(ab)^k &= ab(ab\cdots ab) = 0 \Rightarrow a\delta(b)(ab\cdots ab) = (a\delta(b)a)b(ab\cdots ab) = 0\\ &\Rightarrow (a\delta(b)a)\delta(b)(ab\cdots ab) = 0 \Rightarrow \cdots\\ &\Rightarrow (a\delta(b))^{k-1}ab1 = 0 \Rightarrow (a\delta(b))^k = 0.\end{aligned}$$

This implies that $a\delta(b) \in \mathrm{nil}(R)$, and hence $a\delta^m(b) \in \mathrm{nil}(R)$ for any $a, b \in R$ and any positive integer m.

(5) $ab \in \mathrm{nil}(R) \Longrightarrow a\alpha^i(b), a\delta^j(b) \in \mathrm{nil}(R) \Longrightarrow a\delta^j\alpha^i(b), a\alpha^i\delta^j(b) \in \mathrm{nil}(R)$ for all $i \geqslant 0$ and $j \geqslant 0$ by (3) and (4). Since R is NI, we have $af_i^j(b) \in \mathrm{nil}(R)$ for all $0 \leqslant i \leqslant j$.

(6) Since R is a weak symmetric α-ring, $ab\alpha(c) \in \mathrm{nil}(R) \Longrightarrow ac\alpha(b) \in \mathrm{nil}(R) \Longrightarrow ac\alpha^n(b) \in \mathrm{nil}(R)$ by (4), for $a, b, c \in R$ and any positive integer n. □

Lemma 4.3 *Let R be a weakly 2-primal ring. If R is (α, δ)-compatible, and $f(x) = \sum_{i=0}^n a_i x^i \in R[x; \alpha, \delta]$, then $f(x) \in \mathrm{nil}(R[x; \alpha, \delta])$ if and only if $a_i \in \mathrm{nil}(R)$ for each $0 \leq i \leq n$, that is, we have,* $\mathrm{nil}(R[x; \alpha, \delta]) = \mathrm{nil}(R)[x; \alpha, \delta]$.

Proof Let $f(x) = \sum_{i=0}^n a_i x^i \in \mathrm{nil}(R[x; \alpha, \delta])$. Then there exists some positive integer k such that $0 = f(x)^k = (a_0 + a_1 x + \cdots + a_n x^n)^k =$ "lower terms" $+ a_n\alpha^n(a_n)\alpha^{2n}(a_n)\cdots\alpha^{(k-1)n}(a_n)x^{nk}$. Thus, we have that

$$\begin{aligned}
&a_n\alpha^n(a_n)\alpha^{2n}(a_n)\cdots\alpha^{(k-1)n}(a_n) = 0\\
&\quad\Rightarrow a_n\alpha^n((a_n)\alpha^n(a_n)\cdots\alpha^{(k-2)n}(a_n)) = 0\\
&\quad\Rightarrow a_n^2\alpha^n(a_n)\cdots\alpha^{(k-3)n}(a_n)\alpha^{(k-2)n}(a_n) = 0\\
&\quad\Rightarrow a_n^3\alpha^n(a_n)\cdots\alpha^{(k-3)n}(a_n) = 0\\
&\quad\Rightarrow \cdots \Rightarrow a_n^k = 0 \Rightarrow a_n \in \mathrm{nil}(R).
\end{aligned}$$

By Lemma 4.2, $a_n = 1 \cdot a_n \in \mathrm{nil}(R)$ implies $1 \cdot f_s^t(a_n) = f_s^t(a_n) \in \mathrm{nil}(R)$ for all $0 \leqslant s \leqslant t$. Let $Q = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$. Then we have

$$\begin{aligned}
0 &= (Q + a_n x^n)^k = (Q + a_n x^n)(Q + a_n x^n)\cdots(Q + a_n x^n)\\
&= (Q^2 + Q \cdot a_n x^n + a_n x^n \cdot Q + a_n x^n \cdot a_n x^n)(Q + a_n x^n)\cdots(Q + a_n x^n)\\
&= \cdots = Q^k + \Delta,
\end{aligned}$$

where $\Delta \in R[x; \alpha, \delta]$. Notice that the coefficients of Δ can be written as sums of monomials in a_i and $f_u^v(a_j)$ where $a_i, a_j \in \{a_0, a_1, \ldots, a_n\}$ and $0 \leqslant u \leqslant v$ are positive integers, and each monomial has a_n or $f_s^t(a_n)$. Since $\mathrm{nil}(R)$ is an ideal of R, we obtain that each monomial is in $\mathrm{nil}(R)$, and then $\Delta \in \mathrm{nil}(R)[x; \alpha, \delta]$. Thus we obtain $(a_0 + a_1 x + \cdots + a_{n-1}x^{n-1})^k =$ "lower terms" $+ a_{n-1}\alpha^{n-1}(a_{n-1})\cdots\alpha^{(n-1)(k-1)}(a_{n-1})x^{(n-1)k} \in \mathrm{nil}(R)[x; \alpha, \delta]$. By Lemma 4.2, we have

$$\begin{aligned}
&a_{n-1}\alpha^{n-1}(a_{n-1})\cdots\alpha^{(n-1)(k-1)}(a_{n-1}) \in \mathrm{nil}(R)\\
&\quad\Rightarrow a_{n-1}\alpha^{n-1}(a_{n-1}\alpha^{n-1}(a_{n-1})\cdots\alpha^{(n-1)(k-2)}(a_{n-1})) \in \mathrm{nil}(R)\\
&\quad\Rightarrow a_{n-1}^2\alpha^{n-1}(a_{n-1})\cdots\alpha^{(n-1)(k-2)}(a_{n-1}) \in \mathrm{nil}(R)\\
&\quad\Rightarrow a_{n-1}^3\alpha^{n-1}(a_{n-1})\cdots\alpha^{(n-1)(k-3)}(a_{n-1}) \in \mathrm{nil}(R)\\
&\quad\Rightarrow \cdots \Rightarrow a_{n-1}^{k-1} \in \mathrm{nil}(R) \Rightarrow a_{n-1} \in \mathrm{nil}(R).
\end{aligned}$$

Using induction on n, we have $a_i \in \mathrm{nil}(R)$ for all $0 \leqslant i \leqslant n$.

Conversely, consider the finite subset $\{a_0, a_1, \ldots, a_n\}$. Since R is weakly 2-primal, $\mathrm{nil}(R) = L - rad(R)$ and $\langle a_0, a_1, \ldots, a_n\rangle$ is nilpotent subring of R. So, there exists a positive integer k such that any product of k elements $a_{i1}a_{i2}\cdots a_{ik}$ from $\{a_0, a_1, \ldots, a_n\}$ is zero. Note that the coefficients of $f(x)^{k+1} = (\sum_{i=0}^{n} a_i x^i)^{k+1}$ in $R[x;\alpha,\delta]$ can be written as sums of monomials of length $k+1$ in a_i and $f_u^v(a_j)$, where $a_i, a_j \in \{a_0, a_1, \ldots, a_n\}$ and $0 \leqslant u \leqslant v$ are positive integers. For each monomial $a_{i_1} f_{s_2}^{t_2}(a_{i_2}) \cdots f_{s_{k+1}}^{t_{k+1}}(a_{i_{k+1}})$, where $a_{i_1}, a_{i_2}, \ldots, a_{i_{k+1}} \in \{a_0, a_1, \ldots, a_n\}$ and t_j, s_j ($t_j \geqslant s_j, 2 \leqslant j \leqslant k+1$) are nonnegative integers, we obtain $a_{i_1} f_{s_2}^{t_2}(a_{i_2}) \cdots f_{s_{k+1}}^{t_{k+1}}(a_{i_{k+1}}) = 0$ by Lemma 4.2. Thus, we have $f(x)^{k+1} = 0$ and hence $f(x) \in \mathrm{nil}(R[x;\alpha,\delta])$. $\square$

Proposition 4.4 *Let R be an (α,δ)-compatible weakly 2-primal ring. Then for $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$, $h(x) = \sum_{k=0}^{p} c_k x^k \in R[x;\alpha,\delta]$, and $c \in R$, we have the following:*

(1) $fg \in \mathrm{nil}(R[x;\alpha,\delta]) \Leftrightarrow a_i b_j \in \mathrm{nil}(R)$ for all $0 \leqslant i \leqslant m$, $0 \leqslant j \leqslant n$;

(2) $fgc \in \mathrm{nil}(R[x;\alpha,\delta]) \Leftrightarrow a_i b_j c \in \mathrm{nil}(R)$ for all $0 \leqslant i \leqslant m$, $0 \leqslant j \leqslant n$;

(3) $fgh \in \mathrm{nil}(R[x;\alpha,\delta]) \Leftrightarrow a_i b_j c_k \in \mathrm{nil}(R)$ for all $0 \leqslant i \leqslant m$, $0 \leqslant j \leqslant n$ and $0 \leqslant k \leqslant p$.

Proof We refer to the proof of [15, Theorem 2.11] to show the proposition.

(1) ($\Rightarrow$) Let $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x;\alpha,\delta]$ such that $fg \in \mathrm{nil}(R[x;\alpha,\delta])$. Then

$$\begin{aligned} f(x)g(x) =& (\sum_{i=0}^{m} a_i x^i)(\sum_{j=0}^{n} b_j x^j) \\ =& \sum_{i=0}^{m} a_i f_0^i(b_0) + (\sum_{i=1}^{m} a_i f_1^i(b_0) + \sum_{i=0}^{m} a_i f_0^i(b_1))x + \cdots + (\sum_{s+t=k}(\sum_{i=s}^{m} a_i f_s^i(b_t)))x^k + \\ & \cdots + a_m \alpha^m(b_n)x^{m+n} \in \mathrm{nil}(R[x;\alpha,\delta]). \end{aligned}$$

Thus, we have the following system of equations by Lemma 4.3:

$$\Omega_{m+n} = a_m \alpha^m(b_n) \in \mathrm{nil}(R); \tag{1}$$

$$\Omega_{m+n-1} = a_m \alpha^m(b_{n-1}) + a_{m-1}\alpha^{m-1}(b_n) + a_m f_{m-1}^m(b_n) \in \mathrm{nil}(R); \tag{2}$$

$$\Omega_{m+n-2} = a_m \alpha^m(b_{n-2}) + \sum_{i=m-1}^{m} f_{m-1}^i(b_{n-1}) + \sum_{i=m-2}^{m} f_{m-2}^i(b_n) \in \mathrm{nil}(R); \tag{3}$$

$$\vdots$$

$$\Omega_k = \sum_{s+t=k}(\sum_{i=s}^{m} a_i f_s^i(b_t)) \in \mathrm{nil}(R); \tag{4}$$

From Lemma 4.2 and Eq.(1), we have $a_m b_n \in \mathrm{nil}(R)$. Next we show that $a_i b_n \in \mathrm{nil}(R)$ for all $0 \leqslant i \leqslant m$. If we multiply Eq.(2) on the left side by b_n, then $b_n a_{m-1}\alpha^{m-1}(b_n) \in \mathrm{nil}(R)$ since $\mathrm{nil}(R)$ is an ideal of R. Thus by Lemma 4.2, we obtain $b_n a_{m-1} b_n \in \mathrm{nil}(R)$, and so $b_n a_{m-1} \in \mathrm{nil}(R)$, $a_{m-1} b_n \in \mathrm{nil}(R)$. If we multiply Eq.(3) on the left side by b_n, since $\mathrm{nil}(R)$ is an ideal of R, we obtain

$$\begin{aligned} b_n a_{m-2}\alpha^{m-2}(b_n) =& b_n \Omega_{m+n-2} - b_n a_m \alpha^m(b_{n-2}) - b_n a_{m-1}\alpha^{m-1}(b_{n-1}) - b_n a_m f_{m-1}^m(b_{n-1}) - \\ & b_n a_{m-1} f_{m-2}^{m-1}(b_n) - b_n a_m f_{m-2}^m(b_n) \in \mathrm{nil}(R). \end{aligned}$$

Thus we obtain $b_n a_{m-2} \in \mathrm{nil}(R)$ and $a_{m-2} b_n \in \mathrm{nil}(R)$. Continuing this procedure yields $a_i b_n \in \mathrm{nil}(R)$ for all $0 \leqslant i \leqslant m$, and so $a_i f_s^t(b_n) \in \mathrm{nil}(R)$ for any $0 \leqslant s \leqslant t$ and $0 \leqslant i \leqslant m$ by Lemma 4.2.

Thus it is easy to verify that $(\sum_{i=0}^{m} a_i x^i)(\sum_{j=0}^{n-1} b_j x^j) \in \mathrm{nil}(R[x;\alpha,\delta])$. Applying the preceding method repeatedly, we obtain that $a_i b_j \in \mathrm{nil}(R)$ for all $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n$.

($\Leftarrow$) Let $a_i b_j \in \mathrm{nil}(R)$ for all i, j. Then $a_i f_s^i(b_j) \in \mathrm{nil}(R)$ for all i, j and all position integer $0 \leqslant s \leqslant i$ by Lemma 4.2. Thus $\sum_{s+t=k}(\sum_{i=s}^{m} a_i f_s^i(b_t)) \in \mathrm{nil}(R)$, $k = 0, 1, 2, \ldots, m+n$. Hence $fg = \sum_{k=0}^{m}(\sum_{s+t=k}(\sum_{i=s}^{m} a_i f_s^i(b_t)))x^k \in \mathrm{nil}(R[x;\alpha,\delta])$ by Lemma 4.3.

(2) ($\Rightarrow$) We have

$$\begin{aligned} g(x)c &= (\sum_{j=0}^{n} b_j x^j)c = \sum_{j=0}^{n} b_j f_0^j(c) + (\sum_{j=1}^{n} b_j f_1^j(c))x + \cdots + (\sum_{j=s}^{n} b_j f_s^j(c))x^s + \cdots + b_n \alpha^n(c) x^n \\ &= \Delta_0 + \Delta_1 x + \cdots + \Delta_s x^s + \cdots + \Delta_n x^n, \end{aligned}$$

where $\Delta_s = \sum_{j=s}^{n} b_j f_s^j(c)$, $0 \leqslant s \leqslant n$. By (1) we have $a_i \Delta_s = a_i(\sum_{j=s}^{n} b_j f_s^j(c)) \in \mathrm{nil}(R)$ for $0 \leqslant i \leqslant m$ and $0 \leqslant s \leqslant n$.

For $s = n$, we have $a_i \Delta_n = a_i b_n \alpha^n(c) \in \mathrm{nil}(R)$ for all $0 \leqslant i \leqslant m$. Then by Lemma 4.2, we obtain $a_i b_n c \in \mathrm{nil}(R)$ for all $0 \leqslant i \leqslant m$.

For $s = n-1$, we have $a_i \Delta_{n-1} = a_i b_{n-1} \alpha^{n-1}(c) + a_i b_n f_{n-1}^n(c) \in \mathrm{nil}(R)$ for all $0 \leqslant i \leqslant m$. Since $a_i b_n c \in \mathrm{nil}(R)$, we have $a_i b_n f_{n-1}^n(c) \in \mathrm{nil}(R)$ by Lemma 4.2. Thus, $a_i b_{n-1} \alpha^{n-1}(c) = a_i \Delta_{n-1} - a_i b_n f_{n-1}^n(c) \in \mathrm{nil}(R)$, and hence $a_i b_n c \in \mathrm{nil}(R)$ for all $0 \leqslant i \leqslant m$.

Now suppose that k is a positive integer such that $a_i b_j c \in \mathrm{nil}(R)$ for all $0 \leqslant i \leqslant m$ when $j > k$. We show that $a_i b_k c \in \mathrm{nil}(R)$ for all $0 \leqslant i \leqslant m$.

If $s = k$, then for all $0 \leqslant i \leqslant m$, we have

$$a_i \Delta_k = a_i b_k \alpha^k(c) + a_i b_{k+1} f_k^{k+1}(c) + \cdots + a_i b_n f_k^n(c) \in \mathrm{nil}(R).$$

Since $a_i b_j c \in \mathrm{nil}(R)$ for $0 \leqslant i \leqslant m$ and $k < j \leqslant n$, we have $a_i b_j f_k^j(c) \in \mathrm{nil}(R)$ for all $0 \leqslant i \leqslant m$ and $k < j \leqslant n$ by Lemma 4.2. It follows that $a_i b_k \alpha^k(c) \in \mathrm{nil}(R)$, and hence $a_i b_k c \in \mathrm{nil}(R)$ for all $0 \leqslant i \leqslant m$. Therefore, by induction we obtain $a_i b_j c \in \mathrm{nil}(R)$ for all $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n$.

($\Leftarrow$) Suppose $a_i b_j c \in \mathrm{nil}(R)$ for all $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n$. Then $a_i b_j f_s^j(c) \in \mathrm{nil}(R)$ and so $a_i \sum_{j=s}^{n}(b_j f_s^j(c)) \in \mathrm{nil}(R)$ for all $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n$. By (1), we obtain $fgc \in \mathrm{nil}(R[x;\alpha,\delta])$. $\square$

Theorem 4.5 *Let R be a weakly 2-primal ring. If R is (α,δ)-compatible, then R is a weak symmetric α-ring if and only if the Ore extension $R[x;\alpha,\delta]$ of R is a weak symmetric $\bar{\alpha}$-ring.*

Proof Suppose that $R[x;\alpha,\delta]$ is a weak symmetric $\bar{\alpha}$-ring. Since S is a subring of $R[x;\alpha,\delta]$ with $\bar{\alpha}(S) \subseteq S$, and hence is also a weak symmetric $\bar{\alpha}$-ring. Thus R is a weak symmetric α-ring.

Conversely, assume that R is a weak symmetric α-ring. Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n, g(x) = b_0 + b_1 x + \cdots + b_m x^m$, and $h(x) = c_0 + c_1 x + \cdots + c_l x^l \in R[x;\alpha,\delta]$ with $fg\bar{\alpha}(h) \in \mathrm{nil}(R[x;\alpha,\delta])$. Then by Lemma 4.3, we have $a_i b_j \alpha(c_k) \in \mathrm{nil}(R)$ for all i, j, k, and hence $a_i c_k \alpha(b_j) \in \mathrm{nil}(R)$ for all i, j, k since R is a weak symmetric α-ring. This implies $fh\bar{\alpha}(g) \in \mathrm{nil}(R[x;\alpha,\delta])$ by Lemma 4.3, so $R[x;\alpha,\delta]$ is a weak symmetric $\bar{\alpha}$-ring. $\square$

Corollary 4.6 ([15, Theorem 2.12]) *Let R be a reversible ring. If R is α-compatible, then R is a weak symmetric α-ring if and only if the skew polynomial ring $R[x;\alpha]$ is a weak symmetric*

$\bar{\alpha}$-ring.

Corollary 4.7 *Let R be a weakly 2-primal ring. If R is α-compatible, then R is a weak symmetric α-ring if and only if the skew polynomial ring $R[x;\alpha]$ is a weak symmetric $\bar{\alpha}$-ring.*

Corollary 4.8 *Let R be a weakly 2-primal ring. If R is δ-compatible, then R is a weak symmetric α-ring if and only if the differential polynomial ring $R[x;\delta]$ is a weak symmetric $\bar{\alpha}$-ring.*

Corollary 4.9 *Let R be a weakly 2-primal ring. Then R is a weak symmetric α-ring if and only if the polynomial ring $R[x]$ is a weak symmetric $\bar{\alpha}$-ring.*

Corollary 4.10 ([15, Corollary 2.13]) *Let R be a reversible ring. Then we have the following:*

(1) R is weak symmetric if and only if $R[x]$ is weak symmetric;

(2) If R is α-compatible, then R is weak symmetric if and only if $R[x]$ is weak symmetric;

(3) If R is δ-compatible, then R is weak symmetric if and only if differential polynomial ring $R[x,\delta]$ is weak symmetric.

References

[1] D. D. ANDERSON, V. CAMILLO. *Semigroups and rings whose zero products commute*. Comm. Algebra, 1999, **27**(6): 2847–2852.

[2] M. BASER, A. HARMANCI, T. K. KWAK. *Generalized semicommutative rings and their extensions*. Bull. Korean Math. Soc., 2008, **45**(2): 285–297.

[3] M. BASER, F. KAYNARCA, T. K. KWAK. Ring endomorphisms with the reversible condition. Comm. Korean Math. Soc., 2010, **25**(3): 349–364.

[4] Weixin CHEN, Shuying CUI. *On weakly semicommutative rings. Comm. Math. Res.*, 2011, **27**(2): 179–192.

[5] C. Y. HONG, H. K. KIM, N. K. KIM, et al. *Rings whose nilpotent elements form a Levitzki radical ring*. Comm. Algebra, 2007, **35**(4): 1379–1390.

[6] C. Y. HONG, N. K. KIM, T. K. KWAK. *Ore extensions of Baer and p.p.-rings*. J. Pure Appl. Algebra, 2000, **151**(3): 215–226.

[7] C. Y. HONG, T. K. KWAK, S. T. RIZVI. *Extensions of generalized Armendariz rings*. Algebra Colloq., 2006, **13**(2): 253–266.

[8] C. HUH, H. K. KIM, N. K. KIM, et al. *Basic examples and extensions of symmetric rings*. J. Pure Appl. Algebra, 2005, **202**(1-3): 154–167.

[9] N. K. KIM, Y. LEE. *extensions of reverseble rings*. J. Pure Appl. Algebra, 2003, **185**: 207–223.

[10] T. K. KWAK. *Extensions of extended symmetric ring*. Bull. Korean Math. Soc., 2007, **44**(4): 777–788.

[11] G. KAFKAS, B. UNGOR, S. HALICIOGLU, et al. *Generalized symmetric rings*. Algebra Discrete Math., 2011, **12**(2): 72–84.

[12] J. LAMBEK. *On the representation of modules by sheaves of factor modules*. Canad. Math. Bull., 1971, **14**(3): 359–368.

[13] Li LIANG, Limin WANG, Zhongkui LIU. *On a generalization of semicommutative rings*. Taiwanese J. Math., 2007, **11**(5): 1359–1368.

[14] G. MARKS. *Reversible and symmetric rings*. J. Pure Appli. Algebra, 2002, **174**: 311–318.

[15] Lunqun OUYANG, Huanyin CHEN. *On weak symmetric rings*. Comm. Algebra, 2010, **38**(2): 697–713.

[16] H. POURTAHERIAN, I. S. RAKHIMOV. *On skew version of reversible rings*. Int. J. Pure Appl. Math., 2011, **73**(3): 267–280.

[17] Yao WANG, Qing SHEN, Yanli REN. *Rings with the semicommutative endomorphisms*. J. Jilin Univ. (Science Edition), 2013, **51**(6): 997–1003 (in Chinese).

[18] L. B. YAKOUB, M. LOUZARI. *Ore extensions of extended symmetric and reversible rings*. Int. J. Algebra, 2009, **3**(9-12): 423–433.