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Some Properties of Subclasses of Meromorphic Multivalent Functions Defined by Subordination

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Abstract In the present paper, we introduce some new subclasses of meromorphic starlike, convex, close-to-convex and quasi-convex functions of β-reciprocal in terms of the linear operator using subordination. We obtain the coefficient estimates, convolution properties, integral preserving properties and inclusion relationships of the classes. The results presented here include several results as their special cases.

Keywords analytic functions; meromorphic; subordination; reciprocal; coefficient estimate; convolution; integral preserving; inclusion

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1. Introduction and preliminaries

Let f_1 and f_2 be two analytic functions in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

We say that the function f_1 is subordinate to f_2 in $\mathbb{U}$, and write $f_1(z) \prec f_2(z)$ $(z \in \mathbb{U})$, if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1, \quad z \in \mathbb{U},$$

such that $f_1(z) = f_2(\omega(z))$ $(z \in \mathbb{U})$ (see [1]). Furthermore, if the function f_2 is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f_1(z) \prec f_2(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f_1(0) = f_2(0) \quad \text{and} \quad f_1(\mathbb{U}) \subset f_2(\mathbb{U}).$$

Also, let $\mathcal{P}$ be the class of functions of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k, \tag{1.1}$$

which are analytic and convex in $\mathbb{U}$ and satisfy the condition $\Re[p(z)] > 0$ for $z \in \mathbb{U}$.

Let Σ_p denote the class of meromorphic functions of the form

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{k-p}, \quad p \in \mathbb{N} = \{1, 2, \ldots\}, \tag{1.2}$$

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which are analytic and p-valent in the punctured open unit disk

$$\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U}\backslash\{0\}.$$

For functions $f \in \Sigma_p$ of the form (1.2) and g given by $g(z) = z^{-p} + \sum\limits_{k=1}^{\infty} b_k z^{k-p}$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) := z^{-p} + \sum_{k=1}^{\infty} a_k b_k z^{k-p} = (g * f)(z).$$

For complex parameters

$$\alpha_i \in \mathbb{C} \ (i = 1, \ldots, l) \ \text{ and } \ \beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^- \ (j = 1, \ldots, m; \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}),$$

the generalized hypergeometric function ${}_lF_m$ (with l numerator and m denominator parameters) is defined by

$${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)(z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_l)_k}{(\beta_1)_k \cdots (\beta_m)_k} \cdot \frac{z^k}{k!},$$

where $l \le m + 1; l, m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\}$, and $(\lambda)_n$ is the Pochhammer symbol (or the shifted factorial) defined (in terms of the Gamma function) by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1, & n = 0, \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1), & n \in \mathbb{N}. \end{cases}$$

Corresponding to the function

$$h_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) = z^{-p} {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)(z),$$

the linear operator

$$H_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) : \Sigma_p \longrightarrow \Sigma_p$$

is defined by using the following Hadamard product (or convolution):

$$H_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z) = h_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z).$$

For a function f of the form (1.2), we have

$$\begin{aligned} H_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z) =& z^{-p} + \sum_{k=1}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_l)_k}{k!(\beta_1)_k \cdots (\beta_m)_k} a_k z^{k-p} \\ :=& H_{p,l,m}[\alpha_1]f(z). \end{aligned} \tag{1.3}$$

The above-defined operator $H_{p,l,m}[\alpha_1]$ was introduced by Liu and Srivastava [2] and it was the development of the Dziok-Srivastava operator (see [3,4]) for functions belonging to Σ_p.

Using the same method of Srivastava et al. [5], we introduce the generalized Dziok-Srivastava operator in Σ_p as follows:

$$\begin{aligned} L_{\lambda,l,m}^{1,\alpha_1} f(z) =& (1 - \lambda)H_{p,l,m}[\alpha_1]f(z) - \frac{\lambda}{p} z(H_{p,l,m}[\alpha_1]f(z))' := L_{\lambda,l,m}^{\alpha_1} f(z), \quad \lambda \ge 0, \\ L_{\lambda,l,m}^{2,\alpha_1} f(z) =& L_{\lambda,l,m}^{\alpha_1}(L_{\lambda,l,m}^{1,\alpha_1} f(z)), \end{aligned}$$

in general,

$$L_{\lambda,l,m}^{\tau,\alpha_1} f(z) = L_{\lambda,l,m}^{\alpha_1}(L_{\lambda,l,m}^{\tau-1,\alpha_1} f(z)), \quad l \le m + 1; l, m \in \mathbb{N}_0, \tau \in \mathbb{N}. \tag{1.4}$$

If $f(z)$ is given by (1.2), then we see from (1.3) and (1.4) that

$$L_{\lambda,l,m}^{\tau,\alpha_1} f(z) = z^{-p} + \sum_{k=1}^{\infty} \Big(\frac{(1-\frac{k\lambda}{p})(\alpha_1)_k \cdots (\alpha_l)_k}{k!(\beta_1)_k \cdots (\beta_m)_k} \Big)^{\tau} a_k z^{k-p}, \quad \tau \in \mathbb{N}. \tag{1.5}$$

In terms of the Hadamard product (or convolution), we have

$$L_{\lambda,l,m}^{\tau,\alpha_1} f(z) = L_{\lambda,l,m}^{\tau,\alpha_1}(z) * f(z),$$

where

$$L_{\lambda,l,m}^{\tau,\alpha_1}(z) = z^{-p} + \sum_{k=1}^{\infty} \Big(\frac{(1-\frac{k\lambda}{p})(\alpha_1)_k \cdots (\alpha_l)_k}{k!(\beta_1)_k \cdots (\beta_m)_k} \Big)^{\tau} z^{k-p} \tag{1.6}$$

and $\alpha_i \in \mathbb{C}$ $(i = 1, \ldots, l)$, $\beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-$ $(j = 1, \ldots, m)$, $l \le m+1, l, m \in \mathbb{N}_0, \lambda \ge 0, \tau \in \mathbb{N}$.

Remark 1.1 The operator $L_{\lambda,l,m}^{\tau,\alpha_1}$ generalizes several previously familiar operators. Now, we list some of the interesting particular cases as follows:

(i) $H_{p,l,m}[\alpha_1] = L_{0,l,m}^{1,\alpha_1}$, where $H_{p,l,m}[\alpha_1]$ was introduced by Liu et al. [2];

(ii) $D^{n+p-1} = L_{0,1,0}^{1,n+p}$ $(n \ge -p)$, where D^{n+p-1} was introduced by Cho [6];

(iii) $I_p(n,\lambda) = L_{0,2,1}^{n,\lambda-p+1}$ $(\alpha_1 = \lambda-p+1, \alpha_2 = 1, \beta_1 = \lambda-p)$, where $I_p(n,\lambda)$ was introduced by Ali et al. [7];

(iv) $L_p(a,c) = L_{0,2,1}^{1,a}$ $(\alpha_2 = 1, \beta_1 = c)$, where $L_p(a,c)$ is the Liu-Srivastava operator [8].

For $\alpha_i, \beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-$ $(i = 1, \ldots, l; j = 1, \ldots, m), l \le m+1, l, m \in \mathbb{N}_0, \lambda > p$ and $\tau \in \mathbb{N}$, we introduce the following linear operator $J_{\lambda,l,m}^{\mu,\tau,\alpha_1} : \Sigma_p \longrightarrow \Sigma_p$, defined by

$$J_{\lambda,l,m}^{\mu,\tau,\alpha_1} f(z) = J_{\lambda,l,m}^{\mu,\tau,\alpha_1}(z) * f(z), \quad \mu > 0, z \in \mathbb{U}^*, \tag{1.7}$$

where $J_{\lambda,l,m}^{\mu,\tau,\alpha_1}(z)$ is the function defined as follows:

$$L_{\lambda,l,m}^{\tau,\alpha_1}(z) * J_{\lambda,l,m}^{\mu,\tau,\alpha_1}(z) = \frac{1}{z^p(1-z)^{\mu}}, \quad \mu > 0, z \in \mathbb{U}^*. \tag{1.8}$$

Since

$$\frac{1}{z^p(1-z)^{\mu}} = z^{-p} + \sum_{k=1}^{\infty} \frac{(\mu)_k}{k!} z^{k-p}, \quad \mu > 0, z \in \mathbb{U}^*, \tag{1.9}$$

combining (1.6), (1.8) and (1.9), we obtain

$$J_{\lambda,l,m}^{\mu,\tau,\alpha_1}(z) = z^{-p} + \sum_{k=1}^{\infty} \Big(\frac{k!(\beta_1)_k \cdots (\beta_m)_k}{(1-\frac{k\lambda}{p})(\alpha_1)_k \cdots (\alpha_l)_k} \Big)^{\tau} \frac{(\mu)_k}{k!} z^{k-p}, \quad \mu > 0, z \in \mathbb{U}^*. \tag{1.10}$$

For convenience, we write $\mathcal{J}_\mu := J_{\lambda,l,m}^{\mu,\tau,\alpha_1}$.

If f is given by (1.2), then from (1.7) and (1.10) we have that

$$\mathcal{J}_\mu f(z) = z^{-p} + \sum_{k=1}^{\infty} \Big(\frac{k!(\beta_1)_k \cdots (\beta_m)_k}{(1-\frac{k\lambda}{p})(\alpha_1)_k \cdots (\alpha_l)_k} \Big)^{\tau} \frac{(\mu)_k}{k!} a_k z^{k-p}, \tag{1.11}$$

where $\alpha_i, \beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-$ $(i = 1, \ldots, l; j = 1, \ldots, m), l \le m+1, l, m \in \mathbb{N}_0, \lambda > p, \tau \in \mathbb{N}, \mu > 0$ and $z \in \mathbb{U}^*$.

From (1.11), it is easy to verify that

$$z(\mathcal{J}_\mu f(z))' = \mu \mathcal{J}_{\mu+1} f(z) - (p+\mu)\mathcal{J}_\mu f(z). \tag{1.12}$$

On the other hand, for $f(z)$ of the form (1.2), the integral operator F_ζ is defined by

$$F_\zeta(f)(z) := \frac{\zeta}{z^{\zeta+p}}\int_0^z t^{\zeta+p-1}f(t)\mathrm{d}t, \quad \Re(\zeta)>0;\ z\in\mathbb{U}^*, \tag{1.13}$$

that is,

$$F_\zeta(f)(z) = z^{-p} + \sum_{k=1}^{\infty}\frac{\zeta}{k+\zeta}a_k z^{k-p}.$$

From (1.13), we note that

$$F_\zeta\Big(\frac{zf'(z)}{-p}\Big) = \frac{z(F_\zeta(f)(z))'}{-p} \tag{1.14}$$

and

$$\mathcal{J}_\mu F_\zeta f(z) = z^{-p} + \sum_{k=1}^{\infty}\Big(\frac{k!(\beta_1)_k\cdots(\beta_m)_k}{(1-\frac{k\lambda}{p})(\alpha_1)_k\cdots(\alpha_l)_k}\Big)^{\tau}\frac{(\mu)_k}{k!}\frac{\zeta}{k+\zeta}a_k z^{k-p}, \tag{1.15}$$

where $\alpha_i,\beta_j\in\mathbb{C}\backslash\mathbb{Z}_0^-$ $(i=1,\dots,l;j=1,\dots,m), l\le m+1, l,m\in\mathbb{N}_0, \lambda>p, \tau\in\mathbb{N}, \mu>0, \Re(\zeta)>0$ and $z\in\mathbb{U}^*$.

According to (1.15), we get

$$z(\mathcal{J}_\mu F_\zeta(f)(z))' = \zeta\mathcal{J}_\mu f(z) - (p+\zeta)\mathcal{J}_\mu F_\zeta(f)(z). \tag{1.16}$$

By making use of the principle of subordination between analytic functions, we introduce the subclasses $\mathcal{MS}(p;\beta;\phi), \mathcal{MK}(p;\beta;\phi), \mathcal{MCS}(p;\beta;\phi,\psi)$ and $\mathcal{MCK}(p;\beta;\phi,\psi)$ of the class Σ_p as follows.

Definition 1.2 *A function $f\in\Sigma_p$ of the form (1.2) is said to be in the class $\mathcal{MS}(p;\beta;\phi)$ of meromorphic p-valent starlike function of β-reciprocal if and only if*

$$\frac{-p}{1-p\beta}\Big\{\frac{f(z)}{zf'(z)}+\beta\Big\} \prec \phi(z), \tag{1.17}$$

where $p\in\mathbb{N}, \beta\in\mathbb{R}, p\beta\ne 1$ and $\phi\in\mathcal{P}$.

Remark 1.3 We note that the class $\mathcal{MS}(p;\beta;\phi)$ generalizes several previous classes, and we show some of the interesting particular cases as follows.

(i) For $p=1,\beta=0$ and $\phi(z)=\frac{1+(1-2\alpha)z}{1-z}$ $(0\le\alpha<1)$, the class $\mathcal{MS}(1;0;\frac{1+(1-2\alpha)z}{1-z}) = \mathcal{NS}^*(\alpha)$ was considered by Sun et al. [9].

(ii) For $p\beta>1, \phi(z)=\frac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$, the class $\mathcal{MS}(p;\beta;\frac{1+Az}{1+Bz}) = \mathcal{MS}(p;\beta;A,B)$ was considered by Ma et al. [10].

In addition, we have the class $\mathcal{MK}(p;\beta;\phi)$ of meromorphically p-valent convex function of β-reciprocal if and only if $\frac{zf'(z)}{-p}\in\mathcal{MS}(p;\beta;\phi)$.

Definition 1.4 *A function $f\in\Sigma_p$ of the form (1.2) is said to be in the class $\mathcal{MCS}(p;\beta;\phi,\psi)$ of meromorphic p-valent close-to-convex function of β-reciprocal if and only if*

$$\frac{-p}{1-p\beta}\Big\{\frac{g(z)}{zf'(z)}+\beta\Big\} \prec \psi(z), \tag{1.18}$$

where $p\in\mathbb{N}, \beta\in\mathbb{R}, p\beta\ne 1, \phi,\psi\in\mathcal{P}$ and $g\in\mathcal{MS}(p;\beta;\phi)$.

In particular, we have the class $\mathcal{MCK}(p;\beta;\phi,\psi)$ of meromorphic p-valent quasi-convex function of β-reciprocal if and only if $\frac{zf'(z)}{-p} \in \mathcal{MCS}(p;\beta;\phi,\psi)$.

In recent years, more and more researchers are interested in the starlike functions of reciprocal and other cases of reciprocal [11–14].

Next, by using the integral operator $\mathcal{J}_\mu$ and $\mathcal{J}_\mu F_\zeta$ defined by (1.11) and (1.15), respectively, we introduce the following subclasses of Σ_p:

$$\mathcal{MS}_\mu(p;\beta;\phi) := \{f \in \Sigma_p : \mathcal{J}_\mu f \in \mathcal{MS}(p;\beta;\phi)\},$$
$$\mathcal{MK}_\mu(p;\beta;\phi) := \{f \in \Sigma_p : \mathcal{J}_\mu f \in \mathcal{MK}(p;\beta;\phi)\},$$
$$\mathcal{MCS}_\mu(p;\beta;\phi,\psi) := \{f \in \Sigma_p : \mathcal{J}_\mu f \in \mathcal{MCS}(p;\beta;\phi,\psi)\},$$
$$\mathcal{MCK}_\mu(p;\beta;\phi,\psi) := \{f \in \Sigma_p : \mathcal{J}_\mu f \in \mathcal{MCK}(p;\beta;\phi,\psi)\}$$

and

$$\mathcal{MSF}^\zeta_\mu(p;\beta;\phi) := \{f \in \Sigma_p : \mathcal{J}_\mu F_\zeta(f) \in \mathcal{MS}(p;\beta;\phi)\},$$
$$\mathcal{MKF}^\zeta_\mu(p;\beta;\phi) := \{f \in \Sigma_p : \mathcal{J}_\mu F_\zeta(f) \in \mathcal{MK}(p;\beta;\phi)\},$$
$$\mathcal{MCSF}^\zeta_\mu(p;\beta;\phi,\psi) := \{f \in \Sigma_p : \mathcal{J}_\mu F_\zeta(f) \in \mathcal{MCS}(p;\beta;\phi,\psi)\},$$
$$\mathcal{MCKF}^\zeta_\mu(p;\beta;\phi,\psi) := \{f \in \Sigma_p : \mathcal{J}_\mu F_\zeta(f) \in \mathcal{MCK}(p;\beta;\phi,\psi)\},$$

where $\alpha_i, \beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-$ $(i = 1,\ldots,l; j = 1,\ldots,m), l \le m+1, l, m \in \mathbb{N}_0, \lambda > p, \tau \in \mathbb{N}, \mu > 0, \Re(\zeta) > 0, p \in \mathbb{N}, \beta \in \mathbb{R}, p\beta \neq 1$ and $\phi, \psi \in \mathcal{P}$.

In order to prove our main results, we need the following lemmas.

Lemma 1.5 ([15]) *Let $\kappa, \vartheta \in \mathbb{C}$. Suppose that m is convex and univalent in $\mathbb{U}$ with $m(0) = 1$ and $\Re(\kappa m(z) + \vartheta) > 0$ $(z \in \mathbb{U})$. If u is analytic in $\mathbb{U}$ with $u(0) = 1$, then the following subordination:*

$$u(z) + \frac{zu'(z)}{\kappa u(z) + \vartheta} \prec m(z)$$

implies that $u(z) \prec m(z)$.

Lemma 1.6 ([16]) *Let h be convex univalent in $\mathbb{C}$ and ρ be analytic in $\mathbb{U}$ with*

$$\Re(\rho(z)) \ge 0, \quad z \in \mathbb{U}.$$

If q is analytic in $\mathbb{U}$ and $q(0) = h(0)$, then the subordination

$$q(z) + \rho(z)zq'(z) \prec h(z)$$

implies that $q(z) \prec h(z)$.

The main purpose of the present paper is to investigate the coefficient estimates and convolution properties of the subclasses of meromorphic functions. Furthermore, several integral preserving properties and inclusion relationships are also derived.

2. Coefficient estimates

In this section, unless otherwise mentioned, we assume that $p \in \mathbb{N} = \{1, 2, \ldots\}, \beta \in \mathbb{R}, p\beta \neq$

1 and $-1 \le D \le B < A \le C \le 1$.

Lemma 2.1 ([17]) *If $p(z)$ is given by (1.1) and $\frac{1+Az}{1+Bz} = p(z)$, then $|p_n| \le (A-B)$ $(n = 1, 2, \ldots)$.*

Theorem 2.2 *Let f of the form (1.2) be in the class $\mathcal{MS}(p;\beta;A,B)$. For $k \ge 2$, we have*

$$|a_k| \le \frac{p|1-p\beta|(A-B)}{k} \prod_{m=1}^{k-1} \Big[\frac{p|1-p\beta|(A-B)}{m}|1-\frac{m}{p}|+1\Big] \tag{2.1}$$

and

$$1+\sum_{m=1}^{k}|1-\frac{m}{p}||a_m| \le \prod_{m=1}^{k}\Big[\frac{p|1-p\beta|(A-B)}{m}|1-\frac{m}{p}|+1\Big]. \tag{2.2}$$

Proof Suppose that f of the form (1.2) be in the class $\mathcal{MS}(p;\beta;A,B)$. We have

$$\frac{-p}{1-p\beta}\{\frac{f(z)}{zf'(z)}+\beta\} = \frac{1+\sum_{k=1}^{\infty}(1+\frac{k\beta}{1-p\beta})a_k z^k}{1+\sum_{k=1}^{\infty}(1-\frac{k}{p})a_k z^k} \prec \frac{1+Az}{1+Bz}. \tag{2.3}$$

According to the definition of subordination, it follows that

$$\frac{1+\sum_{k=1}^{\infty}(1+\frac{k\beta}{1-p\beta})a_k z^k}{1+\sum_{k=1}^{\infty}(1-\frac{k}{p})a_k z^k} = \frac{1+A\omega(z)}{1+B\omega(z)} = 1+\sum_{k=1}^{\infty}d_k z^k, \tag{2.4}$$

where $\omega(z) = c_1 z + c_2 z^2 + \cdots$, $\omega \in \mathcal{W}$, $\mathcal{W}$ denotes the well-known class of the bounded analytic functions in $\mathbb{U}$ and satisfies the conditions $\omega(0) = 0$ and $|\omega(z)| \le |z|$ $(z \in \mathbb{U}^*)$.

After a simple computation, we get

$$\begin{aligned}
&1+(1+\frac{\beta}{1-p\beta})a_1 z+(1+\frac{2\beta}{1-p\beta})a_2 z^2+(1+\frac{3\beta}{1-p\beta})a_3 z^3+\cdots+(1+\frac{k\beta}{1-p\beta})a_k z^k+\cdots\\
&=1+\Big[(1-\frac{1}{p})a_1+d_1\Big]z+\Big[(1-\frac{2}{p})a_2+(1-\frac{1}{p})a_1 d_1+d_2\Big]z^2+\\
&\Big[(1-\frac{3}{p})a_3+(1-\frac{2}{p})a_2 d_1+(1-\frac{1}{p})a_1 d_2+d_3\Big]z^3+\cdots+\\
&\Big[(1-\frac{k}{p})a_k+(1-\frac{k-1}{p})a_{k-1}d_1+\cdots+(1-\frac{1}{p})a_1 d_{k-1}+d_k\Big]z^k+\cdots.
\end{aligned}$$

Comparing the coefficients of the both sides, we obtain

$$\begin{aligned}
a_1 &= p(1-p\beta)d_1;\\
a_2 &= \frac{p(1-p\beta)}{2}\Big[(1-\frac{1}{p})a_1 d_1+d_2\Big];\\
a_3 &= \frac{p(1-p\beta)}{3}\Big[(1-\frac{2}{p})a_2 d_1+(1-\frac{1}{p})a_1 d_2+d_3\Big];\\
&\vdots\\
a_k &= \frac{p(1-p\beta)}{k}\Big[(1-\frac{k-1}{p})a_{k-1}d_1+\cdots+(1-\frac{1}{p})a_1 d_{k-1}+d_k\Big].
\end{aligned} \tag{2.5}$$

By using Lemma 2.1 and (2.5), we have

$$1+|1-\frac{1}{p}||a_1| \le p|1-p\beta|(A-B)|1-\frac{1}{p}|+1, \tag{2.6}$$

$$1 + \sum_{m=1}^{2} |1 - \frac{m}{p}||a_m| \leq \prod_{m=1}^{2} \Big[\frac{p|1 - p\beta|(A - B)}{m}|1 - \frac{m}{p}| + 1\Big] \tag{2.7}$$

and

$$|a_k| \leq \frac{p|1 - p\beta|(A - B)}{k}\Big(1 + |1 - \frac{1}{p}||a_1| + \cdots + |1 - \frac{k-1}{p}||a_{k-1}|\Big). \tag{2.8}$$

We now prove (2.2) by induction. It follows from (2.7) that (2.2) holds for $k = 2$. We suppose that (2.2) holds for $k = n - 1$. So we have

$$1 + \sum_{m=1}^{n-1} |1 - \frac{m}{p}||a_m| \leq \prod_{m=1}^{n-1} \Big[\frac{p|1 - p\beta|(A - B)}{m}|1 - \frac{m}{p}| + 1\Big]. \tag{2.9}$$

For $k = n$, by using (2.8) and (2.9), we have

$$1 + \sum_{m=1}^{n} |1 - \frac{m}{p}||a_m| \leq \prod_{m=1}^{n} \Big[\frac{p|1 - p\beta|(A - B)}{m}|1 - \frac{m}{p}| + 1\Big]. \tag{2.10}$$

Therefore, (2.2) holds true.

Also, by (2.8) and (2.2), we have

$$|a_k| \leq \frac{p|1 - p\beta|(A - B)}{k} \prod_{m=1}^{k-1} \Big[\frac{p|1 - p\beta|(A - B)}{m}|1 - \frac{m}{p}| + 1\Big], \quad k \geq 2,$$

which completes the proof of Theorem 2.2. □

Corollary 2.3 *Let $f \in \mathcal{MK}(p; \beta; A, B)$. Then for $k \geq 2$ and $k \neq p$, we have*

$$|a_k| \leq \frac{p|1 - p\beta|(A - B)}{k|1 - \frac{k}{p}|} \prod_{m=1}^{k-1} \Big[\frac{p|1 - p\beta|(A - B)}{m}|1 - \frac{m}{p}| + 1\Big]$$

and

$$1 + \sum_{m=1}^{k} |1 - \frac{m}{p}|^2|a_m| \leq \prod_{m=1}^{k} \Big[\frac{p|1 - p\beta|(A - B)}{m}|1 - \frac{m}{p}| + 1\Big].$$

Theorem 2.4 *Let $f \in \mathcal{MCS}(p; \beta; \frac{1+Az}{1+Bz}, \frac{1+Cz}{1+Dz})$. Then for $k \geq 2$ and $k \neq p$, we have*

$$\begin{aligned}
|a_k| \leq & \frac{1}{|1 - \frac{k}{p}|} \frac{p|1 - p\beta|(A - B)}{k} \prod_{m=1}^{k-1} \Big[\frac{p|1 - p\beta|(A - B)}{m}|1 - \frac{m}{p}| + 1\Big] + \\
& \frac{1}{|1 - \frac{k}{p}|}|1 - p\beta|(C - D) \sum_{l=2}^{k-1} [|1 - p\beta|(C - D) + 1]^{k-l-1} \frac{p|1 - p\beta|(A - B)}{l} \\
& \prod_{m=1}^{l-1} \Big(\frac{p|1 - p\beta|(A - B)}{m}|1 - \frac{m}{p}| + 1\Big) + \\
& \frac{1}{|1 - \frac{k}{p}|}|1 - p\beta|(C - D)[|1 - p\beta|(C - D) + 1]^{k-2}\{|1 - p\beta|[p(A - B) + (C - D)] + 1\}
\end{aligned}$$

and

$$1 + \sum_{m=1}^{k} |1 - \frac{m}{p}||a_m|$$

$$\leq \sum_{l=2}^{k}[|1-p\beta|(C-D)+1]^{k-l}\frac{p|1-p\beta|(A-B)}{l}\prod_{k=1}^{l-1}\Big[\frac{p|1-p\beta|(A-B)}{m}|1-\frac{m}{p}|+1\Big]+$$
$$[|1-p\beta|(C-D)+1]^{k-1}\{|1-p\beta|[p(A-B)+(C-D)]+1\}.$$

Proof The proof of Theorem 2.4 is similar to the proof of Theorem 2.2. Hence, the proof of Theorem 2.4 will be omitted here. □

Corollary 2.5 *Let $f\in\mathcal{MCK}(p;\beta;\frac{1+Az}{1+Bz},\frac{1+Cz}{1+Dz})$. Then for $k\geq 2$ and $k\neq p$, we have*

$$|a_k|\leq \frac{1}{|1-\frac{k}{p}|^2}\frac{p|1-p\beta|(A-B)}{k}\prod_{m=1}^{k-1}\Big[\frac{p|1-p\beta|(A-B)}{m}|1-\frac{m}{p}|+1\Big]+$$
$$\frac{1}{|1-\frac{k}{p}|^2}|1-p\beta|(C-D)\sum_{l=2}^{k-1}[|1-p\beta|(C-D)+1]^{k-l-1}\frac{p|1-p\beta|(A-B)}{l}$$
$$\prod_{m=1}^{l-1}\Big[\frac{p|1-p\beta|(A-B)}{m}|1-\frac{m}{p}|+1\Big]+$$
$$\frac{1}{|1-\frac{k}{p}|^2}|1-p\beta|(C-D)[|1-p\beta|(C-D)+1]^{k-2}\{|1-p\beta|[p(A-B)+(C-D)]+1\}$$

and

$$1+\sum_{m=1}^{k}|1-\frac{m}{p}|^2|a_m|$$
$$\leq \sum_{l=2}^{k}[|1-p\beta|(C-D)+1]^{k-l}\frac{p|1-p\beta|(A-B)}{l}\prod_{k=1}^{l-1}\Big[\frac{p|1-p\beta|(A-B)}{m}|1-\frac{m}{p}|+1\Big]+$$
$$[|1-p\beta|(C-D)+1]^{k-1}\{|1-p\beta|[p(A-B)+(C-D)]+1\}.$$

3. Convolution properties

In this section, the convolution properties for the subclasses of Σ_p are obtained. In order to establish our main results, we shall require the following lemma.

Lemma 3.1 *Let $f\in\mathcal{MS}(p;\beta;\phi)$ with $p\in\mathbb{N},\beta\in\mathbb{R},0\leq\beta<\frac{1}{p},\phi\in\mathcal{P}$ and $z\in\mathbb{U}^*$. Then*

$$f(z)=\exp\int_0^z\frac{-p}{\eta[(1-p\beta)\phi(\omega(\eta))+p\beta]}\mathrm{d}\eta, \tag{3.1}$$

where ω is analytic in $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1$ $(z\in\mathbb{U}^*)$.

Proof Suppose that $f\in\mathcal{MS}(p;\beta;\phi)$. It follows that

$$\frac{zf'(z)}{-pf(z)}=\frac{1}{(1-p\beta)\phi(\omega(z))+p\beta}, \tag{3.2}$$

where ω is analytic in $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1$ $(z\in\mathbb{U}^*)$. From (3.2), it is easy to see that

$$\frac{f'(z)}{f(z)}=\frac{-p}{z[(1-p\beta)\phi(\omega(z))+p\beta]}. \tag{3.3}$$

Taking the integration on both sides of (3.3), we get

$$\log f(z)=\int_0^z \frac{-p}{\eta[(1-p\beta)\phi(\omega(\eta))+p\beta]}\mathrm{d}\eta.$$

Thus we conclude that

$$f(z)=\exp\int_0^z \frac{-p}{\eta[(1-p\beta)\phi(\omega(\eta))+p\beta]}\mathrm{d}\eta. \ \square$$

Theorem 3.2 *Let $f\in\mathcal{MK}(p;\beta;\phi)$ with $p\in\mathbb{N},\beta\in\mathbb{R},0\le\beta<\frac{1}{p},\phi\in\mathcal{P}$ and $z\in\mathbb{U}^*$. Then*

$$f(z)=\Big(\exp\int_0^z \frac{-p}{\eta[(1-p\beta)\phi(\omega(\eta))+p\beta]}\mathrm{d}\eta\Big)*\Big(z^{-p}+\sum_{k=1}^{p-1}\frac{p}{p-k}z^{k-p}+\sum_{k=p+1}^{\infty}\frac{p}{p-k}z^{k-p}\Big), \quad (3.4)$$

where ω is analytic in $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1$ $(z\in\mathbb{U}^*)$.

Proof It is clear that $f\in\mathcal{MK}(p;\beta;\phi)\iff\frac{zf'(z)}{-p}\in\mathcal{MS}(p;\beta;\phi)$. According to Lemma 3.1, it follows that

$$\frac{zf'(z)}{-p}=\exp\int_0^z \frac{-p}{\eta[(1-p\beta)\phi(\omega(\eta))+p\beta]}\mathrm{d}\eta. \quad (3.5)$$

On the other hand,

$$\frac{zf'(z)}{-p}=f(z)*\Big(z^{-p}+\sum_{k=1}^{p-1}\frac{p-k}{p}z^{k-p}+\sum_{k=p+1}^{\infty}\frac{p-k}{p}z^{k-p}\Big). \quad (3.6)$$

From (3.5) and (3.6), (3.4) holds true. This completes the proof of Theorem 3.2. $\square$

Similarly, we can easily obtain the following results.

Corollary 3.3 *Let $f\in\mathcal{MS}_\mu(p;\beta;\phi)$ with $\alpha_i,\beta_j\in\mathbb{C}\backslash\mathbb{Z}_0^-$ $(i=1,\dots,l;j=1,\dots,m),l\le m+1,l,m\in\mathbb{N}_0,\tau\in\mathbb{N},p\in\mathbb{N},\beta\in\mathbb{R},0\le\beta<\frac{1}{p},\lambda>p,\mu>0,\phi\in\mathcal{P}$ and $z\in\mathbb{U}^*$. Then*

$$f(z)=\Big(\exp\int_0^z \frac{-p}{\eta[(1-p\beta)\phi(\omega(\eta))+p\beta]}\mathrm{d}\eta\Big)*\Big(z^{-p}+\sum_{k=1}^{\infty}\Big(\frac{(1-\frac{k\lambda}{p})(\alpha_1)_k\cdots(\alpha_l)_k}{k!(\beta_1)_k\cdots(\beta_m)_k}\Big)^\tau\frac{k!}{(\mu)_k}z^{k-p}\Big),$$

where ω is analytic in $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1$ $(z\in\mathbb{U}^*)$.

Corollary 3.4 *Let $f\in\mathcal{MSF}_\mu^\zeta(p;\beta;\phi)$ with $\alpha_i,\beta_j\in\mathbb{C}\backslash\mathbb{Z}_0^-$ $(i=1,\dots,l;j=1,\dots,m),l\le m+1,l,m\in\mathbb{N}_0,\tau\in\mathbb{N},p\in\mathbb{N},\beta\in\mathbb{R},0\le\beta<\frac{1}{p},\lambda>p,\Re(\zeta)>0,\mu>0,\phi\in\mathcal{P}$ and $z\in\mathbb{U}^*$. Then*

$$f(z)=\Big(\exp\int_0^z \frac{-p}{\eta[(1-p\beta)\phi(\omega(\eta))+p\beta]}\mathrm{d}\eta\Big)*\Big(z^{-p}+\sum_{k=1}^{\infty}\Big(\frac{(1-\frac{k\lambda}{p})(\alpha_1)_k\cdots(\alpha_l)_k}{k!(\beta_1)_k\cdots(\beta_m)_k}\Big)^\tau\frac{k!}{(\mu)_k}\frac{k+\zeta}{\zeta}z^{k-p}\Big),$$

where ω is analytic in $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1$ $(z\in\mathbb{U}^*)$.

Corollary 3.5 *Let $f\in\mathcal{MK}_\mu(p;\beta;\phi)$ with $\alpha_i,\beta_j\in\mathbb{C}\backslash\mathbb{Z}_0^-$ $(i=1,\dots,l;j=1,\dots,m),l\le m+1,l,m\in\mathbb{N}_0,\tau\in\mathbb{N},p\in\mathbb{N},\beta\in\mathbb{R},0\le\beta<\frac{1}{p},\lambda>p,\mu>0,\phi\in\mathcal{P}$ and $z\in\mathbb{U}^*$. Then*

$$f(z)=\Big(\exp\int_0^z \frac{-p}{\eta[(1-p\beta)\phi(\omega(\eta))+p\beta]}\mathrm{d}\eta\Big)*\Big(z^{-p}+\sum_{k=1}^{p-1}\frac{p}{p-k}z^{k-p}+\sum_{k=p+1}^{\infty}\frac{p}{p-k}z^{k-p}\Big)*$$
$$\Big(z^{-p}+\sum_{k=1}^{\infty}\Big(\frac{(1-\frac{k\lambda}{p})(\alpha_1)_k\cdots(\alpha_l)_k}{k!(\beta_1)_k\cdots(\beta_m)_k}\Big)^\tau\frac{k!}{(\mu)_k}z^{k-p}\Big)$$

where ω is analytic in $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1$ $(z\in\mathbb{U}^*)$.

Corollary 3.6 *Let $f\in\mathcal{MKF}^{\zeta}_{\mu}(p;\beta;\phi)$ with $\alpha_i,\beta_j\in\mathbb{C}\backslash\mathbb{Z}_0^-$ $(i=1,\dots,l;j=1,\dots,m), l\le m+1, l,m\in\mathbb{N}_0,\tau\in\mathbb{N},p\in\mathbb{N},\beta\in\mathbb{R},0\le\beta<\frac{1}{p},\lambda>p,\mu>0,\Re(\zeta)>0,\phi\in\mathcal{P}$ and $z\in\mathbb{U}^*$. Then*

$$f(z)=\Big(\exp\int_0^z\frac{-p}{\eta[(1-p\beta)\phi(\omega(\eta))+p\beta]}\mathrm{d}\eta\Big)*\Big(z^{-p}+\sum_{k=1}^{p-1}\frac{p}{p-k}z^{k-p}+\sum_{k=p+1}^{\infty}\frac{p}{p-k}z^{k-p}\Big)*$$
$$\Big(z^{-p}+\sum_{k=1}^{\infty}\Big(\frac{(1-\frac{k\lambda}{p})(\alpha_1)_k\cdots(\alpha_l)_k}{k!(\beta_1)_k\cdots(\beta_m)_k}\Big)^{\tau}\frac{k!}{(\mu)_k}\frac{k+\zeta}{\zeta}z^{k-p}\Big),$$

where ω is analytic in $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1$ $(z\in\mathbb{U}^*)$.

Theorem 3.7 *Let $f\in\mathcal{MCS}(p;\beta;\phi,\psi)$ with $p\in\mathbb{N},\beta\in\mathbb{R},0\le\beta<\frac{1}{p},\phi,\psi\in\mathcal{P}$ and $z\in\mathbb{U}^*$. Then*

$$f(z)=\Big(\frac{\exp\int_0^z\frac{-p}{\eta[(1-p\beta)\phi(\omega_2(\eta))+p\beta]}\mathrm{d}\eta}{[(1-p\beta)\psi(\omega_1(z))+p\beta]}\Big)*\Big(z^{-p}+\sum_{k=1}^{p-1}\frac{p}{p-k}z^{k-p}+\sum_{k=p+1}^{\infty}\frac{p}{p-k}z^{k-p}\Big)$$

where ω_1,ω_2 are analytic in $\mathbb{U}$ with $\omega_1(0)=\omega_2(0)=0,|\omega_1(z)|<1$ and $|\omega_2(z)|<1$ $(z\in\mathbb{U}^*)$.

Proof Suppose that $f\in\mathcal{MCS}(p;\beta;\phi,\psi)$. It follows that

$$\frac{-p}{1-p\beta}\Big\{\frac{g(z)}{zf'(z)}+\beta\Big\}\prec\psi(z),\ \ g\in\mathcal{MS}(p;\beta;\phi).$$

According to the definition of subordination and Lemma 3.1, there exist ω_1,ω_2, which are analytic in $\mathbb{U}$ with $\omega_1(0)=\omega_2(0)=0,|\omega_1(z)|<1$ and $|\omega_2(z)|<1$, satisfying

$$\frac{-p}{1-p\beta}\Big\{\frac{g(z)}{zf'(z)}+\beta\Big\}=\psi(\omega_1(z))\text{ and }g(z)=\exp\int_0^z\frac{-p}{\eta[(1-p\beta)\phi(\omega_2(\eta))+p\beta]}\mathrm{d}\eta.$$

It follows that

$$\frac{zf'(z)}{-p}=\frac{\exp\int_0^z\frac{-p}{\eta[(1-p\beta)\phi(\omega_2(\eta))+p\beta]}\mathrm{d}\eta}{[(1-p\beta)\psi(\omega_1(z))+p\beta]}.$$

Therefore, it is easy to show that

$$f(z)=\Big(\frac{\exp\int_0^z\frac{-p}{\eta[(1-p\beta)\phi(\omega_2(\eta))+p\beta]}\mathrm{d}\eta}{[(1-p\beta)\psi(\omega_1(z))+p\beta]}\Big)*\Big(z^{-p}+\sum_{k=1}^{p-1}\frac{p}{p-k}z^{k-p}+\sum_{k=p+1}^{\infty}\frac{p}{p-k}z^{k-p}\Big).\ \ \square$$

Corollary 3.8 *Let $f\in\mathcal{MCK}(p;\beta;\phi,\psi)$ with $p\in\mathbb{N},\beta\in\mathbb{R},0\le\beta<\frac{1}{p},\phi,\psi\in\mathcal{P}$ and $z\in\mathbb{U}^*$. Then*

$$f(z)=\Big(\frac{\exp\int_0^z\frac{-p}{\eta[(1-p\beta)\phi(\omega_2(\eta))+p\beta]}\mathrm{d}\eta}{[(1-p\beta)\psi(\omega_1(z))+p\beta]}\Big)*\Big(z^{-p}+\sum_{k=1}^{p-1}\frac{p^2}{(p-k)^2}z^{k-p}+\sum_{k=p+1}^{\infty}\frac{p^2}{(p-k)^2}z^{k-p}\Big),$$

where ω_1,ω_2 are analytic in $\mathbb{U}$ with $\omega_1(0)=0,\omega_2(0)=0,|\omega_1(z)|<1$ and $|\omega_2(z)|<1$ $(z\in\mathbb{U}^*)$.

Theorem 3.9 *The function f given by (1.2) is in the class $\mathcal{MS}(p;\beta;\phi)$ if and only if*

$$z^p\Big\{f(z)*\frac{1-D(p,\beta,\phi,\theta)z}{z^p(1-z)^2}\Big\}\neq 0, \tag{3.7}$$

for all $z \in \mathbb{U}^*, 0 < \theta < 2\pi, p \in \mathbb{N}, \beta \in \mathbb{R}, p\beta \neq 1, (1-p\beta)\phi(e^{i\theta}) + p\beta \neq \{0, 1\}$ *and* $\phi \in \mathcal{P}$*, where*

$$D(p, \beta, \phi, \theta) = 1 + \frac{(1-p\beta)\phi(e^{i\theta}) + p\beta}{p(1-p\beta)(\phi(e^{i\theta}) - 1)}. \tag{3.8}$$

Proof On the one hand, suppose that $f \in \mathcal{MS}(p; \beta; \phi)$. According to the definition of subordination, there exists a function $\omega(z)$, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that

$$\frac{-pf(z)}{zf'(z)} = (1-p\beta)\phi(\omega(z)) + p\beta,$$

which is equivalent to

$$\frac{zf'(z)}{-pf(z)} \neq \frac{1}{(1-p\beta)\phi(e^{i\theta}) + p\beta} = 1 - \frac{1}{p(D(p, \beta, \phi, \theta) - 1)}, \quad z \in \mathbb{U}^*, 0 < \theta < 2\pi.$$

Using the fact that

$$f(z) = f(z) * \frac{1}{z^p(1-z)} \quad \text{and} \quad \frac{zf'(z)}{-p} = f(z) * \frac{1 - (1 + \frac{1}{p})z}{z^p(1-z)^2},$$

we have

$$\begin{aligned}
&z^p\Big\{f(z) * \frac{1 - D(p, \beta, \phi, \theta)z}{z^p(1-z)^2}\Big\} \\
&= z^p f(z)\Big\{p(D(p, \beta, \phi, \theta) - 1)\frac{zf'(z)}{-pf(z)} - [p(D(p, \beta, \phi, \theta) - 1) - 1]\Big\} \\
&\neq z^p f(z)\Big\{p(D(p, \beta, \phi, \theta) - 1)\Big(1 - \frac{1}{p(D(p, \beta, \phi, \theta) - 1)}\Big) - [p(D(p, \beta, \phi, \theta) - 1) - 1]\Big\} \\
&= 0.
\end{aligned} \tag{3.9}$$

So we complete the proof of the necessary part of Theorem 3.9.

On the other hand, from (3.9) we have

$$z^p\Big\{f(z) * \frac{1 - D(p, \beta, \phi, \theta)z}{z^p(1-z)^2}\Big\} = z^p f(z)\Big\{p(D(p, \beta, \phi, \theta) - 1)\frac{zf'(z)}{-pf(z)} - [p(D(p, \beta, \phi, \theta) - 1) - 1]\Big\}.$$

If the condition (3.7) holds, it follows that

$$z^p f(z)\Big\{p(D(p, \beta, \phi, \theta) - 1)\frac{zf'(z)}{-pf(z)} - [p(D(p, \beta, \phi, \theta) - 1) - 1]\Big\} \neq 0.$$

It is trivial to show that

$$\frac{zf'(z)}{-pf(z)} \neq \frac{1}{(1-p\beta)\phi(e^{i\theta}) + p\beta}, \quad z \in \mathbb{U}^*, 0 < \theta < 2\pi.$$

Consequently, we infer that $f(z) \in \mathcal{MS}(p; \beta; \phi)$. □

Theorem 3.10 *The function* $f(z)$ *given by (1.2) is in the class* $\mathcal{MK}(p; \beta; \phi)$ *if and only if*

$$z^p\Big\{f(z) * \frac{p - [p + 2 + (p-1)D(p, \beta, \phi, \theta)]z + (p+1)D(p, \beta, \phi, \theta)z^2}{pz^p(1-z)^3}\Big\} \neq 0 \tag{3.10}$$

for all $z \in \mathbb{U}^*, p \in \mathbb{N}, \beta \in \mathbb{R}, p\beta \neq 1, 0 < \theta < 2\pi, \phi \in \mathcal{P}$ *and* $(1-p\beta)\phi(e^{i\theta}) + p\beta \neq \{0, 1\}$*, where* $D(p, \beta, \phi, \theta)$ *is given by (3.8).*

Proof Set

$$g(z)=\frac{1-D(p,\beta,\phi,\theta)z}{z^p(1-z)^2}.$$

Taking derivative to the both sides of the equality, we obtain

$$\frac{zg'(z)}{-p}=\frac{p-[p+2+(p-1)D(p,\beta,\phi,\theta)]z+(p+1)D(p,\beta,\phi,\theta)z^2}{pz^p(1-z)^3}.$$

By the definition of the class $\mathcal{MK}(p;\beta;\phi)$ and Theorem 3.9, it follows that $f\in\mathcal{MK}(p;\beta;\phi)$ if and only if

$$z^p\Big\{\frac{zf'(z)}{-p}*g(z)\Big\}\neq 0. \tag{3.11}$$

In addition, we have the identity

$$\frac{zf'(z)}{-p}*g(z)=f(z)*\frac{zg'(z)}{-p}. \tag{3.12}$$

It follows from (3.11) and (3.12) that $f(z)\in\mathcal{MK}(p;\beta;\phi)$ if and only if

$$z^p\Big\{f(z)*\frac{zg'(z)}{-p}\Big\}\neq 0,$$

which completes the proof of Theorem 3.10. □

Similarly, we can obtain the convolution properties of the classes of $\mathcal{MS}_\mu(p;\beta;\phi)$, $\mathcal{MK}_\mu(p;\beta;\phi)$, $\mathcal{MSF}_\mu^\zeta(p;\beta;\phi)$ and $\mathcal{MKF}_\mu^\zeta(p;\beta;\phi)$.

4. Integral-preserving properties

In the following, we discuss the integral-preserving properties for another family of integral operators.

Theorem 4.1 *Let $f(z)\in\mathcal{MS}(p;\beta;\phi)$ with $p\in\mathbb{N},\beta\in\mathbb{R},0\le\beta<\frac{1}{p},\nu\in\mathbb{C},\xi\in\mathbb{C}\backslash\{0\},\phi\in\mathcal{P}$, $z\in\mathbb{U}^*$ and*

$$\Re\Big(\nu+p-\frac{p\xi}{(1-p\beta)\phi(z)+p\beta}\Big)>0. \tag{4.1}$$

Then the function $F_\nu^\xi(f)\in\Sigma_p$ defined by

$$F_\nu^\xi(f)(z):=\Big(\frac{\nu+p-p\xi}{z^{\nu+p}}\int_0^z t^{\nu+p-1}(f(t))^\xi\mathrm{d}t\Big)^{\frac{1}{\xi}},\ \ \Re(\nu+p-p\xi)>0;z\in\mathbb{U}^* \tag{4.2}$$

belongs to the class $\mathcal{MS}(p;\beta;\phi)$.

Proof Let $f\in\mathcal{MS}(p;\beta;\phi)$ and suppose that

$$\mathbb{Y}(z):=\frac{z(F_\nu^\xi(f)(z))'}{-pF_\nu^\xi(f)(z)}. \tag{4.3}$$

By using (4.2), we have

$$\Big(F_\nu^\xi(f)(z)\Big)^\xi=\frac{\nu+p-p\xi}{z^{\nu+p}}\int_0^z t^{\nu+p-1}(f(t))^\xi\mathrm{d}t. \tag{4.4}$$

Differentiating both sides of (4.4) with respect to z and multiplying the resulting equation by z, we get

$$\frac{\xi z(F_\nu^\xi(f)(z))'}{F_\nu^\xi(f)(z)} = \frac{(\nu+p-p\xi)(f(z))^\xi}{(F_\nu^\xi(f)(z))^\xi} - (\nu+p). \tag{4.5}$$

Combining (4.3) and (4.5), we have

$$\nu+p-p\xi\mathbb{Y}(z) = \frac{(\nu+p-p\xi)(f(z))^\xi}{(F_\nu^\xi(f)(z))^\xi}. \tag{4.6}$$

Differentiating both sides of (4.6) with respect to z logarithmically and multiplying the resulting equation by z, we get

$$\frac{zf'(z)}{-pf(z)} = \mathbb{Y}(z) + \frac{z\mathbb{Y}'(z)}{\nu+p-p\xi\mathbb{Y}(z)} \prec \frac{1}{(1-p\beta)\phi(z)+p\beta}. \tag{4.7}$$

Thus, by the condition (4.1) and an application of Lemma 1.5 to (4.7) yields

$$\mathbb{Y}(z) = \frac{z(F_\nu^\xi(f)(z))'}{-pF_\nu^\xi(f)(z)} \prec \frac{1}{(1-p\beta)\phi(z)+p\beta},$$

that is, $F_\nu^\xi(f) \in \mathcal{MS}(p;\beta;\phi)$. This completes the proof of Theorem 4.1. □

By suitable modification to Theorem 4.1, we can show the following corollary.

Corollary 4.2 *Let $f \in \mathcal{MS}_\mu(p;\beta;\phi)$ with $\alpha_i, \beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-$ $(j = 1,\dots,m; i = 1,\dots,l), l \le m+1, l,m \in \mathbb{N}_0, \tau \in \mathbb{N}, p \in \mathbb{N}, \beta \in \mathbb{R}, 0 \le \beta < \frac{1}{p}, \lambda > p, \mu > 0, \nu,\xi \in \mathbb{C}, \xi \ne 0, \phi \in \mathcal{P}$, $z \in \mathbb{U}^*$ and (4.1) holds. Then the function $\mathcal{J}_\mu F_\nu^\xi(f) \in \Sigma_p$ defined by*

$$\mathcal{J}_\mu F_\nu^\xi(f)(z) := \Big(\frac{\nu+p-p\xi}{z^{\nu+p}}\int_0^z t^{\nu+p-1}(\mathcal{J}_\mu f(t))^\xi \mathrm{d}t\Big)^{\frac{1}{\xi}}, \quad \Re(\nu+p-p\xi) > 0; z \in \mathbb{U}^*$$

belongs to the class $\mathcal{MS}_\mu(p;\beta;\phi)$.

5. Inclusion relationships

Using Lemmas 1.5 and 1.6, we obtain the inclusion relationships of the subclasses of Σ_p. An argument similar to one used in Theorem 4.1, the proof of the following results is not difficult but too long to give here.

Theorem 5.1 *Let $\alpha_i, \beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-$ $(i = 1,\dots,l; j = 1,\dots,m), l \le m+1, l,m \in \mathbb{N}_0, \tau \in \mathbb{N}, p \in \mathbb{N}, \beta \in \mathbb{R}, 0 \le \beta < \frac{1}{p}, \lambda > p, \mu > 0, \Re(\zeta) > 0, \phi \in \mathcal{P}, z \in \mathbb{U}^*$ and*

$$\Re\Big(\frac{1}{(1-p\beta)\phi(z)+p\beta}\Big) < \min\Big\{\frac{p+\mu}{p}, \frac{\Re(\zeta)+p}{p}\Big\}. \tag{5.1}$$

Then

$$\mathcal{MS}_{\mu+1}(p;\beta;\phi) \subset \mathcal{MS}_\mu(p;\beta;\phi) \subset \mathcal{MSF}_\mu^\zeta(p;\beta;\phi).$$

Corollary 5.2 *Let $\alpha_i, \beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-$ $(i = 1,\dots,l; j = 1,\dots,m), l \le m+1, l,m \in \mathbb{N}_0, \tau \in \mathbb{N}, p \in \mathbb{N}, \beta \in \mathbb{R}, 0 \le \beta < \frac{1}{p}, \lambda > p, \mu > 0, \Re(\zeta) > 0, \phi \in \mathcal{P}, z \in \mathbb{U}^*$ and (5.1) holds. Then*

$$\mathcal{MK}_{\mu+1}(p;\beta;\phi) \subset \mathcal{MK}_\mu(p;\beta;\phi) \subset \mathcal{MKF}_\mu^\zeta(p;\beta;\phi).$$

Theorem 5.3 *Let* $\alpha_i, \beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-$ $(i = 1, \dots, l; j = 1, \dots, m), l \le m+1, l, m \in \mathbb{N}_0, \tau \in \mathbb{N}, p \in \mathbb{N}, \beta \in \mathbb{R}, 0 \le \beta < \frac{1}{p}, \lambda > p, \mu > 0, \Re(\zeta) > 0, \phi, \psi \in \mathcal{P}, z \in \mathbb{U}^*$ *and (5.1) holds. Then*

$$\mathcal{MCS}_{\mu+1}(p;\beta;\phi,\psi) \subset \mathcal{MCS}_{\mu}(p;\beta;\phi,\psi) \subset \mathcal{MCSF}_{\mu}^{\zeta}(p;\beta;\phi,\psi).$$

Corollary 5.4 *Let* $\alpha_i, \beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-$ $(i = 1, \dots, l; j = 1, \dots, m), l \le m+1, l, m \in \mathbb{N}_0, \tau \in \mathbb{N}, p \in \mathbb{N}, \beta \in \mathbb{R}, 0 \le \beta < \frac{1}{p}, \lambda > p, \mu > 0, \Re(\zeta) > 0, \phi, \psi \in \mathcal{P}, z \in \mathbb{U}^*$ *and (5.1) holds. Then*

$$\mathcal{MCK}_{\mu+1}(p;\beta;\phi,\psi) \subset \mathcal{MCK}_{\mu}(p;\beta;\phi,\psi) \subset \mathcal{MCKF}_{\mu}^{\zeta}(p;\beta;\phi,\psi).$$

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