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On Jeśmanowicz' Conjecture Concerning Pythagorean Triples

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Abstract Let (a, b, c) be a primitive Pythagorean triple. Jeśmanowicz conjectured in 1956 that for any positive integer n, the Diophantine equation $(an)^x + (bn)^y = (cn)^z$ has only the positive integer solution $(x, y, z) = (2, 2, 2)$. Let $p \equiv 3 \pmod 4$ be a prime and s be some positive integer. In the paper, we show that the conjecture is true when $(a, b, c) = (4p^{2s} - 1, 4p^s, 4p^{2s} + 1)$ and certain divisibility conditions are satisfied.

Keywords Jeśmanowicz' conjecture; Diophantine equation

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1. Introduction

Let a, b, c be a primitive Pythagorean triple, that is, a, b, c are relatively prime positive integers such that $a^2 + b^2 = c^2$. In 1956, Jeśmanowicz [2] conjectured that for any positive integer n, the Diophantine equation

$$(an)^x + (bn)^y = (cn)^z \tag{1}$$

has no positive integer solutions other than $(x, y, z) = (2, 2, 2)$.

Whether there are other solutions has been investigated by many authors. Sierpiński [7] showed that Eq. (1) has no other positive integer solutions when $n = 1$ and $(a, b, c) = (3, 4, 5)$. Jeśmanowicz [2] further proved the same conclusion for $n = 1$ and $(a, b, c) = (5, 12, 13)$, $(7, 24, 25)$, $(9, 40, 41)$, $(11, 60, 61)$. For any positive integer k, Lu [4] showed that Eq.(1) has only the positive integer solution $(x, y, z) = (2, 2, 2)$ if $n = 1$ and $(a, b, c) = (4k^2 - 1, 4k, 4k^2 + 1)$. Recently, Tang and Weng [9] proved that for any positive integer m, if $c = 2^{2^m} + 1$ is a Fermat number, then Eq. (1) has only the positive integer solution $(x, y, z) = (2, 2, 2)$. Deng [1] showed that if $k = 2^s$ for some positive integer s and certain divisibility conditions are satisfied, then Jeśmanowicz' conjecture is true. For related problems, we refer to [3,5,6,8,10].

For any positive integer N with $N > 1$, let $P(N)$ denote the product of distinct prime factors of N and let $(\frac{*}{*})$ denote the Legendre symbol. If q^e is a prime power, we write $p^e \| N$ to mean that $p^e | N$ while $p^{e+1} \nmid N$. Let $p \equiv 3 \pmod 4$ be a prime and s be some positive integer.

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In this paper, we consider the case $k=p^s$ and the following results will be proved.

Theorem 1.1 *Let $p\equiv 3 \pmod 4$ be a prime and s be some positive integer. Let $(a,b,c)=(4p^{2s}-1,4p^s,4p^{2s}+1)$ be a primitive Pythagorean triple. Suppose that the positive integer n is such that either $P(a)|n$ or $P(n)\nmid a$. Then Eq. (1) has only the positive integer solution $(x,y,z)=(2,2,2)$.*

Corollary 1.2 *Let s be some positive integer and $(a,b,c)=(4\cdot 3^{2s}-1,4\cdot 3^s,4\cdot 3^{2s}+1)$ be a primitive Pythagorean triple. If a has only two distinct prime divisors, then for any positive integer n, Eq.(1) has only the positive integer solution $(x,y,z)=(2,2,2)$.*

2. Proofs

We shall begin with the following two Lemmas.

Lemma 2.1 ([4, Theorem]) *Let $(a,b,c)=(4k^2-1,4k,4k^2+1)$ and $n=1$. Then Eq. (1) has only the positive integer solution $(x,y,z)=(2,2,2)$.*

Lemma 2.2 ([1, Corollary]) *Let (a,b,c) be any primitive Pythagorean triple such that the Diophantine equation $a^x+b^y=c^z$ has only the positive integer solution $(x,y,z)=(2,2,2)$. If (x,y,z) is a solution of Eq.(1) with $(x,y,z)\neq(2,2,2)$, then one of the following conditions is satisfied:*

(i) $x>z>y$ and $P(n)|b$;

(ii) $y>z>x$ and $P(n)|a$.

Proof of Theorem 1.1 By Lemma 2.1, we may suppose that $n\geqslant 2$. We also suppose that (x,y,z) is a solution of Eq. (1) with $(x,y,z)\neq(2,2,2)$.

Case 1 $P(n)\nmid a$. By Lemma 2.2, we have $x>z>y$ and $P(n)|b$. Then

$$a^x n^{x-y}+b^y=c^z n^{z-y}.$$

Since $b=2^2p^s$, we can write $n=2^\alpha p^\beta$ with $\alpha+\beta\geqslant 1$. Thus

$$2^{\alpha(x-y)}p^{\beta(x-y)}a^x+2^{2y}p^{sy}=2^{\alpha(z-y)}p^{\beta(z-y)}c^z. \tag{2}$$

Subcase 1.1 $\alpha=0,\beta\geqslant 1$. Then

$$p^{\beta(x-y)}a^x+2^{2y}p^{sy}=p^{\beta(z-y)}c^z.$$

Since $\beta(x-y)>\beta(z-y)$, we have $sy=\beta(z-y)$. Thus

$$p^{\beta(x-z)}a^x=c^z-2^{2y}. \tag{3}$$

If $p=3$, then $(-1)^{x+\beta(x-z)}\equiv 1 \pmod 4$, and then $x+\beta(x-z)\equiv 0 \pmod 2$. Thus taking modulo 8 in (3), we have

$$5^z\equiv 2^{2y}+1 \pmod 8 \tag{4}$$

If $y=1$, then $z\equiv 1 \pmod 2$ and $3^{\beta(x-z)}a^x=c^z-4$. Since $3\|c^z-4$ and $3\nmid a$, we have

$\beta(x-z)=1$, thus x is even, $x+\beta(x-z)\equiv 1 \pmod 2$, a contradiction. Thus $y\geqslant 2$. By (4) we have $5^z\equiv 1 \pmod 8$ and $z\equiv 0 \pmod 2$.

If $p>3$, then $a=4p^{2s}-1\equiv 0 \pmod 3, c=4p^{2s}+1\equiv 2 \pmod 3$. Thus $2^z\equiv 1 \pmod 3$ and $z\equiv 0 \pmod 2$.

Now we can write $z=2z_1$. By (3),

$$p^{\beta(x-z)}(2p^s+1)^x(2p^s-1)^x=p^{\beta(x-z)}a^x=(c^{z_1}-2^y)(c^{z_1}+2^y).$$

Noting that $\gcd(c^{z_1}-2^y,c^{z_1}+2^y)=1$, we can write $a=a_1a_2$ with $\gcd(a_1,a_2)=1, a_1^x|c^{z_1}+2^y$ and $a_2^x|c^{z_1}-2^y$. Then either $a_1\geqslant 2p^s+1$ or $a_2\geqslant 2p^s+1$. Otherwise, if $a_1<2p^s+1$ and $a_2<2p^s+1$, then $a_1\leqslant 2p^s-1$ and $a_2\leqslant 2p^s-1$ by $a_1a_2=4p^{2s}-1$. Thus

$$a=a_1a_2\leqslant(2p^s-1)^2<(2p^s-1)(2p^s+1)=a,$$

which is impossible. If $a_1\geqslant 2p^s+1$, then

$$a_1^2\geqslant(2p^s+1)^2=c+4p^s>c+4,$$

thus

$$a_1^x>(a_1^2)^{z_1}>(c+4)^{z_1}\geqslant c^{z_1}+4^{z_1}>c^{z_1}+2^y,$$

but $a_1^x|c^{z_1}+2^y$, a contradiction. If $a_2\geqslant 2p^s+1$, we similarly get $a_2^x>c^{z_1}+2^y>c^{z_1}-2^y$, but $a_2^x|c^{z_1}-2^y$, a contradiction.

Subcase 1.2 $\alpha\geqslant 1,\beta=0$. Then by (2) we have

$$2^{\alpha(x-y)}a^x+2^{2y}p^{sy}=2^{\alpha(z-y)}c^z.$$

Since $\alpha(x-y)>\alpha(z-y)$, we have $2y=\alpha(z-y)$. Thus

$$2^{\alpha(x-z)}a^x=c^z-p^{sy}. \tag{5}$$

If $p=3$, then $(-1)^{x+\alpha(x-z)}\equiv 1 \pmod 3$, thus $x+\alpha(x-z)\equiv 0 \pmod 2$. Moreover, $(-1)^x2^{\alpha(x-z)}\equiv 1-3^{sy} \pmod 4$ and $(-1)^x2^{\alpha(x-z)}\equiv 1-3^{sy} \pmod 9$. If $sy\equiv 1 \pmod 2$, then $1-3^{sy}\equiv 2 \pmod 4$, thus $\alpha(x-z)=1, x\equiv 1 \pmod 2$. It follows that $\alpha=1, z\equiv 0 \pmod 2, y\equiv 1 \pmod 2$, but $2y=\alpha(z-y)$, a contradiction. Then $sy\equiv 0 \pmod 2$ and $(-1)^x2^{\alpha(x-z)}\equiv 1 \pmod 9$. Hence $\alpha(x-z)\geqslant 3$. Further, $5^z\equiv 1 \pmod 8$, thus $z\equiv 0 \pmod 2$.

If $p>3$, then $(-1)^x2^{\alpha(x-z)}\equiv 1 \pmod p$. Since $p\equiv 3 \pmod 4$ and $p>3$, we have $\alpha(x-z)\geqslant 3$. Moreover, $(-1)^{sy}\equiv 1 \pmod 4$ and $5^z\equiv 1 \pmod 8$, thus $sy\equiv 0 \pmod 2, z\equiv 0 \pmod 2$.

Now we can write $z=2z_1, sy=2y_1$. By (5),

$$2^{\alpha(x-z)}a^x=(c^{z_1}-p^{y_1})(c^{z_1}+p^{y_1}).$$

Let $a=a_1a_2$ with $\gcd(a_1,a_2)=1, a_1^x|c^{z_1}+p^{y_1}$ and $a_2^x|c^{z_1}-p^{y_1}$. Then either $a_1\geqslant 2p^s+1$ or $a_2\geqslant 2p^s+1$. Otherwise, if $a_1\leqslant 2p^s-1$ and $a_2\leqslant 2p^s-1$, then

$$a=a_1a_2\leqslant(2p^s-1)^2<(2p^s-1)(2p^s+1)=a,$$

which is impossible. If $a_1 \geqslant 2p^s+1$, then

$$a_1^2 \geqslant (2p^s+1)^2 = c+4p^s > c+p^s.$$

Further, we have

$$a_1^x > (a_1^2)^{z_1} > (c+p^s)^{z_1} \geqslant c^{z_1}+p^{sz_1} > c^{z_1}+p^{y_1},$$

but $a_1^x|c^{z_1}+p^{y_1}$, a contradiction. If $a_2 \geqslant 2p^s+1$, we similarly get $a_2^x > c^{z_1}+p^{y_1} > c^{z_1}-p^{y_1}$, but $a_2^x|c^{z_1}-p^{y_1}$, a contradiction.

Subcase 1.3 $\alpha \geqslant 1, \beta \geqslant 1$. Then $\alpha(x-y) > \alpha(z-y), \beta(x-y) > \beta(z-y)$, so $2y = \alpha(z-y), sy = \beta(z-y)$, thus

$$2^{\alpha(x-z)}p^{\beta(x-z)}a^x = c^z-1. \tag{6}$$

Since $4|c^z-1$ and $p^{2s}|c^z-1$, we have $\alpha(x-z) \geqslant 2$ and $\beta(x-z) \geqslant 2s$. Then $\alpha(x-z) \geqslant 3$. Otherwise, if $\alpha(x-z)=2$, then $\alpha=1, x-z=2$ or $\alpha=2, x-z=1$. By $2y=\alpha(z-y)$, we have $z=3y$ or $z=2y$. By $sy=\beta(z-y)$, we have $s=2\beta$ or $s=\beta$, but $\beta(x-z) \geqslant 2s$, it is impossible.

Taking modulo 8 in (6), we have $5^z \equiv 1 \pmod 8$, thus $z \equiv 0 \pmod 2$. Write $z=2z_1$, we have

$$2^{\alpha(x-z)}p^{\beta(x-z)}a^x = (c^{z_1}-1)(c^{z_1}+1).$$

Let $a=a_1a_2$ with $\gcd(a_1,a_2)=1, a_1^x|c^{z_1}+1$ and $a_2^x|c^{z_1}-1$. Then either $a_1 \geqslant 2p^s+1$ or $a_2 \geqslant 2p^s+1$. Otherwise, if $a_1 \leqslant 2p^s-1$ and $a_2 \leqslant 2p^s-1$, then

$$a = a_1a_2 \leqslant (2p^s-1)^2 < (2p^s-1)(2p^s+1) = a,$$

which is impossible. If $a_1 \geqslant 2p^s+1$, then

$$a_1^2 \geqslant (2p^s+1)^2 = c+4p^s > c+1.$$

Further, we have

$$a_1^x > (a_1^2)^{z_1} > (c+1)^{z_1} \geqslant c^{z_1}+1,$$

but $a_1^x|c^{z_1}+1$, a contradiction. If $a_2 \geqslant 2p^s+1$, we similarly get $a_2^x > c^{z_1}+1 > c^{z_1}-1$, but $a_2^x|c^{z_1}-1$, a contradiction.

Case 2 $P(a)|n$. By Lemma 2.2, we have $y>z>x$. Then

$$a^x+b^yn^{y-x} = c^zn^{z-x}.$$

Since $y-x > z-x > 0$, we have $P(n)|a$ and $n^{z-x}|a^x$, which means that $P(n)=P(a)$ and $n^{z-x}=a^x$. Thus

$$b^yn^{y-z} = c^z-1. \tag{7}$$

It follows that $5^z \equiv 1 \pmod 8$, so $z \equiv 0 \pmod 2$. Write $z=2z_1$. Since $c \equiv 1 \pmod b$, $c^{z_1}+1 \equiv 2 \pmod b$, we have $\gcd(c^{z_1}+1, b)=2$. Then by (7), we have $\frac{b^y}{2}|c^{z_1}-1$. But

$$\frac{b^y}{2} > \frac{b^{2z_1}}{2} = \frac{(c-a)^{z_1}(c+a)^{z_1}}{2} \geqslant c^{z_1}+a^{z_1} > c^{z_1}-1,$$

which is impossible.

This completes the proof of Theorem 1.1 □

Proof of Corollary 1.2 By Lemma 2.1, we may suppose that $n \geqslant 2$ and suppose that (x, y, z) is a solution of (1) with $(x, y, z) \neq (2, 2, 2)$. By Case 2 of Theorem 1.1 and Lemma 2.2, we may suppose that $y > z > x, P(n)|a$ and $P(n) < P(a)$. Since $a = 4 \cdot 3^{2s} - 1$ has only two distinct prime divisors and $\gcd(2 \cdot 3^s - 1, 2 \cdot 3^s + 1) = 1$, we can write $2 \cdot 3^s - 1 = q_1^{\alpha_1}, 2 \cdot 3^s + 1 = q_2^{\alpha_2}$, where q_1 and q_2 are distinct odd primes and $\alpha_1, \alpha_2 \geqslant 1$. Then we have either $n = q_1^{\alpha}$ or $n = q_2^{\beta}$ with $\alpha, \beta \geqslant 1$.

If $n = q_1^{\alpha}$, then we have

$$q_1^{\alpha_1 x} q_2^{\alpha_2 x} + (4 \cdot 3^s)^y q_1^{\alpha(y-x)} = (4 \cdot 3^{2s} + 1)^z q_1^{\alpha(z-x)}.$$

Since $\alpha(y - x) > \alpha(z - x)$, we have $\alpha_1 x = \alpha(z - x)$ and

$$(4 \cdot 3^s)^y q_1^{\alpha(y-z)} = (4 \cdot 3^{2s} + 1)^z - q_2^{\alpha_2 x} = (4 \cdot 3^{2s} + 1)^z - (2 \cdot 3^s + 1)^x. \tag{8}$$

If s is even, then $3^x \equiv 1 \pmod 4$ and $5^z \equiv 3^x \pmod 8$, thus x and z are both even. If s is odd, then $5^z \equiv 7^x \pmod 8$, thus x and z are both even. Now write $z = 2z_1, x = 2x_1$. By (8),

$$(4 \cdot 3^s)^y q_1^{\alpha(y-z)} = \big((4 \cdot 3^{2s} + 1)^{z_1} - (2 \cdot 3^s + 1)^{x_1}\big)\big((4 \cdot 3^{2s} + 1)^{z_1} + (2 \cdot 3^s + 1)^{x_1}\big). \tag{9}$$

Noting that $\gcd((4 \cdot 3^{2s} + 1)^{z_1} - (2 \cdot 3^s + 1)^{x_1}, (4 \cdot 3^{2s} + 1)^{z_1} + (2 \cdot 3^s + 1)^{x_1}) = 2$ and

$$\begin{aligned}\frac{(4 \cdot 3^s)^y}{2} &= \frac{b^y}{2} > \frac{b^z}{2} = \frac{b^{2z_1}}{2} = \frac{2^{z_1}(c + a)^{z_1}}{2} \geqslant (c + a)^{z_1} \geqslant c^{z_1} + a^{z_1} \\ &> (4 \cdot 3^{2s} + 1)^{z_1} + (2 \cdot 3^s + 1)^{x_1} > (4 \cdot 3^{2s} + 1)^{z_1} - (2 \cdot 3^s + 1)^{x_1},\end{aligned}$$

we deduce that (9) cannot hold.

If $n = q_2^{\beta}$, then we have

$$q_1^{\alpha_1 x} q_2^{\alpha_2 x} + (4 \cdot 3^s)^y q_2^{\beta(y-x)} = (4 \cdot 3^{2s} + 1)^z q_2^{\beta(z-x)}.$$

Since $\beta(y - x) > \beta(z - x)$, we have $\alpha_2 x = \beta(z - x)$ and

$$(4 \cdot 3^s)^y q_2^{\beta(y-z)} = (4 \cdot 3^{2s} + 1)^z - q_1^{\alpha_1 x} = (4 \cdot 3^{2s} + 1)^z - (2 \cdot 3^s - 1)^x. \tag{10}$$

It follows that x is even since $(-1)^x \equiv 1 \pmod 3$ and z is even since $5^z \equiv 1 \pmod 8$. Now write $z = 2z_1, x = 2x_1$. By (10),

$$(4 \cdot 3^s)^y q_2^{\beta(y-z)} = \big((4 \cdot 3^{2s} + 1)^{z_1} - (2 \cdot 3^s - 1)^{x_1}\big)\big((4 \cdot 3^{2s} + 1)^{z_1} + (2 \cdot 3^s - 1)^{x_1}\big). \tag{11}$$

Noting that $\gcd((4 \cdot 3^{2s} + 1)^{z_1} - (2 \cdot 3^s - 1)^{x_1}, (4 \cdot 3^{2s} + 1)^{z_1} + (2 \cdot 3^s - 1)^{x_1}) = 2$ and

$$\frac{b^y}{2} > \frac{b^{2z_1}}{2} \geqslant (c + a)^{z_1} \geqslant c^{z_1} + a^{z_1} > (4 \cdot 3^{2s} + 1)^{z_1} + (2 \cdot 3^s - 1)^{x_1} > (4 \cdot 3^{2s} + 1)^{z_1} - (2 \cdot 3^s - 1)^{x_1},$$

we deduce that (11) cannot hold.

This completes the proof of Corollary 1.2 □

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