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On the Proximate Type of Analytic Function Represented by Laplace-Stieltjes Transformation

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Abstract In present paper, we study precisely the growth of analytic functions defined by zero order Laplace-Stieltjes transformation converging in right plane. The coefficient characterizations of generalized logarithmic p-type and generalized lower logarithmic p-type are obtained, which improve the results of logarithmic type and lower logarithmic type.

Keywords Laplace-Stieltjes transform; zero order; logarithmic p-type; lower logarithmic p-type

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1. Introduction

Consider the Laplace-Stieltjes transform

$$F(s)=\int_0^{+\infty}\mathrm{e}^{-sx}\mathrm{d}\alpha(x),\quad s=\sigma+it,\ \sigma,t\in\mathbb{R} \tag{1.1}$$

where $\alpha(x)$ is a defined real-valued or complex-valued function with $x\geq 0$, and it is of bounded variation on any closed interval $[0,X]$ $(0<X<+\infty)$.

Put a sequence

$$0=\lambda_0<\lambda_1<\lambda_2<\cdots<\lambda_n\uparrow+\infty, \tag{1.2}$$

which satisfies the following conditions

$$\overline{\lim_{n\to\infty}}\frac{n}{\lambda_n}=D<+\infty,\quad \overline{\lim_{n\to\infty}}(\lambda_{n+1}-\lambda_n)=h<+\infty. \tag{1.3}$$

It is known [1] that the transform (1.1) represents an analytic function $F(s)$ in the right half plane when the transform satisfies

$$\overline{\lim_{n\to\infty}}\frac{\log A_n^*}{\lambda_n}=0, \tag{1.4}$$

where $A_n^*=\sup_{\lambda_n<x\leq\lambda_{n+1},t\in\mathbb{R}}|\int_{\lambda_n}^{x}\mathrm{e}^{-ity}\mathrm{d}\alpha(y)|$.

Let D_0 denote the class of all functions $F(s)$ represented by (1.1) and satisfying conditions (1.2) to (1.4). Kong [2] defined the order of $F(s)\in D_0$ as

$$\rho=\overline{\lim_{\sigma\to 0}}\frac{\log^+\log^+M_u(\sigma,F)}{-\log\sigma},$$

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where $M_u(\sigma,F)=\sup_{0<x<+\infty,t\in\mathbb{R}}|\int_0^x \mathrm{e}^{-(\sigma+\mathrm{i}t)y}\mathrm{d}\alpha(y)|$ called the maximum modulus of $F(s)$ in the right half-place. A function $F(s)$ is said to be of slow growth if $\rho=0$. To study the growth of the functions of slow growth, the concept of $\rho(h,F)$-order of $F(s)$ was introduced by Luo [3] as

$$\rho(h,F)=\overline{\lim_{\sigma\to 0}}\frac{h(\log^+ M_u(\sigma,F))}{h(-\log\sigma)},$$

where $h\in\Delta$ and Δ is the class of all functions satisfying the following conditions (I) and (II):

(I) $h(x)$ is defined on $[a,\infty)$ and is positive, strictly increasing, differentiable and tends to ∞ as $x\to\infty$;

(II) $\lim_{x\to\infty}\frac{h(cx)}{h(x)}=1$ for all $c, 0<c<\infty$.

In particular, if $h(x)=\log^{[p]}(x)$, $p\geq 1$ ($\log^{[1]}(x)=\log x, \log^{[p]}(x)=\log(\log^{p-1}(x)), p\geq 2$), we shall call $\rho(h,F)$-order as logarithmic p-order of $F(s)$ and denote it ρ_p. In present paper, we have introduced some new growth parameters to compare precisely the growths of two functions belonging to D_0 and having the same logarithmic p-order. For this, we first need the following definition of logarithmic p-proximate order, which, for the case $p=1$, includes the definition of logarithmic proximate order, due to Xu [4].

Definition 1.1 *A real valued function $\rho_p(\sigma)$ is called a logarithmic p-proximate order if it satisfies the following:*

(1) $\rho_p(\sigma)$ is a positive, continuous and piecewise differentiable function for all σ such that $0<\sigma<\sigma_0<\infty$;

(2) $\lim_{\sigma\to\infty}\rho_p(\sigma)=\rho_p$ $(1<\rho_p<\infty)$;

(3) $\lim_{\sigma\to 0}{\rho_p}'(\sigma)\sigma\Pi_{k=1}^{p+1}\log^{[k]}(1/\sigma)=0$,

where ${\rho_p}'(\sigma)$ is either the right or the left hand derivative of $\rho_p(\sigma)$ where they are different.

We now define the generalized logarithmic p-type T_p and generalized lower logarithmic p-type t_p of $F(s)\in D_0$ with respect to a given logarithmic p-proximate order $\rho_p(\sigma)$ as

$$T_p=\overline{\lim_{\sigma\to 0}}\frac{\log^{[p]}M_u(\sigma,F)}{(\log^{[p]}(1/\sigma))^{\rho_p(\sigma)}},\quad t_p=\underline{\lim_{\sigma\to 0}}\frac{\log^{[p]}M_u(\sigma,F)}{(\log^{[p]}(1/\sigma))^{\rho_p(\sigma)}}. \tag{1.5}$$

Definition 1.2 *A logarithmic p-proximate order $\rho_p(\sigma)$ is called a logarithmic p-proximate order of $F(s)\in D_0$ if $0<T_p<\infty$.*

Definition 1.3 *$F(s)\in D_0$ is said to be of perfectly regular logarithmic growth with respect to its logarithmic p-proximate order ρ_p (σ) if $T_p=t_p<\infty$.*

For a function $F(s)\in D_0$, having logarithmic p-order ρ_p $(1<\rho_p<\infty)$, the existence of a logarithmic p-proximate order $\rho_p(\sigma)$ can be established on the lines of those used by Levin [5, p. 35–39].

2. Some lemmas

Lemma 2.1 *Let $\rho_p(\sigma), p\geq 1$ be a logarithmic p-proximate order. Then the function $(\log^{[p]}(1/\sigma))^{\rho_p(\sigma)}$ is a monotonically decreasing function of σ for $0<\sigma<\sigma_0$.*

Proof Let $H(\sigma) = (\log^{[p]}(1/\sigma))^{\rho_p(\sigma)}$. Taking logarithm on both the sides and then differentiating with respect to σ, we get

$$H'(\sigma) = {\rho_p}'(\sigma)(\log^{[p]}(1/\sigma))^{\rho_p(\sigma)}\log^{[p+1]}(1/\sigma) - \frac{\rho_p(\sigma)(\log^{[p]}(1/\sigma))^{\rho_p(\sigma)}}{\sigma\prod_{k=1}^{p}\log^{[k]}(1/\sigma)}.$$

On using the properties (2) and (3) of logarithmic -proximate order, we have

$$H'(\sigma) < \frac{(\varepsilon-\rho_p)(\log^{[p]}(1/\sigma))^{\rho_p(\sigma)}}{\sigma\prod_{k=1}^{p}\log^{[k]}(1/\sigma)} < 0, \quad 0 < \varepsilon < \rho_p.$$

Hence the lemma follows. □

Since $(\log^{[p]}(1/\sigma))^{\rho_p(\sigma)}$ is a monotonically decreasing function of σ for $0 < \sigma < \sigma_0$, a single valued real function $\psi_p(t)$ of t can be defined for $t > t_0$ such that

$$t = \frac{1}{\sigma} \text{ if and only if } \psi_p(t) = (\log^{[p]}(1/\sigma))^{\rho_p(\sigma)}. \tag{2.1}$$

If $\psi_p(t)$ is defined as above, the following lemma follows easily.

Lemma 2.2 *Let $\rho_p(\sigma)$, $p \geq 1$, be a logarithmic p-proximate order and let $\psi_p(t)$ be as defined in (2.1). Then*

$$\lim_{t\to\infty}\frac{\mathrm{dlog}\,\psi_p(t)}{\mathrm{dlog}^{[p+1]}t} = \rho_p \tag{2.2}$$

and for every b $(0 < b < \infty)$

$$\lim_{t\to\infty}\frac{\psi_p(t^b)}{\psi_p(t)} = E(b,\rho_p) \tag{2.3}$$

where $E(b,\rho_p) = b^{\rho_p}$ for $p = 1$ and $E(b,\rho_p) = 1$ for $p \geq 2$.

3. Main results

Theorem 3.1 *Let $F(s) \in D_0$ be given by (1.1). Assume that $F(s)$ has logarithmic p-proximate order $\rho_p(\sigma)$ and logarithmic p-order ρ_p $(1 < \rho_p < \infty)$. Then the generalized logarithmic p-type T_p of $F(s)$ with respect to the logarithmic p-proximate order $\rho_p(\sigma)$ is given by*

$$T_p = \overline{\lim_{n\to\infty}}\frac{\log^{[p]}A_n^*}{\psi_p(\lambda_n)} \tag{3.1}$$

where $\psi_p(\lambda_n)$ is defined by (2.1).

Remark 3.2 For $p = 1$, the above theorem is due to Theorem 3.2 of Xu [4].

Proof From (1.5), given $\varepsilon > 0$, there exists $\sigma_0 = \sigma_0(\varepsilon)$ such that for $0 < \sigma < \sigma_0$, we have

$$\log M_u(\sigma,F) < \exp^{[p-1]}\{(T_p+\varepsilon)(\log^{[p]}(1/\sigma))^{\rho_p(\sigma)}\}.$$

So we have

$$\log A_n^* - \sigma\lambda_n < \exp^{[p-1]}\{(T_p+\varepsilon)(\log^{[p]}(1/\sigma))^{\rho_p(\sigma)}\}$$

for $0<\sigma<\sigma_0$ and all n. Taking, in particular, $\sigma=\frac{1}{\lambda_n}$ in the above inequality gives

$$\log A_n^*-1<\exp^{[p-1]}\{(T_p+\varepsilon)\psi_p(\lambda_n)\}.$$

The above relation gives

$$\theta\le T_p+\varepsilon,$$

where θ denotes the limit superior on the right hand side of (3.1). Since $\varepsilon>0$ is arbitrary, this in turn gives that

$$\theta\le T_p. \tag{3.2}$$

On the other hand, from the definition of θ, for given $\varepsilon>0$, we have for all $n>n_0=n_0(\varepsilon)$,

$$\log^{[p]}A_n^*<(\theta+\varepsilon)\psi_p(\lambda_n). \tag{3.3}$$

Now, since (1.3) holds, we have $n<D_1\lambda_n$ for all $n>n_1=n_1(D_1)$, where $D_1>D$. We can assume without loss of generality that $n_0>n_1$.

For any $x>0$, there exists $n\in\mathbb{N},\lambda_n\le x\le\lambda_{n+1}$, such that

$$\int_0^x \mathrm{e}^{-(\sigma+\mathrm{i}t)y}\mathrm{d}\alpha(y)=\sum_{k=1}^{n-1}\int_{\lambda_k}^{\lambda_{k+1}}\mathrm{e}^{-(\sigma+\mathrm{i}t)y}\mathrm{d}\alpha(y)+\int_{\lambda_n}^{x}\mathrm{e}^{-(\sigma+\mathrm{i}t)y}\mathrm{d}\alpha(y).$$

Let

$$I_k(x,\mathrm{i}t)=\int_{\lambda_k}^{x}\mathrm{e}^{-\mathrm{i}ty}\mathrm{d}\alpha(y),\quad \lambda_k\le x\le\lambda_{k+1}.$$

For any $t\in\mathbb{R}$, we have

$$|I_k(x,\mathrm{i}t)|\le A_k^*\le\mu(\sigma,F)\mathrm{e}^{\lambda_k\sigma},\quad \sigma>0.$$

Hence for any $x\in[\lambda_n,\lambda_{n+1}]$ and $\sigma>0$

$$\begin{aligned}\int_0^x \mathrm{e}^{-(\sigma+\mathrm{i}t)y}\mathrm{d}\alpha(y)=&\sum_{k=1}^{n-1}[\mathrm{e}^{-\lambda_{k+1}\sigma}I_k(\lambda_{k+1},\mathrm{i}t)-\int_{\lambda_k}^{\lambda_{k+1}}I_k(y,\mathrm{i}t)\mathrm{d}\mathrm{e}^{-\sigma y}]+\\ &\mathrm{e}^{-x\sigma}I_n(x,\mathrm{i}t)-\int_{\lambda_n}^{x}I_n(y,\mathrm{i}t)\mathrm{d}\mathrm{e}^{-\sigma y}.\end{aligned}$$

So

$$\begin{aligned}\Big|\int_0^x \mathrm{e}^{-(\sigma+\mathrm{i}t)y}\mathrm{d}\alpha(y)\Big|&\le\sum_{k=1}^{n-1}|A_k^*\mathrm{e}^{-\lambda_{k+1}\sigma}-A_k^*(\mathrm{e}^{-\sigma\lambda_{k+1}}-\mathrm{e}^{-\sigma\lambda_k})|+|A_n^*\mathrm{e}^{-x\sigma}-A_n^*(\mathrm{e}^{-\sigma x}-\mathrm{e}^{-\sigma\lambda_n})|\\ &\le\sum_{n=1}^{\infty}A_n^*\mathrm{e}^{-\lambda_n\sigma}.\end{aligned}$$

Using (3.3), we have

$$\Big|\int_0^x \mathrm{e}^{-(\sigma+\mathrm{i}t)y}\mathrm{d}\alpha(y)\Big|\le Q(n_0)+\sum_{n=n_0+1}^{\infty}\exp\{\exp^{[p-1]}\{(\theta+\varepsilon)\psi_p(\lambda_n)\}-\sigma\lambda_n\}, \tag{3.4}$$

where $Q(n_0)$, the sum of first terms n_0, is bounded.

For each σ ($\sigma>0$), we define a natural number $n(\sigma)$ as

$$\lambda_{n(\sigma)}\le\frac{4}{\sigma^2}<\lambda_{n(\sigma)+1}.$$

It is seen that $\exp^{[p-1]}\{b\psi_p(x)\}/x < 1/\sqrt{x}$, $0 < b < \infty$, for all $x > x_0(b)$. Hence for σ sufficiently close to 0 and all $n > n(\sigma)$, we have

$$\frac{\exp^{[p-1]}\{(\theta+\varepsilon)\psi_p(\lambda_n)\}}{\lambda_n} < \frac{1}{\sqrt{\lambda_{n(\sigma)+1}}} < \frac{\sigma}{2}. \tag{3.5}$$

Using (3.5), for all σ sufficiently close to 0, we have

$$\begin{aligned}
&\sum_{n=n(\sigma)+1}^{\infty} \exp\{\exp^{[p-1]}\{(\theta+\varepsilon)\psi_p(\lambda_n)\} - \sigma\lambda_n\} \\
&\leq \sum_{n=n(\sigma)+1}^{\infty} \exp\{\frac{-\sigma\lambda_n}{2}\} \leq \sum_{n=n(\sigma)+1}^{\infty} \exp\{\frac{-\sigma n}{2D_1}\} \\
&\leq \frac{1}{1-\exp\{\frac{-\sigma}{2D_1}\}} \sim \frac{2D_1}{\sigma}.
\end{aligned} \tag{3.6}$$

Now, consider the function $G(x)$, defines as

$$G(x) = \exp^{[p-1]}\{(\theta+\varepsilon)\psi_p(x)\} - \sigma x.$$

Let x_* be defined as $G(x_*) = \max_{x_0 \leq x \leq \infty} G(x)$. Then

$$(\theta+\varepsilon)\{\prod_{k=1}^{p-1} \exp^{[k]}\{(\theta+\varepsilon)\psi_p(x_*)\}\}\frac{\mathrm{d}\psi_p(x)}{\mathrm{d}x}\Big|_{x=x_*} = \sigma, \tag{3.7}$$

where the quantity inside the curly bracket is assumed to be 1 for $p = 1$. As $\sigma \to 0$, the relation (3,7), in view of (2.2), gives that

$$x_* = (1/\sigma)^{1+o(1)}.$$

Thus,

$$\max_{x_0 \leq x \leq \infty} G(x) \leq \exp^{[p-1]}\{(\theta+\varepsilon)\psi_p((1/\sigma)^{1+o(1)})\}. \tag{3.8}$$

From (3.4), (3.6) and (3.8), we have

$$M_u(\sigma, F) \leq Q(n_0) + n(\sigma)\exp^{[p]}\{(\theta+\varepsilon)\psi_p((1/\sigma)^{1+o(1)})\} \cdot \frac{2D_1}{\sigma}.$$

Now, by the definition of $n(\sigma)$, we have

$$M_u(\sigma, F) \leq \frac{(4D_1)^2}{\sigma^3}\exp^{[p]}\{(\theta+\varepsilon)\psi_p((1/\sigma)^{1+o(1)})\}$$

or

$$\log M_u(\sigma, F) \leq 4\log\frac{1}{\sigma} + \exp^{[p-1]}\{(\theta+\varepsilon)\psi_p((1/\sigma)^{1+o(1)})\}.$$

The above relation, in view of (2.3), gives that

$$T_p \leq \theta + \varepsilon.$$

And since $\varepsilon > 0$ is arbitrary, we have

$$T_p \leq \theta. \tag{3.9}$$

In view of (3.2) and (3.9), the proof of the theorem is completed. □

Theorem 3.3 *Let $F(s)\in D_0$ be given by (1.1). Assume that $F(s)$ has logarithmic p-proximate order $\rho_p(\sigma)$ and logarithmic p-order ρ_p $(1<\rho_p<\infty)$. Let $\varphi(n)=\frac{\log A_n^*-\log A_{n+1}^*}{\lambda_{n+1}-\lambda_n}$ be ultimately a non-decreasing function of. Then the generalized lower logarithmic p-type t_p of $F(s)$ satisfies*

$$t_p\le \varliminf_{n\to\infty}\frac{\log^{[p]}A_n^*}{\psi_p(\lambda_n)} \tag{3.10}$$

where $\psi_p(\lambda_n)$ is defined by (2.1). Further, if $\psi_p(\lambda_n)\sim\psi_p(\lambda_{n+1})$ as $n\to\infty$, then the equality holds in (3.10).

Remark 3.4 For $p=1$, the above theorem is due to Theorem 3.3 of Xu [4].

The proof of the theorem can be constructed by suitably adopting the techniques used in [4] and the present paper and so we omit the proof.

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