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Evaluating Binomial Character Sums Modulo Powers of Two

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Abstract We show that for any mod 2^m characters, χ_1, χ_2, the complete exponential sum, $\sum_{x=1}^{2^m} \chi_1(x)\chi_2(Ax^k + B)$ has a simple explicit evaluation.

Keywords character sums

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1. Introduction

Suppose that χ_1 and χ_2 are mod 2^m multiplicative characters with χ_2 primitive mod 2^m, $m \geq 3$. We are interested here in evaluating the complete character sum

$$S = \sum_{x=1}^{2^m} \chi_1(x)\chi_2(Ax^k + B).$$

Writing $\chi(Ax^K + Bx^L) = \chi^L(x)\chi(Ax^{K-L} + B)$ these sums of course include the binomial character sums. Cases where one can explicitly evaluate an exponential or character sum are unusual and therefore worth investigating.

In [8] we considered the corresponding result for mod p^m characters with $p \geq 3$ and m sufficiently large, using reduction techniques of Cochrane [2] and Cochrane & Zheng [3,4]. Though their results are stated for odd primes, the approach can often be adapted for $p = 2$ as we showed for twisted monomial exponential sums in [7]. When $k = 1$ and $A = -1$, $B = 1$, the mod p^m sum is the classical Jacobi sum (though uninterestingly zero if $p = 2$). See [1] or [5] for an extensive treatment of mod p Jacobi sums and their generalizations over $\mathbb{F}_{p^m}$. In [6] we treated the $k = 1$ case for general A, B, including when $p = 2$, along with generalizations of the multivariable Jacobi sums considered in [9].

Plainly $S = 0$ if A and B are not of opposite parity (otherwise x or $Ax^k + B$ will be even and the individual terms will all be zero). We assume here that A is even and B is odd and write

$$A = 2^n A_1,\ n > 0, \quad k = 2^t k_1, \quad 2 \nmid A_1 k_1 B.$$

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If B is even and A odd, we can use $x \mapsto x^{-1}$ to write S in the form

$$S = \sum_{x=1}^{2^m} \overline{\chi}_1 \overline{\chi}_2^k(x)\chi_2(Bx^k + A).$$

Since $\mathbb{Z}_{2^m}^* = \langle -1, 5\rangle$, the characters χ_1, χ_2 are completely determined by their values on -1 and 5. Since 5 has order 2^{m-2} mod 2^m, we can define integers c_1, c_2 with

$$\chi_i(5) = e_{2^{m-2}}(c_i), \quad 1 \leq c_i \leq 2^{m-2},$$

where $e_n(x) := e^{2\pi i x/n}$. Since χ_2 is primitive, we have $2 \nmid c_2$. We define the odd integers R_i, $i \geq 2$, by

$$5^{2^{i-2}} = 1 + R_i 2^i. \tag{1}$$

Defining

$$N := \begin{cases} \lceil \frac{1}{2}(m-n) \rceil, & \text{if } m-n > 2t+4, \\ t+2, & \text{if } t+2 \leq m-n \leq 2t+4, \end{cases}$$

and

$$C(x) := c_1(Ax^k + B) + c_2 Akx^k R_N R_{N+n}^{-1} \tag{2}$$

(here and throughout the paper y^{-1} denotes the inverse of y mod 2^m) it transpires that the sum S will be zero unless there is a solution x_0 to the characteristic equation

$$C(x_0) \equiv 0 \bmod 2^{\lfloor \frac{1}{2}(m+n) \rfloor + t}, \tag{3}$$

with $2 \nmid x_0(Ax_0^k + B)$, when $m - n > 2t + 4$, and a solution to $C(1)$ or $C(-1) \equiv 0 \bmod 2^{m-2}$ when $t + 2 \leq m - n \leq 2t + 4$.

Theorem 1.1 *Suppose that $m - n \geq t + 2$. The sum $S = 0$ unless $c_1 = 2^{n+t}c_3$, with $2 \nmid c_3$, and $\chi_1(-1) = 1$ when k is even, and the characteristic equation (3) has an odd solution x_0 when $m - n > 2t + 4$. Assuming these conditions do hold.*

When $m - n > 2t + 4$,

$$S = 2^{\frac{1}{2}(m+n)+t+\min\{1,t\}} \chi_1(x_0)\chi_2(Ax_0^k + B) \begin{cases} 1, & \text{if } m-n \text{ is even}, \\ \omega^h(\frac{2}{h}), & \text{if } m-n \text{ is odd}, \end{cases}$$

where $(\frac{2}{x})$ is the Jacobi symbol, $\omega = e^{\pi i/4}$, $C(x_0) = \lambda 2^{\lfloor \frac{1}{2}(m+n) \rfloor + t}$ for some integer λ and $h := 2\lambda + (k_1 - 1) + (2^n - 1)c_3$.

When $t + 3 < m - n \leq 2t + 4$,

$$S = \begin{cases} 2^{m-1}\chi_2(A+B), & \text{if } k \text{ is even and } C(1) \equiv 0 \bmod 2^{m-2}, \\ 2^{m-2}\chi_2(A+B), & \text{if } k \text{ is odd and } C(1) \equiv 0 \bmod 2^{m-2}, \\ 2^{m-2}\chi_1(-1)\chi_2(-A+B), & \text{if } k \text{ is odd and } C(-1) \equiv 0 \bmod 2^{m-2}, \\ 0, & \text{otherwise}. \end{cases}$$

When $m - n = t + 3$,

$$S = \begin{cases} 2^{m-1}\chi_2(A+B), & \text{if } k \text{ is even and } \chi_1(5) = \pm 1,\ \chi_1(-1) = 1, \\ 2^{m-2}\left(\chi_2(A+B) + \chi_1(-1)\chi_2(-A+B)\right), & \text{if } k \text{ is odd and } \chi_1(5) = \pm 1, \\ 0, & \text{otherwise}. \end{cases}$$

When $m-n=t+2$,

$$S=\begin{cases} 2^{m-1}\chi_2(A+B), & \text{if } k \text{ is even and } \chi_1=\chi_0 \text{ or } k \text{ is odd and } \chi_1=\chi_4,\\ 0, & \text{otherwise,}\end{cases}$$

where χ_0 *is the principal character mod* 2^m *and* χ_4 *is the mod* 2^m *character induced by the non-trivial character mod 4 (i.e.,* $\chi_4(x)=\pm 1$ *as* $x\equiv \pm 1$ *mod 4, respectively).*

Note that the restriction $m-n\geq t+2$ is quite natural; for $m-n<t+2$ the odd x will make $Ax^k+B\equiv A+B$ mod 2^m and $S=\chi_2(A+B)\sum_{x=1}^{2^m}\chi_1(x)=2^{m-1}\chi_2(A+B)$ if $\chi_1=\chi_0$ and zero otherwise.

Our original assumption that χ_2 is primitive is also reasonable; if χ_1 and χ_2 are both imprimitive, then one should reduce the modulus, while if χ_1 is primitive and χ_2 imprimitive, then $S=0$ (if χ_1 is primitive then $u=1+2^{m-1}$ must have $\chi_1(u)=-1$, since $x+2^{m-1}\equiv ux$ mod 2^m for any odd x, and $x\mapsto xu$ gives $S=\chi_1(u)S$ when χ_2 is imprimitive).

2. Proof

Initial decomposition

Observing that $\pm 5^{\gamma}$, $\gamma=1,\dots,2^{m-2}$, gives a reduced residue system mod 2^m and writing

$$S(A):=\sum_{\gamma=1}^{2^{m-2}}\chi_1(5^{\gamma})\chi_2(A5^{\gamma k}+B),$$

if k is even we have

$$S=(1+\chi_1(-1))S(A)=\begin{cases}0, & \text{if } \chi_1(-1)=-1,\\ 2S(A), & \text{if } \chi_1(-1)=1,\end{cases} \tag{4}$$

and if k is odd

$$S=S(A)+\chi_1(-1)S(-A). \tag{5}$$

Large m **values:** $m>n+2t+4$

If I_1 is an interval of length $2^{\lceil \frac{m-n}{2}\rceil -t-2}$, then plainly

$$\gamma=u2^{\lceil \frac{m-n}{2}\rceil -t-2}+v,\quad v\in I_1,\quad u\in I_2:=\left[1,2^{\lfloor \frac{m+n}{2}\rfloor +t}\right],$$

runs through a complete set of residues mod 2^{m-2}. Hence, writing $h(x):=Ax^k+B$ and noting that $2\nmid h(5^v)$,

$$\begin{aligned} S(A)&=\sum_{v\in I_1}\chi_1(5^v)\sum_{u\in I_2}\chi_1\big(5^{u2^{\lceil \frac{m-n}{2}\rceil -t-2}}\big)\chi_2\big(A5^{vk}5^{ku2^{\lceil \frac{m-n}{2}\rceil -t-2}}+B\big)\\ &=\sum_{v\in I_1}\chi_1(5^v)\chi_2(h(5^v))\sum_{u\in I_2}\chi_1\big(5^{u2^{\lceil \frac{m-n}{2}\rceil -t-2}}\big)\chi_2(W)\end{aligned}$$

where

$$W=h(5^v)^{-1}A5^{vk}\big(5^{ku2^{\lceil \frac{m-n}{2}\rceil -t-2}}-1\big)+1.$$

Since $n+2\lceil \frac{m-n}{2}\rceil\geq m$ and $2\lceil \frac{m+n}{2}\rceil \geq m$, we have

$$W=A_1 5^{vk}h(5^v)^{-1}2^n\big((1+R_{\lceil \frac{m-n}{2}\rceil}2^{\lceil \frac{m-n}{2}\rceil})^{uk_1}-1\big)+1$$

$$\begin{aligned}
&\equiv 1 + A_1 5^{vk} h(5^v)^{-1} u k_1 R_{\lceil \frac{m-n}{2} \rceil} 2^{\lceil \frac{m+n}{2} \rceil} \bmod 2^m \\
&\equiv \left(1 + R_{\lceil \frac{m+n}{2} \rceil} 2^{\lceil \frac{m+n}{2} \rceil}\right)^{A_1 5^{vk} h(5^v)^{-1} u k_1 R_{\lceil \frac{m-n}{2} \rceil} R^{-1}_{\lceil \frac{m+n}{2} \rceil}} \bmod 2^m \\
&= 5^{A_1 5^{vk} h(5^v)^{-1} u k_1 R_{\lceil \frac{m-n}{2} \rceil} R^{-1}_{\lceil \frac{m+n}{2} \rceil} 2^{\lceil \frac{m+n}{2} \rceil - 2}} \\
&= 5^{A 5^{vk} h(5^v)^{-1} u k R_{\lceil \frac{m-n}{2} \rceil} R^{-1}_{\lceil \frac{m+n}{2} \rceil} 2^{\lceil \frac{m-n}{2} \rceil - t - 2}}.
\end{aligned}$$

So we can write

$$\sum_{u \in I_2} \chi_1\left(5^{u 2^{\lceil \frac{m-n}{2} \rceil - t - 2}}\right) \chi_2(W) = \sum_{u \in I_2} e_{2^{\lfloor \frac{m+n}{2} \rfloor + t}}\left(u(c_1 + c_2 A 5^{vk} h(5^v)^{-1} k R_{\lceil \frac{m-n}{2} \rceil} R^{-1}_{\lceil \frac{m+n}{2} \rceil})\right),$$

which equals $2^{\lfloor \frac{m+n}{2} \rfloor + t}$ for the v with

$$c_1 h(5^v) + c_2 A 5^{vk} k R_{\lceil \frac{m-n}{2} \rceil} R^{-1}_{\lceil \frac{m+n}{2} \rceil} \equiv 0 \bmod 2^{\lfloor \frac{m+n}{2} \rfloor + t} \tag{6}$$

and zero otherwise. Since $m \geq n+2$, equation (6) has no solution (and hence $S = 0$) unless $c_1 = 2^{n+t} c_3$ with $2 \nmid c_3$, in which case (6) becomes

$$\left(c_3 A + c_2 A_1 k_1 R_{\lceil \frac{m-n}{2} \rceil} R^{-1}_{\lceil \frac{m+n}{2} \rceil}\right) 5^{vk} \equiv -c_3 B \bmod 2^{\lfloor \frac{m-n}{2} \rfloor}. \tag{7}$$

If no v satisfies (6), then plainly $S = 0$. So assume that (6) has a solution $v = v_0$ and take $I_1 = [v_0, v_0 + 2^{\lceil \frac{m-n}{2} \rceil - t - 2})$. Now any other v solving (7) must have

$$5^{vk} \equiv 5^{v_0 k} \bmod 2^{\lfloor \frac{m-n}{2} \rfloor} \Rightarrow vk \equiv v_0 k \bmod 2^{\lfloor \frac{m-n}{2} \rfloor - 2} \Rightarrow v \equiv v_0 \bmod 2^{\lfloor \frac{m-n}{2} \rfloor - t - 2}.$$

So if $m - n$ is even, I_1 contains only the solution v_0 and

$$S(A) = 2^{\lfloor \frac{m+n}{2} \rfloor + t} \chi_1(5^{v_0}) \chi_2(A 5^{v_0 k} + B). \tag{8}$$

Observe that a solution $x_0 = 5^{v_0}$ or $x_0 = -5^{v_0}$ of (3) corresponds to a solution v_0 to (6) when k is even and a solution v_0 to (6) for A or $-A$ respectively (both cannot have solutions) if k is odd. The evaluation for S follows at once from (8) and (4) or (5). When $m - n$ is odd, I_1 contains two solutions v_0 and $v_0 + 2^{\lfloor \frac{m-n}{2} \rfloor - t - 2}$ and

$$\begin{aligned}
S(A) &= 2^{\lfloor \frac{m+n}{2} \rfloor + t} \chi_1(5^{v_0}) \left(\chi_2(h(5^{v_0})) + \chi_1(5^{2^{\lfloor \frac{m-n}{2} \rfloor - t - 2}}) \chi_2(A 5^{v_0 k} 5^{k 2^{\lfloor \frac{m-n}{2} \rfloor - t - 2}} + B)\right) \\
&= 2^{\lfloor \frac{m+n}{2} \rfloor + t} \chi_1(5^{v_0}) \chi_2(h(5^{v_0})) \left(1 + \chi_1(5^{2^{\lfloor \frac{m-n}{2} \rfloor - t - 2}}) \chi_2(\xi)\right)
\end{aligned}$$

where, since $3 \lfloor \frac{m-n}{2} \rfloor + n \geq m$ for $m \geq n+3$,

$$\begin{aligned}
\xi &= A 5^{v_0 k} \left(5^{k_1 2^{\lfloor \frac{m-n}{2} \rfloor - 2}} - 1\right) h(5^{v_0})^{-1} + 1 \\
&= A 5^{v_0 k} h(5^{v_0})^{-1} \left((1 + R_{\lfloor \frac{m-n}{2} \rfloor} 2^{\lfloor \frac{m-n}{2} \rfloor})^{k_1} - 1\right) + 1 \\
&\equiv A 5^{v_0 k} h(5^{v_0})^{-1} \left(k_1 R_{\lfloor \frac{m-n}{2} \rfloor} 2^{\lfloor \frac{m-n}{2} \rfloor} + \binom{k_1}{2} R^2_{\lfloor \frac{m-n}{2} \rfloor} 2^{m-n-1}\right) + 1 \bmod 2^m \\
&\equiv \left(A_1 5^{v_0 k} h(5^{v_0})^{-1} k_1 R_{\lfloor \frac{m-n}{2} \rfloor} R^{-1}_{\lfloor \frac{m+n}{2} \rfloor} + \frac{1}{2}(k_1 - 1) 2^{\lfloor \frac{m-n}{2} \rfloor}\right) R_{\lfloor \frac{m+n}{2} \rfloor} 2^{\lfloor \frac{m+n}{2} \rfloor} + 1 \bmod 2^m \\
&\equiv \left(1 + R_{\lfloor \frac{m+n}{2} \rfloor} 2^{\lfloor \frac{m+n}{2} \rfloor}\right)^{A_1 5^{v_0 k} h(5^{v_0})^{-1} k_1 R_{\lfloor \frac{m-n}{2} \rfloor} R^{-1}_{\lfloor \frac{m+n}{2} \rfloor} + \frac{1}{2}(k_1 - 1) 2^{\lfloor \frac{m-n}{2} \rfloor}} \bmod 2^m \\
&= 5^{\left(A_1 5^{v_0 k} h(5^{v_0})^{-1} k_1 R_{\lfloor \frac{m-n}{2} \rfloor} R^{-1}_{\lfloor \frac{m+n}{2} \rfloor} + \frac{1}{2}(k_1 - 1) 2^{\lfloor \frac{m-n}{2} \rfloor}\right) 2^{\lfloor \frac{m+n}{2} \rfloor - 2}}.
\end{aligned}$$

Hence, setting

$$c_3 + c_2 A_1 5^{v_0 k} h(5^{v_0})^{-1} k_1 R_{\lceil \frac{m-n}{2} \rceil} R^{-1}_{\lceil \frac{m+n}{2} \rceil} = \lambda 2^{\lfloor \frac{m-n}{2} \rfloor}$$

(only the parity of λ will be used) and recalling that c_2 is odd, we have

$$\begin{aligned}\chi_1(5^{2^{\lfloor \frac{m-n}{2} \rfloor - t-2}})\chi_2(\xi) &= e_{2^{\lceil \frac{m-n}{2} \rceil}}\big(c_3 + c_2 A_1 5^{v_0 k} h(5^{v_0})^{-1} k_1 R_{\lfloor \frac{m-n}{2} \rfloor} R^{-1}_{\lfloor \frac{m+n}{2} \rfloor}\big)(-1)^{\frac{1}{2}(k_1-1)c_2} \\ &= e_{2^{\lceil \frac{m-n}{2} \rceil}}\big(c_2 A_1 5^{v_0 k} h(5^{v_0})^{-1} k_1 (R_{\lfloor \frac{m-n}{2} \rfloor} R^{-1}_{\lfloor \frac{m+n}{2} \rfloor} - R_{\lceil \frac{m-n}{2} \rceil} R^{-1}_{\lceil \frac{m+n}{2} \rceil})\big)(-1)^{\frac{1}{2}(k_1-1)+\lambda}.\end{aligned}$$

Since $1 + R_{i+1} 2^{i+1} = (1 + R_i 2^i)^2$, we have $R_{i+1} = R_i + 2^{i-1} R_i^2 \equiv R_i + 2^{i-1} \bmod 2^{i+2}$, giving $R_i \equiv 3 \bmod 4$ for $i \geq 3$, and

$$\begin{aligned}R_{\lfloor \frac{m-n}{2} \rfloor} R^{-1}_{\lfloor \frac{m+n}{2} \rfloor} - R_{\lceil \frac{m-n}{2} \rceil} R^{-1}_{\lceil \frac{m+n}{2} \rceil} &\equiv R^{-1}_{\lfloor \frac{m+n}{2} \rfloor} R^{-1}_{\lceil \frac{m+n}{2} \rceil}\big((R_{\lceil \frac{m-n}{2} \rceil} - 2^{\lceil \frac{m-n}{2} \rceil - 2}) R_{\lceil \frac{m+n}{2} \rceil} - \\ &\qquad R_{\lceil \frac{m-n}{2} \rceil}(R_{\lceil \frac{m+n}{2} \rceil} - 2^{\lceil \frac{m-n}{2} \rceil + n-2})\big) \bmod 2^{\lceil \frac{m-n}{2} \rceil} \\ &\equiv (1 - 2^n) 2^{\lceil \frac{m-n}{2} \rceil - 2} \bmod 2^{\lceil \frac{m-n}{2} \rceil}.\end{aligned}$$

From (6) we have $c_2 A_1 5^{v_0 k} h(5^{v_0})^{-1} k_1 \equiv -c_3 \bmod 4$ and

$$S(A) = 2^{\lfloor \frac{m+n}{2} \rfloor + t} \chi_1(5^{v_0}) \chi_2(h(5^{v_0}))\big(1 + i^{(2^n-1)c_3}(-1)^{\frac{1}{2}(k_1-1)+\lambda}\big).$$

The result follows on writing $\frac{1+i^h}{\sqrt{2}} = \omega^h (\frac{2}{h})$.

Small m values: $t + 2 \leq m - n \leq 2t + 4$

Since $n + 2(t+2) \geq m$, we have

$$\begin{aligned}A 5^{\gamma k} + B &= A_1 2^n (1 + R_{t+2} 2^{t+2})^{\gamma k_1} + B \\ &\equiv (A+B)\left(1 + \gamma k_1 A_1 R_{t+2} (A+B)^{-1} 2^{t+n+2}\right) \bmod 2^m \\ &\equiv (A+B)\left(1 + R_{t+n+2} 2^{t+n+2}\right)^{\gamma k_1 A_1 (A+B)^{-1} R_{t+2} R^{-1}_{t+n+2}} \bmod 2^m \\ &= (A+B) 5^{\gamma A k (A+B)^{-1} R_{t+2} R^{-1}_{t+n+2}}.\end{aligned}$$

Hence $\chi_1(5^\gamma)\chi_2(A 5^{\gamma k} + B)$ equals

$$\chi_2(A+B) e_{2^{m-2}}\left(\gamma\left(c_1(A+B) + c_2 A k R_{t+2} R^{-1}_{t+n+2}\right)(A+B)^{-1}\right)$$

and $S(A) = 2^{m-2}\chi_2(A+B)$ if $C(1) \equiv 0 \bmod 2^{m-2}$ and 0 otherwise. Since $m - n \geq t+2$, the congruence $C(1) \equiv 0 \bmod 2^{m-2}$ implies $c_1 = 2^{t+n} c_3$ (with c_3 odd if $m - n > t+2$) and becomes

$$c_3(A+B) + c_2 A_1 k_1 R_{t+2} R^{-1}_{t+n+2} \equiv 0 \bmod 2^{m-n-t-2}. \tag{9}$$

For $m - n = t+2$ or $t+3$ this will automatically hold (for both A and $-A$ when k is odd) and $S = 2^{m-1}\chi_2(A+B)$ for k even and $\chi_1(-1) = 1$, and

$$S = 2^{m-2}(\chi_2(A+B) + \chi_1(-1)\chi_2(-A+B))$$

for k odd. Further for k odd and $m - n = 2$ we have $-A + B \equiv (1 + 2^{m-1})(A+B) \bmod 2^m$ with $\chi_2(1 + 2^{m-1}) = -1$ and $S = 2^{m-2}\chi_2(A+B)(1 - \chi_1(-1)) = 2^{m-1}\chi_2(A+B)$ if $\chi_1(-1) = -1$ and zero otherwise. Note when $m - n = t+2$, we have $c_1 = 2^{m-2}$ and $\chi_1(5) = 1$ and when $m - n = t+3$, we have $c_1 = 2^{m-2}$ or 2^{m-3} and $\chi_1(5) = \pm 1$.

Since c_3B is odd, (9) cannot hold for both A and $-A$ for $m-n>t+3$ and at most one of $S(A)$ or $S(-A)$ is non-zero. When k is odd, the congruence condition for $-A$ becomes $C(-1)\equiv 0$ mod 2^{m-2}.

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