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On the Fourth Power Mean of Generalized Three-Term Exponential Sums

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Abstract The main purpose of this paper is using estimates for trigonometric sums and properties of congruence to study the computation of one kind of fourth power mean of a generalized three-term exponential sum, and give an interesting identity for it.

Keywords generalized three-term exponential sums; fourth power mean; identity

MR(2010) Subject Classification 11L03; 11L07

1. Introduction

Let $q \geq 3$ be a positive integer, χ be any Dirichlet character mod q. For any integers m and n, the generalized three-term exponential sum $C(m,n,k,\chi;q)$ is defined as follows:

$$C(m,n,k,\chi;q)=\sum_{a=1}^{q-1}\chi(a)e\Big(\frac{a^k+ma^2+na}{q}\Big),$$

where $k \geq 3$ is a fixed integer and $e(y)=e^{2\pi iy}$.

Many researchers have studied the various properties of this exponential sums and related sums, and obtained a series of results. Some related contents can be found in references [2–9]. For example, the first author and Han [6] proved that for any integer $k \geq 3$, we have the identity

$$\sum_{m=1}^{p}\sum_{n=1}^{p}\Big|\sum_{a=1}^{p-1}e\Big(\frac{ma^k+na^2+a}{p}\Big)\Big|^4=2p^4-3p^3-p^2\cdot C(k,p), \tag{1}$$

where the constant $C(k,p)$ is defined as follows:

$$C(k,p)=\sum_{\substack{a=1\\ a^k+b^k\equiv c^k+1 \bmod p\\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}1.$$

In particular, if $k=6$, then we have the identity

$$\sum_{m=1}^{p}\sum_{n=1}^{p}\Big|\sum_{a=1}^{p-1}e\Big(\frac{ma^6+na^2+a}{p}\Big)\Big|^4=\begin{cases}2p^4-11p^3+16p^2, & \text{if } p\equiv 3 \bmod 4,\\ 2p^4-15p^3+36p^2, & \text{if } p\equiv 1 \bmod 4.\end{cases}$$

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It seems that the mean value of the generalized three-term exponential sums

$$\sum_{m=1}^{p}\sum_{n=1}^{p}\Big|\sum_{a=1}^{p-1}\chi(a)e\Big(\frac{a^k+ma^2+na}{p}\Big)\Big|^4, \tag{2}$$

has not ever been studied, at least we have not seen any related result before. The problem is interesting, because it can reflect more or less the upper bound estimates of $C(m,n,k,\chi;p)$. It is easy to see that the mean value (1) is the best possible. So we have reason to believe that (2) and (1) have similar asymptotic properties. In fact, we can use the analytic method and properties of the congruences to give an exact computational formula for (2). That is, we shall prove the following:

Theorem 1.1 *Let $p\geq 3$ be a prime, χ be any Dirichlet character* mod p. *Then for any integer $k\geq 3$ with $(k,p-1)=1$, we have the identity*

$$\sum_{m=1}^{p}\sum_{n=1}^{p}\Big|\sum_{a=1}^{p-1}\chi(a)e\Big(\frac{a^k+ma^2+na}{p}\Big)\Big|^4 = 2p^4-5p^3+3p^2.$$

Theorem 1.2 *Let $p\geq 3$ be a prime, χ be any Dirichlet character* mod p. *Then for any integer $k\geq 4$ with $(k,p-1)=1$, we have the identity*

$$\sum_{m=1}^{p}\sum_{n=1}^{p}\Big|\sum_{a=1}^{p-1}\chi(a)e\Big(\frac{a^k+ma^3+na}{p}\Big)\Big|^4$$

$$= \begin{cases} 3p^2(p-1)(p-2), & \text{if } \chi \text{ is a real character mod } p,\\ p^2(p-1)(2p-5), & \text{if } \chi \text{ is not a real character mod } p.\end{cases}$$

It is very strange that the value we obtain in Theorem 1.1 has nothing to do with the character χ mod p.

2. Proofs of the theorems

In this section, we will give the proofs of our theorems directly. Hereinafter, we will use many properties of trigonometric sums and congruences, all of which can be found in [1] and [7], so they will not be repeated here.

Proof of Theorem 1.1 Note that $(k,p-1)=1$. From the trigonometric identity

$$\sum_{a=0}^{p-1}e\Big(\frac{na}{p}\Big) = \begin{cases} p, & \text{if } (n,p)=p,\\ 0, & \text{if } (n,p)=1,\end{cases} \tag{3}$$

and properties of reduced residue systems mod p we have

$$\sum_{m=1}^{p}\sum_{n=1}^{p}\Big|\sum_{a=1}^{p-1}\chi(a)e\Big(\frac{a^k+ma^2+na}{p}\Big)\Big|^4$$

$$=\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\chi(ab\bar{c}\bar{d})\sum_{m=1}^{p}\sum_{n=1}^{p}e\Big(\frac{m(a^2+b^2-c^2-d^2)+n(a+b-c-d)}{p}\Big)\times$$

$$e\Big(\frac{a^k+b^k-c^k-d^k}{p}\Big)$$

$$=\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\chi(ab\overline{c})\sum_{m=1}^{p}\sum_{n=1}^{p}e\Big(\frac{md^2(a^2+b^2-c^2-1)+nd(a+b-c-1)}{p}\Big)\times e\Big(\frac{d^k(a^k+b^k-c^k-1)}{p}\Big)$$

$$=p^2\cdot\sum_{\substack{a=1\\ a+b\equiv c+1 \bmod p\\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi(ab\overline{c})\sum_{d=1}^{p-1}e\Big(\frac{d^k(a^k+b^k-c^k-1)}{p}\Big)$$

$$=p^2\cdot\sum_{\substack{a=1\\ a+b\equiv c+1 \bmod p\\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi(ab\overline{c})\sum_{d=1}^{p-1}e\Big(\frac{d(a^k+b^k-c^k-1)}{p}\Big)$$

$$=p^3\sum_{\substack{a=1\\ a+b\equiv c+1 \bmod p\\ a^2+b^2\equiv c^2+1 \bmod p\\ a^k+b^k\equiv c^k+1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi(ab\overline{c})-p^2\cdot\sum_{\substack{a=1\\ a+b\equiv c+1 \bmod p\\ a^2+b^2\equiv c^2+1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi(ab\overline{c})$$

$$=p^3\sum_{\substack{a=1\\ a+b\equiv c+1 \bmod p\\ ab\equiv c \bmod p\\ a^k+b^k\equiv c^k+1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi(ab\overline{c})-p^2\cdot\sum_{\substack{a=1\\ a+b\equiv c+1 \bmod p\\ ab\equiv c \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi(ab\overline{c})$$

$$=p^3\sum_{\substack{a=1\\ a+b\equiv ab+1 \bmod p\\ a^k+b^k\equiv (ab)^k+1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}1-p^2\cdot\sum_{\substack{a=1\\ a+b\equiv ab+1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}1$$

$$=p^3\sum_{\substack{a=1\\ (a-1)(b-1)\equiv 0 \bmod p\\ (a^k-1)(b^k-1)\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}1-p^2\cdot\sum_{\substack{a=1\\ (a-1)(b-1)\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}1$$

$$=p^2(p-1)(2p-3)=2p^4-5p^3+3p^2.$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2 From identity (3) and the method of proving Theorem 1.1 we have

$$\sum_{m=1}^{p}\sum_{n=1}^{p}\Big|\sum_{a=1}^{p-1}\chi(a)e\Big(\frac{a^k+ma^3+na}{p}\Big)\Big|^4$$

$$=\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\chi(ab\overline{c}\overline{d})\sum_{m=1}^{p}\sum_{n=1}^{p}e\Big(\frac{m(a^3+b^3-c^3-d^3)+n(a+b-c-d)}{p}\Big)\times e\Big(\frac{a^k+b^k-c^k-d^k}{p}\Big)$$

$$= p^2 \cdot \sum_{\substack{a=1 \\ a+b\equiv c+1 \bmod p \\ a^3+b^3\equiv c^3+1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi(ab\overline{c}) \sum_{d=1}^{p-1} e\Big(\frac{d(a^k+b^k-c^k-1)}{p}\Big)$$

$$= p^3 \sum_{\substack{a=1 \\ a+b\equiv c+1 \bmod p \\ a^3+b^3\equiv c^3+1 \bmod p \\ a^k+b^k\equiv c^k+1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi(ab\overline{c}) - p^2 \cdot \sum_{\substack{a=1 \\ a+b\equiv c+1 \bmod p \\ a^3+b^3\equiv c^3+1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi(ab\overline{c})$$

$$= p^3 \sum_{\substack{a=1 \\ a+b\equiv c+1 \bmod p \\ ab(a+b)\equiv c(c+1) \bmod p \\ a^k+b^k\equiv c^k+1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi(ab\overline{c}) - p^2 \cdot \sum_{\substack{a=1 \\ a+b\equiv c+1 \bmod p \\ ab(a+b)\equiv c(c+1) \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi(ab\overline{c})$$

$$\equiv p^3 \cdot A - p^2 \cdot B. \tag{4}$$

We turn to compute A and B. From the orthogonality of characters mod p we have the identity

$$\sum_{a=1}^{p-1} \chi\left(a^2\right) = \begin{cases} p-1, & \text{if } \chi \text{ is a real character mod } p, \\ 0, & \text{if } \chi \text{ is not a real character mod } p. \end{cases} \tag{5}$$

Note that $(k, p-1) = 1$, so k is an odd number. Therefore, from (5) we have

$$A = \sum_{\substack{a=1 \\ a+b\equiv c+1 \bmod p \\ ab(a+b)\equiv c(c+1) \bmod p \\ a^k+b^k\equiv c^k+1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi(ab\overline{c}) = \sum_{\substack{a=1 \\ a+b\equiv 0 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \chi(-ab) + \sum_{\substack{a=1 \\ (a-1)(b-1)\equiv 0 \bmod p \\ (a+b,p)=1}}^{p-1} \sum_{b=1}^{p-1} 1$$

$$= \sum_{a=1}^{p-1} \chi^2(a) + 2(p-2) - 1$$

$$= \begin{cases} 3p-6, & \text{if } \chi \text{ is a real character mod } p, \\ 2p-5, & \text{if } \chi \text{ is not a real character mod } p. \end{cases} \tag{6}$$

Similarly, from (5) we also have the identity

$$B = \sum_{\substack{a=1 \\ a+b\equiv c+1 \bmod p \\ ab(a+b)\equiv c(c+1) \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi(ab\overline{c})$$

$$= \begin{cases} 3p-6, & \text{if } \chi \text{ is a real character mod } p, \\ 2p-5, & \text{if } \chi \text{ is not a real character mod } p. \end{cases} \tag{7}$$

Now combining (4), (6) and (7) we immediately get the identity

$$\sum_{m=1}^{p} \sum_{n=1}^{p} \Big| \sum_{a=1}^{p-1} \chi(a) e\Big(\frac{a^k+ma^2+na}{p}\Big)\Big|^4$$

$$= \begin{cases} 3p^2(p-1)(p-2), & \text{if } \chi \text{ is a real character mod } p, \\ p^2(p-1)(2p-5), & \text{if } \chi \text{ is not a real character mod } p. \end{cases}$$

This completes the proof of Theorem 1.2. □

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