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Life Span of Solutions with Large Initial Data for a Semilinear Parabolic System

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Abstract The paper deals with heat equations coupled via exponential nonlinearities. We are interested in the life span (or blow-up time) and obtain the maximal existence time of blow-up solutions. Our proof is based on the comparison principle and Kaplan's method.

Keywords semilinear parabolic system; blow-up; life span

MR(2010) Subject Classification 35K05; 35K60

1. Introduction

In this paper, we consider the following nonlinear parabolic system

$$u_t = \Delta u + e^{pv}, \quad x \in \Omega,\ t > 0, \tag{1.1}$$

$$v_t = \Delta v + e^{qu}, \quad x \in \Omega,\ t > 0, \tag{1.2}$$

$$u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \tag{1.3}$$

$$u(x,0) = \lambda\varphi(x),\ v(x,0) = \lambda\psi(x), \quad x \in \Omega, \tag{1.4}$$

where $p, q > 0$, Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$. $\lambda > 0$ is a parameter, φ and ψ are nonnegative continuous functions on $\bar{\Omega}$.

In [1], it was shown that the problem (1.1)–(1.4) with nonnegative continuous initial data has a unique classical solution. We denote by T_λ^* the maximal existence time of a classical solution (u, v) of problem (1.1)–(1.4), that is

$$T_\lambda^* = \sup\big\{T > 0,\ \sup_{0 \le t \le T}(\|u(\cdot,t)\|_\infty + \|v(\cdot,t)\|_\infty) < \infty\big\},$$

and we call T_λ^* the life span of (u, v). If $T_\lambda^* < \infty$, then we have

$$\lim_{t \to T_\lambda^*} \sup \|u(\cdot,t)\|_\infty = \lim_{t \to T_\lambda^*} \sup \|v(\cdot,t)\|_\infty = \infty.$$

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We are interested in the asymptotic behavior of the life span T_λ^* as $\lambda \to \infty$.

Since Fujita's classic work [2], the single equation

$$u_t = \Delta u + u^p \tag{1.5}$$

has been studied extensively in various directions. Friedman and Lacey [3] gave a result on the life span of solutions of (1.5) in the case of small diffusion. Subsequently, Gui and Wang [4], Lee and Ni [5] obtained the leading term of the expansion of the life span T_ρ of the solution for (1.5) with the initial data $\rho\varphi(x)$. They proved that T_ρ is expanded as

$$T_\rho = \frac{1}{p-1}\|\varphi\|_\infty^{1-p}\rho^{1-p} + o(\rho^{1-p})$$

as $\rho \to \infty$. Later, Mizoguchi and Yanagida [6] extended the result and determined the second term of the expansion of T_ρ. They proved that when φ attains the maximum at only one point $a \in \Omega$, T_ρ is expanded as

$$T_\rho = \frac{1}{p-1}\|\varphi\|_\infty^{1-p}\rho^{1-p} + \frac{2}{p-1}\|\varphi\|_\infty^{2(1-p)-1}|\Delta\varphi(a)|\rho^{2(1-p)} + o(\rho^{2(1-p)})$$

as $\rho \to \infty$. Moreover, Mizoguchi and Yanagida [7] extended the result on the life span of solutions of (1.5) in the case of small diffusion. In [8], Sato extended the results to general nonlinearities $f(u)$ in the case of large initial data. Parabolic systems of the following form

$$u_t = \Delta u + f(v), \quad v_t = \Delta v + g(u) \tag{1.6}$$

have also been studied in several directions. In [9], Sato investigated (1.6) with $f(v)$ and $g(u)$ replaced by v^p and u^q. In their article, they obtained the life span of (u, v) with large initial data. For other results on system (1.6), we refer the reader to the survey [10], the recent monograph [11] and the references therein.

On the other hand, much effort has been devoted to the study of parabolic system in the form (1.6), local and global existence, finite time blowup and blowup rate estimates, etc. In [1], Zheng and Zhao considered the radially symmetric solutions for the parabolic system

$$u_t = \Delta u + e^{mu+pv}, \quad v_t = \Delta v + e^{qu+nv}.$$

The parabolic equations (1.6) with the nonlinearities $f(v) = u^m e^{pv}$, $g(u) = u^q e^{nv}$ subject to null Dirichlet boundary conditions were considered in [12] by Liu and Li.

However, to the author's best knowledge, there is little literature on the study of the life span of solutions for problem (1.1)–(1.4). The aim of this paper is to obtain the leading term of the expansion of life span T_λ^* as $\lambda \to \infty$. In the following, we denote by M_φ and M_ψ the maximum of φ and ψ on $\overline{\Omega}$. Then our main results of this paper will be summarized as the following theorem.

Theorem 1.1 *Let $p, q > 0$. Suppose $\varphi, \psi \in C(\bar{\Omega})$ satisfy $\varphi, \psi \geq 0$ in Ω, $\varphi = \psi = 0$ on $\partial\Omega$, $\varphi + \psi \not\equiv 0$.*

(i) If $qM_\varphi > pM_\psi$, then we have

$$\lim_{\lambda\to\infty} T^*_\lambda \frac{e^{qM_\varphi\lambda}}{\lambda} = \frac{qM_\varphi - pM_\psi}{p}. \tag{1.7}$$

(ii) If $qM_\varphi < pM_\psi$, then we have

$$\lim_{\lambda\to\infty} T^*_\lambda \frac{e^{pM_\psi\lambda}}{\lambda} = \frac{pM_\psi - qM_\varphi}{q}. \tag{1.8}$$

2. Preliminaries

In this section we first consider the ODE system

$$z_t = e^{pw},\ w_t = e^{qz},\quad t > 0; \tag{2.1}$$

$$z(0) = \alpha,\ w(0) = \beta, \tag{2.2}$$

where α and β are nonnegative constants.

Here, for constants α and β with $(\alpha,\beta) \not\equiv (0,0)$, we define by $(z(t;\alpha,\beta), w(t;\alpha,\beta))$ the solution for problem (2.1)–(2.2). It is well known that $(z(t;\alpha,\beta), w(t;\alpha,\beta))$ exists and blows up in finite time. We then give the following lemma.

Lemma 2.1 *Let $p, q > 0$. Suppose that α, β are nonnegative constants and $(\alpha,\beta) \not\equiv (0,0)$. Then the life span of the solution (z,w) for problem (2.1)–(2.2) is*

$$T^*_{\alpha,\beta} = \int_\alpha^\infty \frac{\mathrm{d}s}{\frac{p}{q}\left(e^{qs} - e^{q\alpha}\right) + e^{p\beta}} = \int_\beta^\infty \frac{\mathrm{d}s}{\frac{q}{p}\left(e^{ps} - e^{p\beta}\right) + e^{q\alpha}}. \tag{2.3}$$

Proof Multiplying the first equation in (2.1) by e^{qz} and the second equation by e^{pw}, we obtain the equality $e^{qz}z_t = e^{pw}w_t$. Integrating this equality over $(0,t)$, we have

$$\frac{1}{q}(e^{qz} - e^{q\alpha}) = \frac{1}{p}(e^{pw} - e^{p\beta}).$$

Hence we get

$$e^{qz} = \frac{q}{p}\left(e^{pw} - e^{p\beta}\right) + e^{q\alpha},\quad e^{pw} = \frac{p}{q}\left(e^{qz} - e^{q\alpha}\right) + e^{p\beta}.$$

Substituting those equalities in the equations of (2.1), we see that (z,w) satisfies the initial-value problem

$$z_t = \frac{p}{q}\left(e^{qz} - e^{q\alpha}\right) + e^{p\beta},\quad t > 0,\ z(0) = \alpha, \tag{2.4}$$

$$w_t = \frac{p}{q}\left(e^{qz} - e^{q\alpha}\right) + e^{p\beta},\quad t > 0,\ w(0) = \beta. \tag{2.5}$$

Integrating equations in (2.4), (2.5) over $(0,t)$ yields

$$\int_\alpha^{z(t)} \frac{\mathrm{d}s}{\frac{p}{q}\left(e^{qs} - e^{q\alpha}\right) + e^{p\beta}} = t,\quad \int_\beta^{w(t)} \frac{\mathrm{d}s}{\frac{q}{p}\left(e^{ps} - e^{p\beta}\right) + e^{q\alpha}} = t.$$

This implies that the life span of (z,w) is

$$T^*_{\alpha,\beta} = \min\left\{\int_\beta^\infty \frac{\mathrm{d}s}{\frac{p}{q}\left(e^{qs} - e^{q\alpha}\right) + e^{p\beta}},\ \int_\alpha^\infty \frac{\mathrm{d}s}{\frac{q}{p}\left(e^{ps} - e^{p\beta}\right) + e^{q\alpha}}\right\}.$$

By using the change of variables $e^{p\xi} = \frac{p}{q}(e^{qs} - e^{q\alpha}) + e^{p\beta}$, we see that

$$\int_\beta^\infty \frac{\mathrm{d}s}{\frac{p}{q}(e^{qs} - e^{q\alpha}) + e^{p\beta}} = \int_\alpha^\infty \frac{\mathrm{d}\xi}{\frac{q}{p}(e^{p\xi} - e^{p\beta}) + e^{q\alpha}}.$$

We end this section by giving the following comparison principle which will play a key role in proving Theorem 1.1 in the next section. This Lemma can be proved in the same way as in [10,11]. Since the proof is more or less standard, and is therefore omitted here. □

Lemma 2.2 *Let $(\hat{u}, \hat{v})$ and $(\tilde{u}, \tilde{v})$ be a pair of upper-lower solutions of problem (1.1)–(1.4). Then the problem (1.1)–(1.4) has a unique solution (u, v) satisfying $(\tilde{u}, \tilde{v}) \le (u, v) \le (\hat{u}, \hat{v})$.*

3. Proof of Theorem 1.1

We divide Theorem 1.1 into Lemmas 3.1 and 3.2, in which we derive upper and lower estimates of T^*_λ.

Lemma 3.1 *Assume the assumptions of Theorem 1.1.*

(i) If $qM_\varphi > pM_\psi$, then we have

$$\liminf_{\lambda\to\infty} T^*_\lambda \frac{e^{qM_\varphi\lambda}}{\lambda} \ge \frac{qM_\varphi - pM_\psi}{p}. \tag{3.1}$$

(ii) If $qM_\varphi < pM_\psi$, then we have

$$\liminf_{\lambda\to\infty} T^*_\lambda \frac{e^{pM_\psi\lambda}}{\lambda} \ge \frac{pM_\psi - qM_\varphi}{q}. \tag{3.2}$$

Proof We give the proof of (i). It is obvious that the solution $(z(t;\lambda M_\varphi, \lambda M_\psi), w(t;\lambda M_\varphi, \lambda M_\psi))$, is a supersolution of problem (1.1)–(1.4), so we have

$$u(x,t) \le z(t;\lambda M_\varphi, \lambda M_\psi), \quad v(x,t) \le w(t;\lambda M_\varphi, \lambda M_\psi),$$

for $x \in \Omega$ and $0 < t < \min\{T^*_{\lambda M_\varphi,\lambda M_\psi}, T^*_\lambda\}$. This implies

$$T^*_\lambda \ge T^*_{\lambda M_\varphi,\lambda M_\psi}. \tag{3.3}$$

First we assume that $\varphi \ne 0$. Then by (3.4) and Lemma 2.1, a routine computation shows

$$T^*_\lambda \ge \int_{\lambda M_\varphi}^\infty \frac{\mathrm{d}s}{\frac{q}{p}(e^{ps} - e^{p\lambda M_\psi}) + e^{q\lambda M_\varphi}},$$

this yields

$$T^*_\lambda \ge \frac{(qM_\varphi - pM_\psi)\lambda + \ln p - \ln q}{pe^{q\lambda M_\varphi} - qe^{p\lambda M_\psi}}.$$

Hence, taking $\lambda \to \infty$, we have

$$\liminf_{\lambda\to\infty} T^*_\lambda \frac{e^{qM_\varphi\lambda}}{\lambda} \ge \frac{qM_\varphi - pM_\psi}{p}.$$

So we obtain (3.1). Slightly revising the above process, one can prove (ii). □

Next, we give an upper estimate of T^*_λ.

Lemma 3.2 *Assume the assumptions of Theorem 1.1.*

(i) If $qM_\varphi > pM_\psi$, then we have

$$\limsup_{\lambda\to\infty} T^*_\lambda \frac{e^{qM_\varphi\lambda}}{\lambda} \le \frac{qM_\varphi - pM_\psi}{p}. \tag{3.4}$$

(ii) If $qM_\varphi < pM_\psi$, then we have

$$\limsup_{\lambda\to\infty} T^*_\lambda \frac{e^{pM_\psi\lambda}}{\lambda} \le \frac{pM_\psi - qM_\varphi}{q}. \tag{3.5}$$

Proof We prove this lemma by using Kaplan's method [13]. We only give the proof of (i), and the other case (ii) can be proved similarly. First, we consider the case of $\varphi \ne 0$. Without loss of generality, we may assume that $\varphi(0) = M_\varphi$. We define by μ_R the first eigenvalue of $-\Delta$ in the ball $B_R = B_R(0)$ and ϕ_R the corresponding eigenfunction. Thus, we have

$$-\Delta\phi_R = \mu_R\phi_R \ \text{ in } \ B_R, \tag{3.6}$$

$$\phi_R = 0 \ \text{ on } \ \partial B_R, \tag{3.7}$$

we further assume that $\int_{B_R}\phi_R(x)\mathrm{d}x = 1$. We note that

$$\mu_R = \frac{\mu_1}{R^2}, \quad \phi_R(x) = R^{-N}\phi_1(\frac{x}{R}).$$

Let $B_R \subset \Omega$. We set

$$z(t) = \int_{B_R} u(x,t)\phi_R(x)\mathrm{d}x, \quad w(t) = \int_{B_R} v(x,t)\phi_R(x)\mathrm{d}x, \tag{3.8}$$

$$\alpha(R) = \int_{B_R} \varphi(x)\phi_R(x)\mathrm{d}x, \quad \beta(R) = \int_{B_R} \psi(x)\phi_R(x)\mathrm{d}x. \tag{3.9}$$

By $\varphi, \psi \in C(\bar\Omega)$, $\int_{B_1}\phi_1(x)\mathrm{d}x = 1$, we have

$$\lim_{R\to 0}\alpha(R) = \varphi(0), \quad \lim_{R\to 0}\beta(R) = \psi(0).$$

Multiplying the equations (1.1) and (1.4) by ϕ_R, integrating by parts and using Jensen's inequality, we obtain

$$z_t \ge -\mu_R z + e^{pw}, \quad t > 0, \tag{3.10}$$

$$w_t \ge -\mu_R w + e^{qz}, \quad t > 0, \tag{3.11}$$

$$z(0) = \lambda\alpha(R), \quad w(0) = \lambda\beta(R). \tag{3.12}$$

Hence, we have

$$(e^{\mu_R t}z)_t \ge e^{\mu_R t + pw}, \quad (e^{\mu_R t}w)_t \ge e^{\mu_R t + qz}.$$

Integrating these inequalities over $(0,t)$, we see that

$$e^{\mu_R t}z - \lambda\alpha \ge \int_0^t e^{\mu_R s + pw(s)}\mathrm{d}s, \quad e^{\mu_R t}w - \lambda\beta \ge \int_0^t e^{\mu_R s + qz(s)}\mathrm{d}s.$$

Substituting the second inequality into the first inequality, it follows that

$$e^{\mu_R t}z - \lambda\alpha \ge \int_0^t \exp\Big\{\mu_R s + \lambda p\beta e^{-\mu_R s} + pe^{-\mu_R s}\int_0^s e^{\mu_R y + qz(y)}\mathrm{d}y\Big\}\mathrm{d}s.$$

Moreover, we have

$$z(t) \geq \lambda\alpha e^{-\mu_R t} + e^{-\mu_R t}\int_0^t \exp\Big\{\mu_R s + \lambda p\beta e^{-\mu_R s} + pe^{-\mu_R s}\int_0^s e^{\mu_R y + qz(y)}\mathrm{d}y\Big\}\mathrm{d}s.$$

We fix $0 < \epsilon < 1$ and take $T_R > 0$ such that $e^{-\mu_R T_R} > 1 - \epsilon$. Then we have

$$z(t) \geq (1-\epsilon)\lambda\alpha + (1-\epsilon)\int_0^t \exp\Big\{(1-\epsilon)\lambda p\beta + p(1-\epsilon)\int_0^s e^{qz(y)}\mathrm{d}y\Big\}\mathrm{d}s.$$

We set

$$h(t) = (1-\epsilon)\lambda\alpha + (1-\epsilon)\int_0^t \exp\Big\{(1-\epsilon)\lambda p\beta + p(1-\epsilon)\int_0^s e^{qz(y)}\mathrm{d}y\Big\}\mathrm{d}s,$$

then we have

$$h'(t) = (1-\epsilon)\exp\Big\{(1-\epsilon)\lambda p\beta + p(1-\epsilon)\int_0^t e^{qz(s)}\mathrm{d}s\Big\},$$

$$h''(t) = (1-\epsilon)\exp\Big\{(1-\epsilon)\lambda p\beta + p(1-\epsilon)\int_0^t e^{qz(s)}\mathrm{d}s\Big\}\cdot p(1-\epsilon)e^{qz}.$$

After a careful computation, we see that

$$h''(t) \geq h'(t)p(1-\epsilon)e^{qh(t)}.$$

Integrating this inequality over $(0,t)$, it follows that

$$h'(t) \geq \frac{p}{q}(1-\epsilon)e^{qh(t)} + (1-\epsilon)e^{(1-\epsilon)\lambda p\beta} - \frac{p}{q}(1-\epsilon)e^{(1-\epsilon)\lambda q\alpha}.$$

Dividing the left-hand side by the right-hand side and integrating over $(0,t)$, we obtain

$$\int_{(1-\epsilon)\lambda\alpha}^{h(t)} \frac{\mathrm{d}s}{\frac{p}{q}(1-\epsilon)e^{qs} + (1-\epsilon)e^{(1-\epsilon)\lambda p\beta} - \frac{p}{q}(1-\epsilon)e^{(1-\epsilon)\lambda q\alpha}} \geq t,$$

We take λ large such that

$$T_{\epsilon,R} = \int_{(1-\epsilon)\lambda\alpha}^{\infty} \frac{\mathrm{d}s}{\frac{p}{q}(1-\epsilon)e^{qs} + (1-\epsilon)e^{(1-\epsilon)\lambda p\beta} - \frac{p}{q}(1-\epsilon)e^{(1-\epsilon)\lambda q\alpha}} \leq T_R.$$

Then $\bar{z}$ blows up at some $T \leq T_{\epsilon,R}$, and

$$T_{\epsilon,R} = \frac{\ln p - \ln q + \lambda(1-\epsilon)(q\alpha - p\beta)}{p(1-\epsilon)e^{(1-\epsilon)\lambda q\alpha} - q(1-\epsilon)e^{(1-\epsilon)\lambda p\beta}},$$

hence we get

$$T_\lambda^* \leq \frac{\ln p - \ln q + \lambda(1-\epsilon)(q\alpha - p\beta)}{p(1-\epsilon)e^{(1-\epsilon)\lambda q\alpha} - q(1-\epsilon)e^{(1-\epsilon)\lambda p\beta}}.$$

Therefore, taking $R \to 0$ and then $\epsilon \to 0$, paying attention to $qM_\varphi > pM_\psi$, it follows that

$$\limsup_{\lambda\to\infty} T_\lambda^* \frac{e^{qM_\varphi\lambda}}{\lambda} \leq \frac{qM_\varphi - pM_\psi}{p},$$

so we get the inequality (3.4). □

References

[1] Sining ZHENG, Lizhong ZHAO, Feng CHEN. *Blow-up rates in a parabolic system of ignition model.* Nonlinear Anal., 2002, **51**(4): 663–672.

[2] H. FUJITA. *On the blowing up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\alpha}$. J. Fac. Sci. Univ. Tokyo Sect. I, 1966, **13**: 109–124.
[3] A. FRIEDMAN, A. LACEY. *The blow-up time of solutions of nonlinear heat equation with small diffusion*. SIAM J. Math. Anal., 1987, **18**: 711–721.
[4] Changfeng GUI, Xuefeng WANG. *Life spans of solutions of the Cauchy problem for a semilinear heat equation*. J. Differential Equations, 1995, **115**(1): 166–172.
[5] T. Y. LEE, Weiming NI. *Global existence, large time behavior and life span of solutions of a semilinear parabolic Cauchy problem*. Trans. Amer. Math. Soc., 1992, **333**(1): 365–378.
[6] N. MIZOGUCHI, E. YANAGIDA. *Life span of solutions with large initial data in a semilinear parabolic equation*. Indiana Univ. Math. J., 2001, **50**(1): 591–610.
[7] N. MIZOGUCHI, E. YANAGIDA. *Life span of solutions for a semilinear parabolic problem with small diffusion*. J. Math. Anal. Appl., 2001, **261**(1): 350–368.
[8] S. SATO. *Life span of solutions with large initial data for a superlinear heat equation*. J. Math. Anal. Appl., 2008, **343**(2): 1061–1074.
[9] S. SATO. *Life span of solutions with large initial data for a semilinear parabolic system*. J. Math. Anal. Appl., 2011, **380**(2): 632–641.
[10] Keng DENG, H. A. LEVINE. *The role of critial exponents in blow-up theorems: The sequel*. J. Math. Anal. Appl., 2000, **243**: 85–126.
[11] P. QUITTNER, P. SOUPLET. *Superlinear Parabolic Problems. Blow-up, Global Existence and Steady States*. Birkhäuser Verlag, Basel, 2007.
[12] Bingchen LIU, Fengjie LI. *Blow-up properrties for heat equations coupled via different nonlinearities*. J. Math. Anal. Appl., 2008, **347**: 294–303.
[13] S. KAPLAN. *On the growth of solutions of quasi-linear parabolic equations*. Comm. Pure Appl. Math., 1963, **16**: 305–330.