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# Ordering Graphs by the Augmented Zagreb Indices

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**Abstract** Recently, Furtula et al. proposed a valuable predictive index in the study of the heat of formation in octanes and heptanes, the augmented Zagreb index (AZI index) of a graph $G$, which is defined as

$$\mathrm{AZI}(G)=\sum_{uv\in E(G)}\Big(\frac{d_ud_v}{d_u+d_v-2}\Big)^3,$$

where $E(G)$ is the edge set of $G$, $d_u$ and $d_v$ are the degrees of the terminal vertices $u$ and $v$ of edge $uv$, respectively. In this paper, we obtain the first five largest (resp., the first two smallest) AZI indices of connected graphs with $n$ vertices. Moreover, we determine the trees of order $n$ with the first three smallest AZI indices, the unicyclic graphs of order $n$ with the minimum, the second minimum AZI indices, and the bicyclic graphs of order $n$ with the minimum AZI index, respectively.

**Keywords** augmented Zagreb index; connected graphs; trees; unicyclic graphs; bicyclic graphs

**MR(2010) Subject Classification** 05C35; 05C50

## 1. Introduction

Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. Let $n=|V(G)|$ and $m=|E(G)|$. Let $N(u)$ be the set of all neighbors of $u\in V(G)$ in $G$, and let $d_u=|N(u)|$ be the degree of vertex $u$. A vertex $u$ is called a pendent vertex if $d_u=1$. A connected graph $G$ is called a tree (resp., unicyclic graph and bicyclic graph) if $m=n-1$ (resp., $m=n$ and $m=n+1$).

Molecular descriptors have found a wide application in QSPR/QSAR studies [1]. Among them, topological indices have a prominent place. Inspired by recent work on the atom-bond connectivity index [2,3], Furtula et al. [4] proposed a valuable predictive index whose prediction power is better than atom-bond connectivity index in the study of the heat of formation in octanes and heptanes, the augmented Zagreb index (AZI index for short) of a graph $G$, which is defined as

$$\mathrm{AZI}(G)=\sum_{uv\in E(G)}\Big(\frac{d_ud_v}{d_u+d_v-2}\Big)^3.$$

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Basic properties of AZI index have been studied in [5]. Besides, by using different graph parameters, some attained upper and lower bounds and the corresponding extremal graphs on the AZI indices for various classes of connected graphs have been given in [4,5].

In this paper, we obtain the first five largest (resp., the first two smallest) AZI indices of connected graphs with $n$ vertices. Moreover, we determine the trees of order $n$ with the first three smallest AZI indices, the unicyclic graphs of order $n$ with the minimum, the second minimum AZI indices, and the bicyclic graphs of order $n$ with the minimum AZI index, respectively.

## 2. The first five largest AZI indices of connected graphs

Denote by $P_n$, $C_n$, $K_n$ and $S_n$ the path, cycle, complete graph and star of order $n$, respectively. Let $G_1 \vee G_2$ denote the graph obtained from two graphs $G_1$ and $G_2$ by connecting the vertices of $G_1$ with the vertices of $G_2$. Let $\overline{G}$ be the complement of a graph $G$. Let $G+e$ denote the graph obtained from a graph $G$ by inserting an edge $e \notin E(G)$. Let $G-e$ denote the graph obtained from a graph $G$ by deleting the edge $e \in E(G)$. Let $S_n^+ = S_n + e$.

Let $\mathbb{G}_n$ be the set of connected graphs of order $n$, and let $\mathbb{G}_{n,m}$ be the set of connected graphs with $n$ vertices and $m$ edges, where $n-1 \le m \le \binom{n}{2}$. Obviously, $\mathbb{G}_1 = \{K_1\}$, $\mathbb{G}_2 = \{K_2\}$ and $\mathbb{G}_n = \cup_{n-1 \le m \le \binom{n}{2}} \mathbb{G}_{n,m}$. Now we shall investigate the AZI index of $G \in \mathbb{G}_n$ for $n \ge 3$. To begin with, a key lemma to obtain our main results is given as follows.

**Lemma 2.1** ([5]) *Let $G \in \mathbb{G}_n$ and $G \ncong K_n$, where $n \ge 3$. Then for $e \notin E(G)$,* $\mathrm{AZI}(G) < \mathrm{AZI}(G+e)$.

It follows from Lemma 2.1 that

**Corollary 2.2** *Let $n, m_1, m_2$ be integers with $n \ge 3$ and $n-1 \le m_1 < m_2 \le \binom{n}{2}$.*

*(1) Let $G_1 \in \mathbb{G}_{n,m_1}$. Then there exists a graph $G_2 \in \mathbb{G}_{n,m_2}$ such that* $\mathrm{AZI}(G_2) > \mathrm{AZI}(G_1)$.

*(2) Let $G_2 \in \mathbb{G}_{n,m_2}$. Then there exists a graph $G_1 \in \mathbb{G}_{n,m_1}$ such that* $\mathrm{AZI}(G_1) < \mathrm{AZI}(G_2)$.

Observe that $\mathbb{G}_3 = \{K_3, P_3\}$ and $\mathbb{G}_4 = \{K_4, K_4 - e, C_4, S_4^+, P_4, S_4\}$. By Corollary 2.2 and simply calculating, we immediately get $\mathrm{AZI}(K_3) > \mathrm{AZI}(P_3)$ and

$$\mathrm{AZI}(K_4) > \mathrm{AZI}(K_4 - e) > \mathrm{AZI}(C_4) > \mathrm{AZI}(S_4^+) > \mathrm{AZI}(P_4) > \mathrm{AZI}(S_4).$$

For $n \ge 5$, observe that $\mathbb{G}_{n,\binom{n}{2}} = \{K_n\}$, $\mathbb{G}_{n,\binom{n}{2}-1} = \{K_n - e\}$, $\mathbb{G}_{n,\binom{n}{2}-2} = \{\overline{S_3} \vee K_{n-3}, C_4 \vee K_{n-4}\}$ and $\mathbb{G}_{n,\binom{n}{2}-3} = \{\overline{S_4} \vee K_{n-4}, \overline{K_3} \vee K_{n-3}, P_4 \vee K_{n-4}, \overline{S_3} \vee (K_{n-3} - e), C_4 \vee (K_{n-4} - e)(n \ge 6)\}$.

**Lemma 2.3** *Let $G \in \mathbb{G}_{n,\binom{n}{2}-3}$ and $G \ncong \overline{S_4} \vee K_{n-4}$. Then for $n \ge 5$,*

$$\mathrm{AZI}(\overline{S_3} \vee K_{n-3}) > \mathrm{AZI}(C_4 \vee K_{n-4}) > \mathrm{AZI}(\overline{S_4} \vee K_{n-4}) > \mathrm{AZI}(G).$$

**Proof** By direct computation, for $n \ge 5$, we have

$$\begin{aligned}\mathrm{AZI}(\overline{S_3} \vee K_{n-3}) = &\frac{(n-3)(n-4)(n-1)^6}{2(2n-4)^3} + \frac{2(n-3)(n-1)^3(n-2)^3}{(2n-5)^3} + \\ &\frac{(n-2)^6 + (n-3)^4(n-1)^3}{(2n-6)^3},\end{aligned}$$

$$\begin{aligned}
\mathrm{AZI}(C_4 \vee K_{n-4}) =& \frac{(n-4)(n-5)(n-1)^6}{2(2n-4)^3} + \frac{4(n-4)(n-1)^3(n-2)^3}{(2n-5)^3} + \frac{4(n-2)^6}{(2n-6)^3},\\
\mathrm{AZI}(\overline{S_4} \vee K_{n-4}) =& \frac{(n-4)(n-5)(n-1)^6}{2(2n-4)^3} + \frac{3(n-4)(n-1)^3(n-2)^3}{(2n-5)^3} + \\
& \frac{3(n-2)^6}{(2n-6)^3} + \frac{(n-1)^3(n-4)^4}{(2n-7)^3},\\
\mathrm{AZI}(\overline{K_3} \vee K_{n-3}) =& \frac{(n-3)(n-4)(n-1)^6}{2(2n-4)^3} + \frac{3(n-1)^3(n-3)^4}{(2n-6)^3},\\
\mathrm{AZI}(P_4 \vee K_{n-4}) =& \frac{(n-4)(n-5)(n-1)^6}{2(2n-4)^3} + \frac{2(n-4)(n-1)^3(n-2)^3}{(2n-5)^3} + \\
& \frac{2(n-4)(n-1)^3(n-3)^3 + (n-2)^6}{(2n-6)^3} + \frac{2(n-2)^3(n-3)^3}{(2n-7)^3},
\end{aligned}$$

$$\begin{aligned}
\mathrm{AZI}(\overline{S_3} \vee (K_{n-3}-e)) =& \frac{(n-5)(n-6)(n-1)^6}{2(2n-4)^3} + \frac{4(n-5)(n-1)^3(n-2)^3}{(2n-5)^3} + \\
& \frac{(n-5)(n-1)^3(n-3)^3 + 5(n-2)^6}{(2n-6)^3} + \frac{2(n-2)^3(n-3)^3}{(2n-7)^3},
\end{aligned}$$

$$\begin{aligned}
\mathrm{AZI}(C_4 \vee (K_{n-4}-e))(n \geq 6) =& \frac{(n-6)(n-7)(n-1)^6}{2(2n-4)^3} + \frac{12(n-2)^6}{(2n-6)^3} + \\
& \frac{6(n-6)(n-1)^3(n-2)^3}{(2n-5)^3}.
\end{aligned}$$

It can be checked by calculator that for $n \geq 5$, $\mathrm{AZI}(\overline{S_3} \vee K_{n-3}) - \mathrm{AZI}(C_4 \vee K_{n-4}) > 0$, $\mathrm{AZI}(C_4 \vee K_{n-4}) - \mathrm{AZI}(\overline{S_4} \vee K_{n-4}) > 0$ and $\mathrm{AZI}(\overline{S_4} \vee K_{n-4}) - \mathrm{AZI}(G) > 0$, where $G \in \{\overline{K_3} \vee K_{n-3}, P_4 \vee K_{n-4}, \overline{S_3} \vee (K_{n-3}-e), C_4 \vee (K_{n-4}-e)\ (n \geq 6)\}$. $\square$

The following theorem gives the first five largest AZI indices of connected graphs with $n$ vertices, where $n \geq 5$.

**Theorem 2.4** *Let $G \in \mathbb{G}_n$ and $G \notin \{K_n, K_n - e, \overline{S_3} \vee K_{n-3}, C_4 \vee K_{n-4}, \overline{S_4} \vee K_{n-4}\}$, where $n \geq 5$. Then $\mathrm{AZI}(K_n) > \mathrm{AZI}(K_n - e) > \mathrm{AZI}(\overline{S_3} \vee K_{n-3}) > \mathrm{AZI}(C_4 \vee K_{n-4}) > \mathrm{AZI}(\overline{S_4} \vee K_{n-4}) > \mathrm{AZI}(G)$.*

**Proof** Since $G \in \mathbb{G}_n$ $(n \geq 5)$ and $G \notin \{K_n, K_n - e, \overline{S_3} \vee K_{n-3}, C_4 \vee K_{n-4}, \overline{S_4} \vee K_{n-4}\}$, we have $G \in \cup_{n-1 \leq m \leq \binom{n}{2}-3} \mathbb{G}_{n,m}$. If $G \in \cup_{n-1 \leq m \leq \binom{n}{2}-4} \mathbb{G}_{n,m}$, then by Corollary 2.2, there exists a graph $G^* \in \mathbb{G}_{n,\binom{n}{2}-3}$ such that $\mathrm{AZI}(G) < \mathrm{AZI}(G^*)$. It follows from Lemma 2.3 that

$$\mathrm{AZI}(G) < \mathrm{AZI}(G^*) \leq \mathrm{AZI}(\overline{S_4} \vee K_{n-4}). \tag{2.1}$$

If $G \in \mathbb{G}_{n,\binom{n}{2}-3}$, since $G \ncong \overline{S_4} \vee K_{n-4}$, then we also have

$$\mathrm{AZI}(G) < \mathrm{AZI}(\overline{S_4} \vee K_{n-4}) \tag{2.2}$$

by Lemma 2.3. Moreover, $K_n \cong (K_n - e) + e$ and $K_n - e \cong (\overline{S_3} \vee K_{n-3}) + e$, then by Lemma 2.1, we obtain that

$$\mathrm{AZI}(K_n) > \mathrm{AZI}(K_n - e) > \mathrm{AZI}(\overline{S_3} \vee K_{n-3}). \tag{2.3}$$

Combining inequalities (2.1)–(2.3) with the inequality

$$\mathrm{AZI}(\overline{S_3} \vee K_{n-3}) > \mathrm{AZI}(C_4 \vee K_{n-4}) > \mathrm{AZI}(\overline{S_4} \vee K_{n-4})$$

in Lemma 2.3, we obtain the desired results. $\square$

## 3. Ordering trees by the AZI indices

Let $x_{ij}$ be the number of edges of a graph $G$ connecting vertices of degrees $i$ and $j$, and let $A_{ij} = (\frac{ij}{i+j-2})^3$, where $i, j$ are positive integers. Obviously, $x_{ij} = x_{ji}$ and $A_{ij} = A_{ji}$. Then the augmented Zagreb index of a graph $G$ can be rewritten as $\mathrm{AZI}(G) = \sum_{i \le j} x_{ij} A_{ij}$.

**Lemma 3.1** *(1) $A_{1j}$ is decreasing for $j \ge 2$.*

*(2) $A_{2j} = 8$ for $j \ge 1$.*

*(3) If $i$ $(\ge 3)$ is fixed, then $A_{ij}$ is increasing for $j \ge 2$.*

**Proof** Clearly, $A_{2j} = (\frac{2j}{2+j-2})^3 = 8$ for $j \ge 1$. Note that

$$\frac{\partial(A_{ij})}{\partial j} = \frac{3i^3 j^2 (i-2)}{(i+j-2)^4}.$$

Hence $A_{1j}$ is decreasing and $A_{ij}$ is increasing for $j \ge 2$, where $i$ $(\ge 3)$ is fixed. $\square$

Let $\mathbb{T}_n$ be the set of trees of order $n \ge 3$, and let $\mathbb{T}_{n,p}$ be the set of trees with $n$ vertices and $p$ pendent vertices, where $2 \le p \le n-1$. Then $\mathbb{T}_n = \cup_{2 \le p \le n-1} \mathbb{T}_{n,p}$. Let $DS_n(p_1, p_2)$ be the tree of order $n$ formed from the path of order $n - p_1 - p_2$ by attaching $p_1$ and $p_2$ pendent vertices to its end vertices respectively, where $p_2 \ge p_1 \ge 1$ and $p_1 + p_2 \le n-2$. Clearly, $\mathbb{T}_{n,n-1} = \{S_n\}$, $\mathbb{T}_{n,n-2} = \{DS_n(p_1, n-2-p_1) | 1 \le p_1 \le \lfloor \frac{n-2}{2} \rfloor\}$ and $\mathbb{T}_{n,2} = \{DS_n(1,1)\} = \{P_n\}$.

**Theorem 3.2** *Let $T \in \mathbb{T}_{n,p}$, where $2 \le p \le n-3$. Then*

$$\mathrm{AZI}(T) \ge \frac{(\lfloor \frac{p}{2} \rfloor + 1)^3}{\lfloor \frac{p}{2} \rfloor^2} + \frac{(\lceil \frac{p}{2} \rceil + 1)^3}{\lceil \frac{p}{2} \rceil^2} + 8(n-1-p)$$

*with equality if and only if $T \cong DS_n(\lfloor \frac{p}{2} \rfloor, \lceil \frac{p}{2} \rceil)$.*

**Proof** The case of $p = 2$ is trivial since $\mathbb{T}_{n,2} = \{DS_n(1,1)\} = \{P_n\}$. Notice that there are $t$ vertices, denoted by $v_1, v_2, \ldots, v_t$, such that $\cup_{i=1}^t N(v_i)$ contains all pendent vertices of $T$. Suppose that there are $p_i$ pendent vertices in $N(v_i)$, where $i = 1, 2, \ldots, t$ and $\sum_{i=1}^t p_i = p$. Without loss of generality, we may assume that $p_i \ge 1$ for $1 \le i \le t$. Since $p \ne n-1$ (namely, $T$ is not a star), then $t \ge 2$. Hence

$$\mathrm{AZI}(T) = \sum_{i=1}^{t} p_i A_{1,d_{v_i}} + \sum_{2 \le i \le j \le n-1} x_{ij} A_{ij}. \tag{3.1}$$

Note that the terminal vertices of a diameter-achieving path $P$ of $T$ are two pendent vertices. Without loss of generality, suppose that the neighbors of the terminal vertices are $v_1$ and $v_2$, respectively. By the choice of the diameter-achieving path $P$, we have $d_{v_1} = p_1 + 1$ and $d_{v_2} =$

$p_2+1$. Note that $d_{v_i} \geq 2$ for $1 \leq i \leq t$ and

$$\sum_{i=1}^{t} d_{v_i} \leq 2(n-1) - p - 2(n-p-t) = p + 2t - 2.$$

We claim that $d_{v_i} \leq p+2-d_{v_1}$ for $2 \leq i \leq t$. Otherwise, if $d_{v_i} > p+2-d_{v_1}$ for some $i \neq 1$, then

$$p + 2t - 2 \geq \sum_{i=1}^{t} d_{v_i} > d_{v_1} + (p+2-d_{v_1}) + 2(t-2) = p + 2t - 2,$$

which is a contradiction. Therefore, by Lemma 3.1, we have

$$\begin{aligned}\sum_{i=1}^{t} p_i A_{1,d_{v_i}} &= p_1 A_{1,d_{v_1}} + \sum_{i=2}^{t} p_i A_{1,d_{v_i}} \geq p_1 A_{1,d_{v_1}} + \sum_{i=2}^{t} p_i A_{1,p+2-d_{v_1}} \\ &= p_1 A_{1,p_1+1} + (p-p_1) A_{1,p-p_1+1}.\end{aligned}$$

If $\sum_{i=1}^{t} p_i A_{1,d_{v_i}} = p_1 A_{1,p_1+1} + (p-p_1) A_{1,p-p_1+1}$ and $t \geq 3$, then we get

$$p + 2t - 2 \geq \sum_{i=1}^{t} d_{v_i} \geq d_{v_1} + 2(p+2-d_{v_1}) + 2(t-3) = (p+2t-2) + (p - d_{v_1}),$$

equivalently, $d_{v_1} = p_1 + 1 \geq p$, which is a contradiction. Consequently, we conclude that

$$\sum_{i=1}^{t} p_i A_{1,d_{v_i}} \geq p_1 A_{1,p_1+1} + (p-p_1) A_{1,p-p_1+1} = \frac{(p_1+1)^3}{p_1^2} + \frac{(p-p_1+1)^3}{(p-p_1)^2}$$

with equality if and only if $t = 2$ and $d_{v_2} = p+2-d_{v_1} = p-p_1+1$, namely, $p_1 + p_2 = p$. Moreover, the function $f(x) = \frac{(x+1)^3}{x^2}$ is convex increasing for $x \geq 2$, since

$$f'(x) = \frac{(x+1)^2(x-2)}{x^3} \geq 0 \text{ and } f''(x) = \frac{6(x+1)}{x^4} > 0.$$

Besides, $f(1) = 8 > f(2) = \frac{27}{4}$, and then

$$f(1) + f(p-1) > f(2) + f(p-2) \geq \cdots \geq f(\lfloor \frac{p}{2} \rfloor) + f(\lceil \frac{p}{2} \rceil).$$

It leads to

$$\sum_{i=1}^{t} p_i A_{1,d_{v_i}} \geq \frac{(p_1+1)^3}{p_1^2} + \frac{(p-p_1+1)^3}{(p-p_1)^2} \geq \frac{(\lfloor \frac{p}{2} \rfloor + 1)^3}{\lfloor \frac{p}{2} \rfloor^2} + \frac{(\lceil \frac{p}{2} \rceil + 1)^3}{\lceil \frac{p}{2} \rceil^2}.$$

The equality holds if and only if $t = 2$, $p_1 = \lfloor \frac{p}{2} \rfloor$ and $p_2 = \lceil \frac{p}{2} \rceil$.

On the other hand, it follows from Lemma 3.1 that

$$\sum_{2 \leq i \leq j \leq n-1} x_{ij} A_{ij} \geq \sum_{2 \leq i \leq j \leq n-1} x_{ij} A_{2j} = 8(n-1-p)$$

with equality holding if and only if all edges of $T$ are pendent edges or the edges with one end vertex of degree 2.

All in all, it follows from Equation (3.1) that

$$\mathrm{AZI}(T) \geq \frac{(\lfloor \frac{p}{2} \rfloor + 1)^3}{\lfloor \frac{p}{2} \rfloor^2} + \frac{(\lceil \frac{p}{2} \rceil + 1)^3}{\lceil \frac{p}{2} \rceil^2} + 8(n-1-p)$$

with equality if and only if $T \cong DS_n(\lfloor \frac{p}{2} \rfloor, \lceil \frac{p}{2} \rceil)$. This completes the proof. $\Box$

**Corollary 3.3** *Let $T \in \cup_{2 \le p \le n-3} \mathbb{T}_{n,p}$. Then*

$$\mathrm{AZI}(T) \ge \frac{(\lfloor \frac{n-3}{2} \rfloor + 1)^3}{\lfloor \frac{n-3}{2} \rfloor^2} + \frac{(\lceil \frac{n-3}{2} \rceil + 1)^3}{\lceil \frac{n-3}{2} \rceil^2} + 16$$

*with equality if and only if $T \cong DS_n(\lfloor \frac{n-3}{2} \rfloor, \lceil \frac{n-3}{2} \rceil)$.*

**Proof** By Theorem 3.2, it will suffice to show that $\mathrm{AZI}(DS_n(\lfloor \frac{p-1}{2} \rfloor, \lceil \frac{p-1}{2} \rceil)) > \mathrm{AZI}(DS_n(\lfloor \frac{p}{2} \rfloor, \lceil \frac{p}{2} \rceil))$, where $3 \le p \le n-3$. By Lemma 3.1, we have

$$\begin{aligned}
\mathrm{AZI}(DS_n(\lfloor \frac{p}{2} \rfloor, \lceil \frac{p}{2} \rceil)) &= \lfloor \frac{p}{2} \rfloor A_{1, \lfloor \frac{p}{2} \rfloor + 1} + \lceil \frac{p}{2} \rceil A_{1, \lceil \frac{p}{2} \rceil + 1} + 8(n-1-p) \\
&= \lceil \frac{p-1}{2} \rceil A_{1, \lceil \frac{p-1}{2} \rceil + 1} + \lfloor \frac{p-1}{2} \rfloor A_{1, \lceil \frac{p}{2} \rceil + 1} + A_{1, \lceil \frac{p}{2} \rceil + 1} + 8(n-1-p) \\
&< \lceil \frac{p-1}{2} \rceil A_{1, \lceil \frac{p-1}{2} \rceil + 1} + \lfloor \frac{p-1}{2} \rfloor A_{1, \lfloor \frac{p-1}{2} \rfloor + 1} + 8 + 8(n-1-p) \\
&= \mathrm{AZI}(DS_n(\lfloor \frac{p-1}{2} \rfloor, \lceil \frac{p-1}{2} \rceil)). \quad \Box
\end{aligned}$$

An order of trees in $\mathbb{T}_{n,n-2}$ $(n \ge 4)$ by their AZI indices is given as follows.

**Lemma 3.4** *Observe that $\mathbb{T}_{n,n-2} = \{DS_n(p_1, n-2-p_1) | 1 \le p_1 \le \lfloor \frac{n-2}{2} \rfloor\}$. Then*

$$\mathrm{AZI}(DS_n(\lfloor \frac{n-2}{2} \rfloor, \lceil \frac{n-2}{2} \rceil)) > \cdots > \mathrm{AZI}(DS_n(2, n-4)) > \mathrm{AZI}(DS_n(1, n-3)).$$

**Proof** For $1 \le p_1 \le \lfloor \frac{n-2}{2} \rfloor$, note that

$$\mathrm{AZI}(DS_n(p_1, n-2-p_1)) = \frac{(p_1+1)^3}{p_1^2} + \frac{(n-1-p_1)^3}{(n-2-p_1)^2} + \frac{(p_1+1)^3(n-1-p_1)^3}{(n-2)^3} := f(p_1).$$

The result follows since the function $f(p_1)$ is increasing for $1 \le p_1 \le \lfloor \frac{n-2}{2} \rfloor$. $\Box$

Let $T_6^*, T_7^*, T_7^{**}, T_7^{***}$ be the trees as shown in Figure 1.

$T_6^*$ $T_7^*$ $T_7^{**}$ $T_7^{***}$

Figure 1 $T_6^*, T_7^*, T_7^{**}, T_7^{***}$

Now we obtain an order of $\mathbb{T}_n$ for $3 \le n \le 7$ by their AZI indices. Observe that $\mathbb{T}_3 = \{S_3\}$, $\mathbb{T}_4 = \{P_4, S_4\}$, $\mathbb{T}_5 = \{P_5, DS_5(1,2), S_5\}$,

$$\mathrm{AZI}(P_4) > \mathrm{AZI}(S_4) \quad \text{and} \quad \mathrm{AZI}(P_5) > \mathrm{AZI}(DS_5(1,2)) > \mathrm{AZI}(S_5). \tag{3.2}$$

Note that $\mathbb{T}_6 = \{P_6, T_6^*, DS_6(1,2), DS_6(2,2), DS_6(1,3), S_6\}$,

$$\begin{aligned}
\mathrm{AZI}(P_6) >& \mathrm{AZI}(T_6^*) > \mathrm{AZI}(DS_6(1,2)) \\
&> \mathrm{AZI}(DS_6(2,2)) > \mathrm{AZI}(DS_6(1,3)) > \mathrm{AZI}(S_6),
\end{aligned} \tag{3.3}$$

and $\mathbb{T}_7 = \{P_7, T_7^*, DS_7(1,2), T_7^{**}, T_7^{***}, DS_7(1,3), DS_7(2,2), DS_7(2,3), DS_7(1,4), S_7\}$,

$$\mathrm{AZI}(P_7) > \mathrm{AZI}(T_7^*) > \mathrm{AZI}(DS_7(1,2)) > \mathrm{AZI}(T_7^{**})$$

$$
\begin{aligned}
&> \mathrm{AZI}(T_7^{***}) > \mathrm{AZI}(DS_7(1,3)) > \mathrm{AZI}(DS_7(2,2)) \\
&> \mathrm{AZI}(DS_7(2,3)) > \mathrm{AZI}(DS_7(1,4)) > \mathrm{AZI}(S_7).
\end{aligned} \tag{3.4}
$$

Moreover, the trees of order $n \geq 8$ with the first three smallest AZI indices are determined.

**Theorem 3.5** *Let $T \in \mathbb{T}_n$ and $T \ncong S_n, DS_n(1, n-3), DS_n(\lfloor \frac{n-3}{2} \rfloor, \lceil \frac{n-3}{2} \rceil)$, where $n \geq 8$. Then* $\mathrm{AZI}(S_n) < \mathrm{AZI}(DS_n(1, n-3)) < \mathrm{AZI}(DS_n(\lfloor \frac{n-3}{2} \rfloor, \lceil \frac{n-3}{2} \rceil)) < \mathrm{AZI}(T)$.

**Proof** It is obvious that

$$
\begin{aligned}
\mathrm{AZI}(S_n) &= (n-3)A_{1,n-1} + 2A_{1,n-1} < (n-3)A_{1,n-2} + 16 = \mathrm{AZI}(DS_n(1, n-3)) \\
&< \lceil \frac{n-3}{2} \rceil A_{1,\lceil \frac{n-3}{2} \rceil + 1} + \lfloor \frac{n-3}{2} \rfloor A_{1,\lfloor \frac{n-3}{2} \rfloor + 1} + 16 \\
&= \mathrm{AZI}(DS_n(\lfloor \frac{n-3}{2} \rfloor, \lceil \frac{n-3}{2} \rceil)).
\end{aligned}
$$

Since $T \in \mathbb{T}_n$ $(n \geq 8)$ and $T \ncong S_n, DS_n(1, n-3), DS_n(\lfloor \frac{n-3}{2} \rfloor, \lceil \frac{n-3}{2} \rceil)$, we consider the following two cases.

**Case 1** $T \in \mathbb{T}_{n,n-2} = \{DS_n(p_1, n-2-p_1) | 1 \leq p_1 \leq \lfloor \frac{n-2}{2} \rfloor\}$. By Lemma 3.4, we need to prove that $\mathrm{AZI}(DS_n(2, n-4)) > \mathrm{AZI}(DS_n(\lfloor \frac{n-3}{2} \rfloor, \lceil \frac{n-3}{2} \rceil))$ for $n \geq 8$. By Theorem 3.2,

$$
\begin{aligned}
\mathrm{AZI}(DS_n(\lfloor \frac{n-3}{2} \rfloor, \lceil \frac{n-3}{2} \rceil)) &< \mathrm{AZI}(DS_n(2, n-5)) \\
&= 2A_{1,3} + 16 + (n-5)A_{1,n-4} \\
&< 2A_{1,3} + A_{3,n-3} + (n-4)A_{1,n-3} \\
&= \mathrm{AZI}(DS_n(2, n-4)).
\end{aligned}
$$

**Case 2** $T \in \cup_{2 \leq p \leq n-3} \mathbb{T}_{n,p}$. By Corollary 3.3, we immediately get

$$
\mathrm{AZI}(DS_n(\lfloor \frac{n-3}{2} \rfloor, \lceil \frac{n-3}{2} \rceil)) < \mathrm{AZI}(T). \quad \square
$$

By using Theorem 3.5, we obtain the first two smallest AZI indices of connected graphs with $n \geq 5$ vertices as follows.

**Theorem 3.6** *Let $G \in \mathbb{G}_n$ and $G \ncong S_n, DS_n(1, n-3)$, where $n \geq 5$. Then*

$$
\mathrm{AZI}(S_n) < \mathrm{AZI}(DS_n(1, n-3)) < \mathrm{AZI}(G).
$$

**Proof** From inequalities (3.2)–(3.4) and Theorem 3.5, the inequality $\mathrm{AZI}(S_n) < \mathrm{AZI}(DS_n(1, n-3))$ holds for $n \geq 5$. Note that $G \in \cup_{n-1 \leq m \leq \binom{n}{2}} \mathbb{G}_{n,m}$. We have the following two cases.

**Case 1** $G \in \mathbb{G}_{n,n-1} = \mathbb{T}_n$ $(n \geq 5)$. By inequalities (3.2)–(3.4) and Theorem 3.5,

$$
\mathrm{AZI}(DS_n(1, n-3)) < \mathrm{AZI}(G).
$$

**Case 2** $G \in \cup_{n \leq m \leq \binom{n}{2}} \mathbb{G}_{n,m}$. By Corollary 2.2, there exists a graph $G^* \in \mathbb{G}_{n,n-1}$ such that $\mathrm{AZI}(G^*) < \mathrm{AZI}(G)$. If $G^* \ncong S_n$, then we immediately get $\mathrm{AZI}(DS_n(1, n-3)) \leq \mathrm{AZI}(G^*) < \mathrm{AZI}(G)$. If $G^* \cong S_n$, then by Lemma 2.1, we conclude that $G$ is obtained from $S_n$ by inserting some edges. It follows that

$$\begin{aligned}\mathrm{AZI}(G) &\geq \mathrm{AZI}(S_n^+) = 24 + (n-3)A_{1,n-1}\\ &> 16 + (n-3)A_{1,n-2} = \mathrm{AZI}(DS_n(1, n-3)). \quad \square\end{aligned}$$

## 4. Unicyclic graphs with the first two smallest AZI indices

Denote by $C_{n,p}$ the unicyclic graph of order $n$ formed by attaching $p$ pendent vertices to a vertex of the cycle $C_{n-p}$, where $0 \leq p \leq n-3$. Let $C_{n,p}^{p_1,p_2,\ldots,p_{n-p}}$ denote the unicyclic graph of order $n$ obtained from the cycle $C_{n-p} = v_1v_2\cdots v_{n-p}v_1$ by attaching $p_i$ pendent vertices to vertex $v_i$, where $p_i \geq 0$, $i = 1, 2, \ldots, n-p$ and $\sum_{i=1}^{n-p} p_i = p$. Clearly, $C_{n,0} \cong C_n$, $C_{n,n-3} \cong S_n^+$ and $C_{n,p}^{p,0,\ldots,0} \cong C_{n,p}$.

Let $U_5^*$ be the unicyclic graph obtained by identifying one vertex of $C_3$ and one end vertex of $P_3$. Let $\mathbb{U}_n$ be the set of unicyclic graphs of order $n \geq 3$. Obviously, $\mathbb{U}_3 = \{K_3\}$, $\mathbb{U}_4 = \{C_4, S_4^+\}$ and $\mathbb{U}_5 = \{C_5, S_5^+, C_{5,1}, C_{5,2}^{1,1,0}, U_5^*\}$. By simply calculating, we get that $\mathrm{AZI}(S_4^+) < \mathrm{AZI}(C_4)$ and $\mathrm{AZI}(S_5^+) < \mathrm{AZI}(C_{5,2}^{1,1,0}) < \mathrm{AZI}(C_{5,1}) < \mathrm{AZI}(C_5) = \mathrm{AZI}(U_5^*)$.

Let $\mathbb{U}_{n,p}$ be the set of unicyclic graphs with $n$ vertices and $p$ pendent vertices, where $0 \leq p \leq n-3$. Then $\mathbb{U}_n = \cup_{0\leq p\leq n-3}\mathbb{U}_{n,p}$.

**Lemma 4.1** ([5]) *Let $U \in \mathbb{U}_{n,p}$, where $0 \leq p \leq n-3$. Then*

$$\mathrm{AZI}(U) \geq \frac{p(p+2)^3}{(p+1)^3} + 8(n-p)$$

*with equality if and only if $U \cong C_{n,p}$.*

**Lemma 4.2** *Let $C_{n,p}$ be the unicyclic graph of order $n$ defined above, where $0 \leq p \leq n-3$. Then $\mathrm{AZI}(C_{n,0}) > \mathrm{AZI}(C_{n,1}) > \cdots > \mathrm{AZI}(C_{n,n-4}) > \mathrm{AZI}(C_{n,n-3})$.*

**Proof** Note that $\mathrm{AZI}(C_{n,p}) = \frac{p(p+2)^3}{(p+1)^3} + 8(n-p)$. Let $f(x) = \frac{x(x+2)^3}{(x+1)^3} + 8(n-x)$. Then

$$f'(x) = -\frac{x(7x^3 + 28x^2 + 42x + 24)}{(x+1)^4} \leq 0.$$

Thus $f(x)$ is decreasing for $x \geq 0$. This completes the proof. $\square$

By Lemmas 4.1 and 4.2, it is easy to obtain the following corollary.

**Corollary 4.3** *Let $U \in \cup_{0\leq p\leq n-4}\mathbb{U}_{n,p}$. Then*

$$\mathrm{AZI}(U) \geq \frac{(n-4)(n-2)^3}{(n-3)^3} + 32$$

*with equality if and only if $U \cong C_{n,n-4}$.*

**Lemma 4.4** *Let $U \in \mathbb{U}_{n,n-3}$ and $U \ncong S_n^+$, where $n \geq 6$. Then $\mathrm{AZI}(U) > \mathrm{AZI}(C_{n,n-4}) > \mathrm{AZI}(S_n^+)$.*

**Proof** Since $U \in \mathbb{U}_{n,n-3}$, we may assume that $G \cong C_{n,n-3}^{p_1,p_2,p_3}$, where $p_1 \geq p_2 \geq p_3 \geq 0$ and $\sum_{i=1}^{3} p_i = n-3$. Notice that $U \ncong S_n^+$, then $p_2 \geq 1$. Let $r = (p_1-1)(A_{1,p_1+2} - A_{1,n-2}) +$

$p_2(A_{1,p_2+2} - A_{1,n-2}) + p_3(A_{1,p_3+2} - A_{1,n-2})$.

**Case 1** $p_3 = 0$. Since $n \geq 6$, then $p_1 \geq 2$. By Lemma 3.1, we have $r > 0$ and

$$\mathrm{AZI}(U) - \mathrm{AZI}(C_{n,n-4}) = r + A_{1,p_1+2} + A_{p_1+2,p_2+2} - 16. \tag{4.1}$$

**Subcase 1.1** $p_1 = 2$. It follows from (4.1) and Lemma 3.1 that

$$\mathrm{AZI}(U) - \mathrm{AZI}(C_{n,n-4}) > 0 + A_{1,4} + A_{4,3} - 16 = \frac{4^3}{3^3} + \frac{12^3}{5^3} - 16 > 0.$$

**Subcase 1.2** $p_1 \geq 3$. Then by Lemma 3.1 and (4.1), we have

$$\mathrm{AZI}(U) - \mathrm{AZI}(C_{n,n-4}) > 0 + 1 + A_{5,3} - 16 = 1 + \frac{15^3}{6^3} - 16 > 0.$$

**Case 2** $p_3 \geq 1$. By Lemma 3.1, we obtain that $r > 0$ and

$$\begin{aligned}\mathrm{AZI}(U) - \mathrm{AZI}(C_{n,n-4}) &= r + A_{1,p_1+2} + A_{p_1+2,p_2+2} + A_{p_1+2,p_3+2} + A_{p_2+2,p_3+2} - 32 \\ &> 0 + 1 + 3A_{3,3} - 32 = 1 + 3 \cdot \frac{9^3}{4^3} - 32 > 0.\end{aligned}$$

Combining the above cases, we get that $\mathrm{AZI}(U) > \mathrm{AZI}(C_{n,n-4})$. Moreover, it is easy to obtain that $\mathrm{AZI}(C_{n,n-4}) = (n-4)A_{1,n-2} + 32 > \mathrm{AZI}(S_n^+) = (n-3)A_{1,n-1} + 24$. $\square$

It follows from Corollary 4.3 and Lemma 4.4 that the unicyclic graphs of order $n \geq 6$ with the minimum and the second minimum AZI indices are determined.

**Theorem 4.5** *Let $U \in \mathbb{U}_n$ and $G \ncong S_n^+, C_{n,n-4}$, where $n \geq 6$. Then*

$$\mathrm{AZI}(S_n^+) < \mathrm{AZI}(C_{n,n-4}) < \mathrm{AZI}(U).$$

## 5. Bicyclic graphs with the minimum AZI index

Let $\mathbb{B}_n$ be the set of bicyclic graphs of order $n \geq 4$. Clearly, $\mathbb{B}_4 = \{K_4 - e\}$. Let $\mathbb{B}_{n,p}$ be the set of bicyclic graphs with $n$ vertices and $p$ pendent vertices, where $0 \leq p \leq n-4$. Then $\mathbb{B}_n = \cup_{0 \leq p \leq n-4} \mathbb{B}_{n,p}$.

Denote by $D_{n,r,s,p}$ the bicyclic graph of order $n$ by identifying one vertex of two cycles $C_r$ and $C_s$, and attaching $p$ pendent vertices to the common vertex, where $r \geq s \geq 3$ and $0 \leq p = n+1-r-s \leq n-5$.

**Lemma 5.1** ([5]) *Let $B \in \mathbb{B}_{n,p}$, where $0 \leq p \leq n-5$. Then*

$$\mathrm{AZI}(B) \geq \frac{p(p+4)^3}{(p+3)^3} + 8(n+1-p)$$

*with equality if and only if $B \cong D_{n,r,s,p}$, where $r \geq s \geq 3$ and $r+s = n+1-p$.*

**Lemma 5.2** *Let $D_{n,r,s,p}$ be the bicyclic graph of order $n$ defined above, where $r \geq s \geq 3$ and $0 \leq p = n+1-r-s \leq n-5$. Then $\mathrm{AZI}(D_{n,r,s,0}) > \mathrm{AZI}(D_{n,r,s,1}) > \cdots > \mathrm{AZI}(D_{n,r,s,n-5})$.*

**Proof** Observe that

$$\mathrm{AZI}(D_{n,r,s,p}) = \frac{p(p+4)^3}{(p+3)^3} + 8(n+1-p) := g(p).$$

Then

$$g'(p) = -\frac{7p^4 + 84p^3 + 372p^2 + 704p + 456}{(p+3)^4} < 0.$$

Hence $g(p)$ is decreasing for $p \geq 0$. The proof is completed. □

It can be seen from Lemmas 5.1 and 5.2 that

**Corollary 5.3** *Let $B \in \cup_{0 \leq p \leq n-5} \mathbb{B}_{n,p}$. Then* $\mathrm{AZI}(B) \geq \frac{(n-5)(n-1)^3}{(n-2)^3} + 48$ *with equality if and only if $B \cong D_{n,3,3,n-5}$.*

Now we consider the set $\mathbb{B}_{n,n-4}$, where $n \geq 5$. Let $E_n^{p_1,p_2,p_3,p_4}$ be the bicyclic graph obtained from $K_4 - e$ by attaching $p_i$ pendent vertices to vertex $v_i \in V(K_4 - e)$ for $1 \leq i \leq 4$, where $d_{v_1} = d_{v_2} = 3$, $d_{v_3} = d_{v_4} = 2$, $p_1 \geq p_2 \geq 0$, $p_3 \geq p_4 \geq 0$ and $\sum_{i=1}^4 p_i = n-4$. Then $\mathbb{B}_{n,n-4} = \{E_n^{p_1,p_2,p_3,p_4} | p_1 \geq p_2 \geq 0, p_3 \geq p_4 \geq 0 \text{ and } \sum_{i=1}^4 p_i = n-4\}$.

**Lemma 5.4** *Let $B \in \mathbb{B}_{n,n-4}$, where $n \geq 5$. Then*

$$\mathrm{AZI}(B) \geq \frac{(n-4)(n-1)^3}{(n-2)^3} + \frac{27(n-1)^3}{n^3} + 32$$

*with equality if and only if $B \cong E_n^{n-4,0,0,0}$.*

**Proof** Let $B \cong E_n^{p_1,p_2,p_3,p_4}$, where $p_1 \geq p_2 \geq 0, p_3 \geq p_4 \geq 0$ and $\sum_{i=1}^4 p_i = n-4$. Note that

$$\begin{aligned}\mathrm{AZI}(E_n^{p_1,p_2,p_3,p_4}) =& p_1 A_{1,p_1+3} + p_2 A_{1,p_2+3} + p_3 A_{1,p_3+2} + \\ & p_4 A_{1,p_4+2} + A_{p_1+3,p_2+3} + A_{p_1+3,p_3+2} + \\ & A_{p_1+3,p_4+2} + A_{p_2+3,p_3+2} + A_{p_2+3,p_4+2}.\end{aligned}$$

Let $r = p_1(A_{1,p_1+3} - A_{1,n-1}) + p_2(A_{1,p_2+3} - A_{1,n-1}) + p_3(A_{1,p_3+2} - A_{1,n-1}) + p_4(A_{1,p_4+2} - A_{1,n-1})$. Then by Lemma 3.1, we have $r \geq 0$ with equality holding if and only if $p_1 = n-4$ and $p_2 = p_3 = p_4 = 0$. Now we discuss the following cases.

**Case 1** $p_2 \geq 1$. Then $p_1 \geq p_2 \geq 1$.

**Subcase 1.1** $p_3 \geq 1$. It follows from Lemma 3.1 that

$$\begin{aligned}\mathrm{AZI}(B) - \mathrm{AZI}(E_n^{n-4,0,0,0}) =& r + A_{p_1+3,p_2+3} + A_{p_1+3,p_3+2} + A_{p_1+3,p_4+2} + \\ & A_{p_2+3,p_3+2} + A_{p_2+3,p_4+2} - A_{3,n-1} - 32 \\ >& 0 + A_{4,4} + 2A_{4,3} + 2A_{4,2} - A_{3,n-1} - 32 \\ >& 0 + \frac{16^3}{6^3} + 2 \cdot \frac{12^3}{5^3} + 16 - 27 - 32 > 0.\end{aligned}$$

**Subcase 1.2** $p_3 = 0$. Then $p_4 = 0$. Hence by Lemma 3.1, we have

$$\begin{aligned}\mathrm{AZI}(B) - \mathrm{AZI}(E_n^{n-4,0,0,0}) &= r + A_{p_1+3,p_2+3} - A_{3,n-1} \\ &> 0 + \frac{[p_1p_2 + 3(n-1)]^3 - [3(n-1)]^3}{n^3} > 0.\end{aligned}$$

**Case 2** $p_2 = 0$. Let $q(x) = A_{3,x}$. Then $q(x)$ is concave increasing for $x \geq 2$ since

$$q'(x) = \frac{81x^2}{(x+1)^4} > 0 \text{ and } q''(x) = -\frac{162x(x-1)}{(x+1)^5} < 0.$$

It follows that

$$A_{3,\lceil\frac{n}{2}\rceil}+A_{3,\lfloor\frac{n}{2}\rfloor}>\cdots>A_{3,n-2}+A_{3,2}, \tag{5.1}$$

$$A_{3,\lceil\frac{n+1}{2}\rceil}+A_{3,\lfloor\frac{n+1}{2}\rfloor}>\cdots>A_{3,n-1}+A_{3,2}. \tag{5.2}$$

**Subcase 2.1** $p_4\ge 1$. Then $p_3\ge p_4\ge 1$. If $p_1\ge 1$, then by Lemma 3.1,

$$\begin{aligned}\mathrm{AZI}(B)-\mathrm{AZI}(E_n^{n-4,0,0,0})&\ge r+3A_{4,3}+2A_{3,3}-A_{3,n-1}-32\\&>0+3\cdot\frac{12^3}{5^3}+2\cdot\frac{9^3}{4^3}-27-32>0.\end{aligned}$$

If $p_1=0$, then by Lemma 3.1 and the inequality (5.1), for $n\ge 5$ we have

$$\begin{aligned}&\mathrm{AZI}(B)-\mathrm{AZI}(E_n^{n-4,0,0,0})\\&=r+A_{3,3}+2(A_{3,p_3+2}+A_{3,p_4+2})-A_{3,n-1}-32\\&>0+A_{3,3}+2(A_{3,n-2}+A_{3,2})-A_{3,n-1}-32\\&=\frac{(n-4)(1433n^5-3751n^4-337n^3+5859n^2-2484n+432)}{64n^3(n-1)^3}>0.\end{aligned}$$

**Subcase 2.2** $p_4=0$. It follows from Lemma 3.1 and the inequality (5.2) that

$$\begin{aligned}&\mathrm{AZI}(B)-\mathrm{AZI}(E_n^{n-4,0,0,0})\\&=r+(A_{3,p_1+3}+A_{3,p_3+2})+A_{p_1+3,p_3+2}-A_{3,n-1}-16\\&\ge 0+(A_{3,n-1}+A_{3,2})+A_{p_1+3,p_3+2}-A_{3,n-1}-16\\&=A_{p_1+3,p_3+2}-8\ge 0\end{aligned}$$

with equality if and only if $p_1=n-4$ and $p_2=p_3=p_4=0$, that is, $B\cong E_n^{n-4,0,0,0}$. □

The bicyclic graph of order $n\ge 5$ with the minimum AZI index is characterized in the following theorem.

**Theorem 5.5** *Let $B\in\mathbb{B}_n$ and $B\ncong D_{n,3,3,n-5}$, where $n\ge 5$. Then $\mathrm{AZI}(D_{n,3,3,n-5})<\mathrm{AZI}(B)$.*

**Proof** Note that $\mathbb{B}_n=\cup_{0\le p\le n-4}\mathbb{B}_{n,p}$. Then by Corollary 5.3 and Lemma 5.4, it will suffice to prove that for $n\ge 5$, $\mathrm{AZI}(D_{n,3,3,n-5})<\mathrm{AZI}(E_n^{n-4,0,0,0})$. It is obvious that

$$\mathrm{AZI}(E_n^{n-4,0,0,0})-\mathrm{AZI}(D_{n,3,3,n-5})=\frac{27(n-1)^3}{n^3}+\frac{(n-1)^3}{(n-2)^3}-16>0.$$

This completes the proof of Theorem 5.5. □

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## References


[1] R. TODESCHINI, V. CONSONNI. *Handbook of Molecular Descriptors*. Wiley-VCH, Weinheim, 2000.
[2] K. C. DAS. *Atom-bond connectivity index of graphs*. Discrete Appl. Math., 2010, **158**(11): 1181–1188.
[3] B. FURTULA, A. GRAOVAC, D. VUKIČEVIĆ. *Atom-bond connectivity index of trees*. Discrete Appl. Math., 2009, **157**(13): 2828–2835.
[4] B. FURTULA, A. GRAOVAC, D. VUKIČEVIĆ. *Augmented Zagreb index*. J. Math. Chem., 2010, **48**(2): 370–380.
[5] Yufei HUANG, Bolian LIU, Lu GAN. *Augmented Zagreb index of connected graphs*. MATCH Commun. Math. Comput. Chem., 2012, **67**(2): 483–494.