The Commutants of the Toeplitz Operators with Radial Symbols on the Pluriharmonic Bergman Space

Jun YANG

Department of Mathematics, Shanghai Maritime University, Shanghai 201306, *P. R. China*

Abstract In this paper, we study the commutativity of Toeplitz operators with radial symbols on the pluriharmonic Bergman space. We obtain the necessary and sufficient conditions for the commutativity of bounded Toeplitz operator and Toeplitz operator with radial symbol on the pluriharmonic Bergman space.

Keywords pluriharmonic Bergman space; radial symbols; Toeplitz operators; commutants; unit ball

MR(2010) Subject Classification 47B35; 47B38; 46E15

1. Introduction

Throughout the paper, we fix a positive integer n and let

$$\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$$

denote the n-dimensional complex Euclidean space. For $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we write

$$\langle z, w \rangle = z_1 \overline{w}_1 + \cdots + z_n \overline{w}_n$$

and

$$|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2} = \sqrt{\langle z, z \rangle}.$$

The open unit ball in $\mathbb{C}^n$ is the set

$$\mathbb{B}_n = \{ z \in \mathbb{C}^n : |z| < 1 \}.$$

The boundary of $\mathbb{B}_n$ is the set

$$\mathbb{S}_n = \{ \zeta \in \mathbb{C}^n : |\zeta| = 1 \}.$$

We denote by V the measure on $\mathbb{B}_n$. The Bergman space $L_a^2(\mathbb{B}_n)$ is the space of analytic functions on $\mathbb{B}_n$ which are square-integrable with respect to measure V on $\mathbb{B}_n$.

The reproducing kernel on $L_a^2(\mathbb{B}_n)$ is given by

$$K_z(w) = \frac{1}{(1 - \langle w, z \rangle)^{n+1}},$$

for $z, w \in \mathbb{B}_n$, then for every $f \in L_a^2(\mathbb{B}_n)$,

$$f(z) = \langle f, K_z \rangle.$$

The Bergman space $L_a^2(\mathbb{B}_n)$ is the closed subspace of $L^2(\mathbb{B}_n, \mathrm{d}V)$. Then there exists the orthogonal projection from $L^2(\mathbb{B}_n, \mathrm{d}V)$ to $L_a^2(\mathbb{B}_n)$

$$Pf(z) = \langle f, K_z \rangle, \quad f \in L^2(\mathbb{B}_n, \mathrm{d}V).$$

From [1], for a function f, we call f a pluriharmonic function if

$$D_j \overline{D}_k f = 0, \quad j, k = 1, 2, \dots, n,$$

where $D_j = \frac{\partial}{\partial z_j}$, $\overline{D}_j = \frac{\partial}{\partial \overline{z}_j}$.

The pluriharmonic Bergman space $L_h^2(\mathbb{B}_n)$ consists of all pluriharmonic functions on $L^2(\mathbb{B}_n, \mathrm{d}V)$. Denote by R_z the reproducing kernel on $L_h^2(\mathbb{B}_n)$, then for every $f \in L_h^2(\mathbb{B}_n)$,

$$f(z) = \langle f, R_z \rangle.$$

Then we have

$$R_z(w) = K_z(w) + \overline{K_z(w)} - 1, \quad z, w \in \mathbb{B}_n.$$

Obviously, $L_h^2(\mathbb{B}_n)$ is the closed subspace of $L^2(\mathbb{B}_n, \mathrm{d}V)$. The orthogonal projection from $L^2(\mathbb{B}_n, \mathrm{d}V)$ to $L_h^2(\mathbb{B}_n)$ is as follows,

$$Qf(z) = \langle f, R_z \rangle, \quad f \in L^2(\mathbb{B}_n, \mathrm{d}V).$$

The relation of P to Q is

$$Qf(z) = Pf(z) + \overline{P\overline{f}(z)} - Pf(0).$$

2. Preliminaries

Let Q be the orthogonal projection from $L^2(\mathbb{B}_n, \mathrm{d}V)$ to $L_h^2(\mathbb{B}_n)$. We give the definition of Toeplitz operator through the orthogonal projection Q.

Definition 2.1 Let $\varphi \in L^\infty(\mathbb{B}_n)$. Define operator $T_\varphi : L_h^2(\mathbb{B}_n) \to L_h^2(\mathbb{B}_n)$ by,

$$T_\varphi f = Q(\varphi f), \quad f \in L_h^2(\mathbb{B}_n).$$

T_φ is called a Toeplitz operator with the symbol φ on $L_h^2(\mathbb{B}_n)$.

Using the reproducing kernel $R_z(w)$, we have

$$T_\varphi f(z) = \int_{\mathbb{B}_n} f(w)\varphi(w)\overline{R_z(w)}\mathrm{d}V(w).$$

The boundedness, compactness, concerning Toeplitz operators with radial symbol on the holomorphic spaces, and some algebraic properties of the Toeplitz operators, such as the (zero) product of two Toeplitz operators with radial symbol and the corresponding commuting problem of Toeplitz operators have been of interest to mathematicians working in operator theory and the theory of holomorphic spaces for many years, we recommend the interested readers to refer

to the papers [2–15]. In this paper, we will consider the Toeplitz operators with radial symbols on pluriharmonic Bergman space.

3. Radial function and quasihomogeneous function on Bergman space

Let $\varphi \in L^1(\mathbb{D}, \mathrm{d}A)$ be a radial function, i.e., suppose that:

$$\varphi(z) = \varphi(|z|), \quad z \in \mathbb{D}.$$

If φ satisfies

$$f(re^{i\theta}) = e^{ik\theta}\psi(r), \quad k \in \mathbb{Z},$$

then, f is a quasihomogeneous function of quasihomogeneous degree k.

For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, where $\mathbb{N}$ denotes the set of all nonnegative integers, we write

$$|\alpha| = \alpha_1 + \cdots + \alpha_n$$

and

$$\alpha! = \alpha_1! \cdots \alpha_n!.$$

We will also write

$$z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$$

for $z = (z_1, \ldots, z_n) \in \mathbb{B}_n$.

For any two muliti-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{N}^n$, we say $\alpha \succeq \beta$, if for any $1 \le i \le n$, $\alpha_i \ge \beta_i$, and $\alpha \perp \beta$ means $\alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_n\beta_n = 0$. Denote $\alpha - \beta = (\alpha_1 - \beta_1, \alpha_2 - \beta_2, \ldots, \alpha_n - \beta_n)$. If $\alpha \succeq \beta$, then $|\alpha - \beta| = |\alpha| - |\beta|$.

Definition 3.1 *Let $\varphi \in L^1(\mathbb{B}_n, \mathrm{d}V)$. If for any unitary transformation U, $\varphi(Uz) = \varphi(z)$, then we call φ is a radial function. Let $p, s \in \mathbb{N}^n$ and $p \perp s$, $f \in L^1(\mathbb{B}_n, \mathrm{d}V)$. Then we call f is a quasihomogeneous function of quasihomogeneous degree (p, s), if for any $\xi \in \mathbb{S}$, $r = (r_1, r_2, \ldots, r_n)$ and $|r| = \sqrt{|r_1|^2 + |r_2|^2 + \cdots + |r_n|^2}$, f can be decomposed into*

$$f(|r|\xi) = \xi^p \overline{\xi}^s \varphi(|r|),$$

where $r_i = |z_i|$, $1 \le i \le n$, φ is a radial function.

Definition 3.2 *Let $\varphi \in L^1(\mathbb{B}_n, \mathrm{d}V)$. If for any diagonal unitary transformation U, $\varphi(Uz) = \varphi(z)$, then we call φ a separately radial function. Let $p, s \in \mathbb{N}^n$ and $p \perp s$, $f \in L^1(\mathbb{B}_n, \mathrm{d}V)$. Then we call f a separately quasihomogeneous function, if for any $\xi \in \mathbb{S}$, $r = (r_1, r_2, \ldots, r_n)$ and $|r| = \sqrt{|r_1|^2 + |r_2|^2 + \cdots + |r_n|^2}$, f can be decomposed into*

$$f(|r|\xi) = \xi^p \overline{\xi}^s \varphi(|r|),$$

where $r_i = |z_i|$, $1 \le i \le n$, φ is a separately radial function.

From the definition, the separately radial function satisfies

$$\varphi(z) = \varphi(|z_1|, |z_2|, \ldots, |z_n|).$$

Let
$$\tau(\mathbb{B}_n) = \{r = (r_1, r_2, \ldots, r_n) = (|z_1|, |z_2|, \ldots, |z_n|) : z = (z_1, z_2, \ldots, z_n) \in \mathbb{B}_n\}.$$

If φ is a bounded separately radial function, then
$$\int_{\mathbb{B}_n} \varphi(z) \mathrm{d}V(z) = 2^n n! \int_{\tau(\mathbb{B}_n)} \varphi(r) r \mathrm{d}r,$$

where $r\mathrm{d}r = \prod_{i=1}^n r_i \mathrm{d}r_i$.

Let $\Re = \{\varphi : \mathbb{B}_n \to \mathbb{C}$ be separately radial functions and $\int_{\tau(\mathbb{B}_n)} |\varphi(r)|^2 r \mathrm{d}r < \infty\}$. From [7], for any $f \in L^2(\mathbb{B}_n, \mathrm{d}V)$, we have
$$f(|r|\xi) = \sum_{p \perp s, p, s \in \mathbb{N}^n} \xi^p \overline{\xi}^s f_{p,s}(r), \quad f_{p,s} \in \Re.$$

Lemma 3.3 ([7]) *Let*
$$f(z) = \sum_{p \perp s, p, s \in \mathbb{N}^n} \xi^p \overline{\xi}^s f_{p,s}(r) \in L^\infty(\mathbb{B}_n, \mathrm{d}V).$$

Then for $p, s \in \mathbb{N}^n$, $p \perp s$, $\xi^p \overline{\xi}^s f_{p,s}(r)$ is bounded on $\mathbb{B}_n$.

Definition 3.4 For any $\varphi \in L^1([0,1], r\mathrm{d}r)$, the Mellin transform is:
$$\widehat{\varphi}(z) = \int_0^1 \varphi(r) r^{z-1} \mathrm{d}r.$$

It is clear that, for these functions, the Mellin transform is defined on $\{z : \operatorname{Re} z \geq 2\}$ and analytic on $\{z : \operatorname{Re} z > 2\}$.

Lemma 3.5 ([16]) *Let φ be bounded analytic function on $\{z : \operatorname{Re} z > 0\}$ and $z_1, z_2, \ldots, z_n$ be the zero points of φ. If*
  (a) $\inf\{|z_n|\} > 0$ and
  (b) $\sum_{n \geq 1} \operatorname{Re}(\frac{1}{z_n}) = \infty$. Then φ will be 0 on $\{z : \operatorname{Re} z > 0\}$.

Definition 3.6 Let f, g be defined on $[0,1)$. Then the Mellin convolution of two functions f and g, denoted by $f_{*M}g$, is defined as
$$(f_{*M}g)(r) = \int_r^1 f(\frac{r}{t}) g(t) \frac{\mathrm{d}t}{t}, \quad 0 \leq r < 1.$$

It is easy to see that the Mellin transform converts the convolution product into a pointwise product, i.e.,
$$(\widehat{f_{*M}g})(s) = \widehat{f}(s)\widehat{g}(s),$$

and that, if f and g are in $L^1([0,1], r\mathrm{d}r)$, then so is $f_{*M}g$.

4. Main results

Lemma 4.1 ([17]) *Let φ be the integrable radial function on $\mathbb{B}_n$ such that T_φ is a bounded*

Toeplitz operator. Then for any muliti-index $\alpha \in \mathbb{N}^n$,

$$T_\varphi(z^\alpha) = 2(n + |\alpha|)\widehat{\varphi}(2n + 2|\alpha|)z^\alpha,$$

$$T_\varphi(\overline{z}^\alpha) = 2(n + |\alpha|)\widehat{\varphi}(2n + 2|\alpha|)\overline{z}^\alpha.$$

Proof For any muliti-index β,

$$\langle T_\varphi z^\alpha, z^\beta \rangle = \langle \varphi z^\alpha, z^\beta \rangle$$

$$= \begin{cases} 0, & \alpha \neq \beta; \\ \dfrac{2n!\alpha!}{(n - 1 + |\alpha|)!}\widehat{\varphi}(2n + 2|\alpha|), & \alpha = \beta, \end{cases}$$

$$\langle z^\alpha, z^\beta \rangle = \begin{cases} 0, & \alpha \neq \beta; \\ \dfrac{n!\alpha!}{(n + |\alpha|)!}, & \alpha = \beta. \end{cases}$$

Then

$$\langle T_\varphi z^\alpha, z^\beta \rangle = 2(n + |\alpha|)\widehat{\varphi}(2n + 2|\alpha|)\langle z^\alpha, z^\beta \rangle.$$

If $\beta \succ 0$, obviously,

$$\langle T_\varphi z^\alpha, \overline{z}^\beta \rangle = \langle z^\alpha, \overline{z}^\beta \rangle = 0.$$

Since $\{z^\alpha\}_{\alpha \succeq 0} \cup \{\overline{z}^\alpha\}_{\alpha \succ 0}$ is the orthonormal basis on the pluriharmonic Bergman space $L_h^2(\mathbb{B}_n, \mathrm{d}V)$, we have

$$T_\varphi(z^\alpha) = 2(n + |\alpha|)\widehat{\varphi}(2n + 2|\alpha|)z^\alpha.$$

Analogously, we can prove another part of this lemma. $\square$

Lemma 4.2 ([18]) *Let p, s be two muliti-indices, and $\varphi \in \Re$, $\xi^p\overline{\xi}^s f_{p,s}(r) \in L^\infty(\mathbb{B}_n, \mathrm{d}V)$. Then for any muliti-index $\alpha \in \mathbb{N}^n$,*

$$T_{\xi^p\overline{\xi}^s\varphi}(z^\alpha) = \begin{cases} ((2^n(n + |\alpha| + |p| - |s|)! \displaystyle\int_{\tau(\mathbb{B}_n)} \varphi(r)r^{2\alpha + 2p} \times \\ \quad |r|^{-(|p| + |s|)}r\mathrm{d}r)((p + \alpha - s)!)^{-1})z^{p+\alpha-s}, & p + \alpha \succeq s, \\ ((2^n(n + |s| - |\alpha| - |p|)! \displaystyle\int_{\tau(\mathbb{B}_n)} \varphi(r)r^{2s} \times \\ \quad |r|^{-(|p| + |s|)}r\mathrm{d}r)((s - \alpha - p)!)^{-1})\overline{z}^{s-\alpha-p}, & s \succeq p + \alpha, \\ 0, & s \neq p + \alpha, s \nsucceq p + \alpha, p + \alpha \nsucceq s; \end{cases}$$

$$T_{\xi^p\overline{\xi}^s\varphi}(\overline{z}^\alpha) = \begin{cases} ((2^n(n + |\alpha| + |s| - |p|)! \displaystyle\int_{\tau(\mathbb{B}_n)} \varphi(r)r^{2\alpha + 2s} \times \\ \quad |r|^{-(|p| + |s|)}r\mathrm{d}r)((s + \alpha - p)!)^{-1})\overline{z}^{s+\alpha-p}, & s + \alpha \succeq p, \\ ((2^n(n - |\alpha| + |p| - |s|)! \displaystyle\int_{\tau(\mathbb{B}_n)} \varphi(r)r^{2p} \times \\ \quad |r|^{-(|p| + |s|)}r\mathrm{d}r)((p - \alpha - s)!)^{-1})z^{s-\alpha-p}, & p \succeq s + \alpha, \\ 0, & p \neq s + \alpha, p \nsucceq s + \alpha, s + \alpha \nsucceq p. \end{cases}$$

Especially, if $p \perp s$, then

$$T_{\xi^p \overline{\xi}^s \varphi}(z^\alpha) = \begin{cases} ((2^n(n + |\alpha| + |p| - |s|)! \int_{\tau(\mathbb{B}_n)} \varphi(r) r^{2\alpha + 2p} \times \\ \quad |r|^{-(|p|+|s|)} r \mathrm{d}r)((p + \alpha - s)!)^{-1}) z^{p+\alpha-s}, & p + \alpha \succeq s, \\ 0, & p + \alpha \nsucceq s; \end{cases}$$

$$T_{\xi^p \overline{\xi}^s \varphi}(\overline{z}^\alpha) = \begin{cases} ((2^n(n + |\alpha| + |s| - |p|)! \int_{\tau(\mathbb{B}_n)} \varphi(r) r^{2\alpha + 2s} \times \\ \quad |r|^{-(|p|+|s|)} r \mathrm{d}r)((s + \alpha - p)!)^{-1}) \overline{z}^{s+\alpha-p}, & s + \alpha \succeq p, \\ 0, & s + \alpha \nsucceq p. \end{cases}$$

Proof We only need to prove the first equality of this lemma, since the proof of the second one is similar to the first one. For any muliti-index $\beta \in \mathbb{N}^n$, if $p + \alpha \nsucceq s$, then there exists i such that $\alpha_i + p_i < s_i$. Then $p + \alpha \neq \beta + s$. By calculating, we have

$$\langle P[\xi^p \overline{\xi}^s \varphi z^\alpha], z^\beta \rangle = \int_{\mathbb{B}_n} \xi^p \overline{\xi}^s \varphi(z) z^\alpha \overline{z}^\beta \mathrm{d}V(z) = 0.$$

If $p + \alpha \succeq s$, we also have

$$\langle P[\xi^p \overline{\xi}^s \varphi z^\alpha], z^\beta \rangle = \int_{\mathbb{B}_n} \xi^p \overline{\xi}^s \varphi(z) z^\alpha \overline{z}^\beta \mathrm{d}V(z)$$
$$= \begin{cases} 0, & \beta \neq p + \alpha - s, \\ 2^n n! \int \tau(\mathbb{B}_n) \varphi(r) r^{2\alpha + 2p} |r|^{-(|p|+|s|)} r \mathrm{d}r, & \beta = p + \alpha - s; \end{cases}$$

$$\langle z^{p+\alpha-s}, z^\beta \rangle = \begin{cases} 0, & \beta \neq p + \alpha - s, \\ \dfrac{n!(p + \alpha - s)}{(n + |\alpha| + |p| - |s|)!}), & \beta = p + \alpha - s. \end{cases}$$

For any $\beta \succeq 0$, $\beta \neq 0$, then

$$\langle P[\xi^p \overline{\xi}^s \varphi z^\alpha], \overline{z}^\beta \rangle = \langle z^{p+\alpha-s}, z^\beta \rangle = 0.$$

Thus,

$$P[\xi^p \overline{\xi}^s \varphi z^\alpha] = \begin{cases} ((2^n(n + |\alpha| + |p| - |s|)! \int_{\tau(\mathbb{B}_n)} \varphi(r) r^{2\alpha + 2p} \times \\ \quad |r|^{-(|p|+|s|)} r \mathrm{d}r)((p + \alpha - s)!)^{-1}) z^{p+\alpha-s}, & p + \alpha \succeq s, \\ 0, & p + \alpha \nsucceq s. \end{cases}$$

Similarly, we have

$$P[\xi^p \overline{\xi}^s \varphi z^\alpha] = \begin{cases} ((2^n(n + |s| - |\alpha| - |p|)! \int_{\tau(\mathbb{B}_n)} \varphi(r) r^{2s} \times \\ \quad |r|^{-(|p|+|s|)} r \mathrm{d}r)((s - p - \alpha)!)^{-1}) \overline{z}^{s-p-\alpha}, & s \succeq p + \alpha, \\ 0, & s \nsucceq p + \alpha. \end{cases}$$

From two equalities above, we have

$$
T_{\xi^p\overline{\xi}^s\varphi}(z^\alpha) = \begin{cases} ((2^n(n+|\alpha|+|p|-|s|)! \int_{\tau(\mathbb{B}_n)} \varphi(r)r^{2\alpha+2p} \times \\ \quad |r|^{-(|p|+|s|)}r\mathrm{d}r)((p+\alpha-s)!)^{-1})z^{p+\alpha-s}, \qquad p+\alpha \succeq s, \\ ((2^n(n+|s|-|\alpha|-|p|)! \int_{\tau(\mathbb{B}_n)} \varphi(r)r^{2s} \times \\ \quad |r|^{-(|p|+|s|)}r\mathrm{d}r)((s-\alpha-p)!)^{-1})\overline{z}^{s-\alpha-p}, \qquad s \succeq p+\alpha, \\ 0, \qquad\qquad\qquad\qquad s \neq p+\alpha, s \not\succeq p+\alpha, p+\alpha \not\succeq s. \end{cases}
$$

If $p\perp s$, then there exists $i \in \{1, 2, \ldots, n\}$ such that $p_i > 0$ and $s_i = 0$. Thus, there does not exist $\alpha \in \mathbb{N}^n$ such that $s \succeq p + \alpha$. So we can prove the second part of this lemma.

Theorem 4.3 *Let φ be the bounded radial function on $\mathbb{B}_n$, and function f satisfy*

$$
f(z) = f(|r|\xi) = \sum_{p\perp s, p, s\in\mathbb{N}^n} \xi^p\overline{\xi}^s f_{p,s}(r) \in L^\infty(\mathbb{B}_n, \mathrm{d}V).
$$

Then for all $p, s \in \mathbb{N}^n$ and $p\perp s$,

$$
T_\varphi T_f = T_f T_\varphi \Leftrightarrow T_\varphi T_{\xi^p\overline{\xi}^s f_{p,s}(r)} = T_{\xi^p\overline{\xi}^s f_{p,s}(r)} T_\varphi.
$$

Proof Sufficiency is obvious. We only need to prove the necessary condition. From Lemmas 4.1 and 4.2, we have

$$
T_\varphi T_f(z^\alpha) = \sum_{p+\alpha\succeq s} \lambda_1 z^{p+\alpha-s}, \quad T_f T_\varphi(z^\alpha) = \sum_{p+\alpha\succeq s} \lambda_2 z^{p+\alpha-s},
$$

here

$$
\lambda_1 = (2^n(n+|\alpha|+|p|-|s|)! \int_{\tau(\mathbb{B}_n)} \varphi(r)r^{2\alpha+2p}|r|^{-(|p|+|s|)}r\mathrm{d}r)((p+\alpha-s)!)^{-1}) \times
$$
$$
2(n+|p+\alpha-s|)\widehat{\varphi}(2n+2|p+\alpha-s|)
$$

and

$$
\lambda_2 = (2^n(n+|\alpha|+|p|-|s|)! \int_{\tau(\mathbb{B}_n)} \varphi(r)r^{2\alpha+2p}|r|^{-(|p|+|s|)}r\mathrm{d}r)((p+\alpha-s)!)^{-1}) \times
$$
$$
2(n+|\alpha|)\widehat{\varphi}(2n+2|\alpha|)
$$

are all constants only dependent on p, s, α. Thus, for any $\beta \succeq 0$, we get

$$
\langle T_f T_\varphi z^\alpha, z^\beta \rangle = \begin{cases} \langle T_{\xi^p\overline{\xi}^s f_{p,s}(r)} T_\varphi z^\alpha, z^\beta \rangle, & p+\alpha = s+\beta, \\ 0, & p+\alpha \neq s+\beta, \end{cases}
$$

$$
\langle T_\varphi T_f z^\alpha, z^\beta \rangle = \begin{cases} \langle T_\varphi T_{\xi^p\overline{\xi}^s f_{p,s}(r)} z^\alpha, z^\beta \rangle, & p+\alpha = s+\beta, \\ 0, & p+\alpha \neq s+\beta. \end{cases}
$$

For any $\beta \succeq 0$, we get

$$
\langle T_{\xi^p\overline{\xi}^s f_{p,s}(r)} T_\varphi z^\alpha, \overline{z}^\beta \rangle = \langle T_\varphi T_{\xi^p\overline{\xi}^s f_{p,s}(r)} z^\alpha, \overline{z}^\beta \rangle = 0.
$$

So, for any $\alpha \succeq 0$, we get that $T_{\xi^p\overline{\xi}^s f_{p,s}(r)} T_\varphi(z^\alpha) = T_\varphi T_{\xi^p\overline{\xi}^s f_{p,s}(r)}(z^\alpha)$. Similarly, for any $\alpha \succeq 0$, we have $T_{\xi^p\overline{\xi}^s f_{p,s}(r)} T_\varphi(\overline{z}^\alpha) = T_\varphi T_{\xi^p\overline{\xi}^s f_{p,s}(r)}(\overline{z}^\alpha)$. Since $\{z^\alpha\}_{\alpha\succeq 0} \bigcup \{\overline{z}^\alpha\}_{\alpha\succ 0}$ are basis on $L_h^2(\mathbb{B}_n)$,

we have

$$T_{\xi^p \overline{\xi}^s f_{p,s}(r)} T_\varphi = T_\varphi T_{\xi^p \overline{\xi}^s f_{p,s}(r)}. \quad \square$$

Lemma 4.4 ([18]) *Let $p, s \in \mathbb{N}^n$, $p \perp s$ be two muliti-indices, $p \neq 0$, $s \neq 0$. If φ is a bounded radial function which is not equal to zero, $\psi \in \Re$ such that $\xi^p \overline{\xi}^s \psi \in L^\infty(\mathbb{B}_n, \mathrm{d}V)$, then $T_\varphi T_{\xi^p \overline{\xi}^s \psi} = T_{\xi^p \overline{\xi}^s \psi} T_\varphi$ if and only if $|p| = |s|$ or $\psi = 0$.*

The theorem below is the main theorem. This theorem gives a complete description of a Toeplitz operator which commutes with a Toeplitz operator with a radial symbol.

Theorem 4.5 *Let φ be a bounded radial function which is not a constant on $\mathbb{B}_n$. If $f \in L^\infty(\mathbb{B}_n, \mathrm{d}V)$, then $T_\varphi T_f = T_f T_\varphi$ if and only if f satisfies:*

$$f(e^{i\theta} z) = f(z),$$

for any $\theta \in \mathbb{R}$ and almost all $z \in \mathbb{B}_n$.

Proof From Theorem 4.3, T_φ commutes with T_f if and only if

$$T_\varphi T_{\xi^p \overline{\xi}^s f_{p,s}(r)} = T_{\xi^p \overline{\xi}^s f_{p,s}(r)} T_\varphi, \quad p \perp s, p, s \in \mathbb{N}^n.$$

From Lemma 4.4, the condition above is equivalent to

$$|p| = |s| \quad \text{or} \quad f_{p,s} = 0,$$

that is

$$e^{i\theta}(|p| - |s|) f_{p,s}(r) = f_{p,s}(r),$$

which is equivalent to

$$(\xi e^{i\theta})^p (\overline{\xi e^{i\theta}})^s f_{p,s}(r) = \xi^p \overline{\xi}^s f_{p,s}(r),$$

that means

$$(\xi^p \overline{\xi}^s f_{p,s})(e^{i\theta} z) = (\xi^p \overline{\xi}^s f_{p,s})(z).$$

The equivalent conditions above hold for all $\theta \in \mathbb{R}$ and almost all $z \in \mathbb{B}_n$. Thus, for all $\theta \in \mathbb{R}$ and almost all $z \in \mathbb{B}_n$, $f(e^{i\theta} z) = f(z)$. $\square$

Remark 4.6 If $n = 1$, from Theorem 4.5, we can see $f(z)$ is a radial function (see [19,20]). If $n > 1$, then $f(z)$ is not necessarily a radial function. For example, when $|p| = |s|$ and φ is a radial function, $f(z) = \varphi(r) \xi^p \overline{\xi}^s$ satisfies $f(e^{i\theta} z) = f(z)$, but $f(z)$ is not a radial function.

References

[1] W. RUDIN. *Function Theory in the Unit Ball of $\mathbb{C}^n$*. Springer-Verlag, Berlin, 2008.

[2] P. AHERN. *On the range of the Berezin transform*. J. Funct. Anal., 2004, **215**(1): 206–216.

[3] P. AHERN, Ž. ČUČKOVIĆ. *A theorem of Brown-Halmos type for Bergman space Toeplitz operators*. J. Funct. Anal., 2001, **187**(1): 200–210.

[4] S. AXLER, Dechao ZHENG. *Compact operators via the Berezin transform.* Indiana Univ. Math. J., 1998, **47**(2): 387–400.

[5] S. AXLER, Dechao ZHENG. *The Berezin transform on the Toeplitz algebra.* Stud. Math., 1998, **127**(2): 113–130.

[6] Ž. ČUČKOVIĆ, N. V. RAO. *Mellin transform, monomial symbols and commuting Toeplitz operators.* J. Funct. Anal., 1998, **154**(1): 195–214.

[7] Xingtang DONG, Zehua ZHOU. *Algebraic properties of Toeplitz operators with separately quasihomogeneous symbols on the Bergman space of the unit ball.* J. Operator Theory, 2011, **66**(1): 193–207.

[8] Xingtang DONG, Zehua ZHOU. *Products of Toeplitz operators on the harmonic Bergman space.* Proc. Amer. Math. Soc., 2010, **138**(5): 1765–1773.

[9] M. ENGLIS. *Function invariant under the Berezin transform.* J. Funct. Anal., 1994, **121**(1): 223–254.

[10] I. LOUHICHI, E. STROUSE, L. ZAKARIASY. *Products of Toeplitz operators on the Bergman space.* Integral Equations Operator Theory, 2006, **54**(4): 525–539.

[11] I. LOUHICHI, L. ZAKARIASY. *On Toeplitz operators with quasihomogeneous symbols.* Arch. Math. (Basel), 2005, **85**(3): 248–257.

[12] N. VASILEVSKI. *Toeplitz operator on the Bergman spaces: inside-the-dommain effects.* Contemp. Math., 2001, **289**(1): 79–146.

[13] N. VASILEVSKI. *Bergman space structure, commutative algebras of Toeplitz operators and hyperbolic geometry.* Integral Equations Operator Theory, 2003, **46**(2): 235–251.

[14] Zehua ZHOU, Xingtang DONG. *Algebraic properties of Toeplitz operators with radial symbols on the Bergman space of the unit ball.* Integral Equations Operator Theory, 2009, **64**(1): 137–154.

[15] Kehe ZHU. *Operator Theory in Function Space.* Marcel Dekker, Inc., New York, 1990.

[16] R. REMMERT. *Classical Topics in Complex Function Theory.* Springer-Verlag, New York, 1998.

[17] Jingyu YANG, Liu LIU, Yufeng LU. *Algebraic properties of Toeplitz operators on the Pluriharmonic Bergman space.* J. Funct. Spaces Appl., 2013, Art. ID 578436, 12 pp.

[18] Hongyan GUAN, Liu LIU, Yufeng LU. *Algebraic properties of quasihomogeneous and separately quasihomogeneous Toeplitz operators on the Pluriharmonic Bergman space.* Abstr. Appl. Anal. 2013, Art. ID 252037, 12 pp.

[19] Dechao ZHENG. *Commuting Toeplitz operators with pluriharmonic symbols.* Trans. Amer. Math. Soc., 1998, **350**: 1595–1618.

[20] I. LOUHICHI, L. ZAKARIASY. *Quasihomogeneous Toeplitz operators on the harmonic Bergman space.* Arch. der Math., 2012, **98**(1): 49-60.