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Maximal Graded Subalgebras of the General Linear Lie Superalgebras over Superrings

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Abstract In this paper, we determine all maximal graded subalgebras of the general linear Lie superalgebras containing the standard Cartan subalgebras over a unital supercommutative superring with 2 invertible.

Keywords maximal graded subalgebras; general linear Lie superalgebras; supercommutative superrings

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1. Introduction

In 1952, Dynkin [1,2] classified the semisimple subalgebras and the maximal subalgebras of the finite-dimensional simple Lie algebras over the field of complex numbers. In 2003, Shechepochkina [3] also investigated the maximal subalgebras of linear Lie superalgebras over the field of complex numbers. In 2004, Elduque, Laliena and Saristan [4,5] described the maximal subalgebras of associative superalgebras and Jordan superalgebras. In 2012, Wang, Ge and Li determined the maximal subalgebras containing the standard Cartan subalgebras for the general linear Lie algebra over a commutative ring. In this paper, we determine all maximal graded subalgebras containing the standard Cartan subalgebras for the general linear Lie superalgebras over a supercommutative superring and the corresponding results for Lie algebras in [6] are covered.

2. Basics

Let us recall certain basic concepts and facts with respect to supercommutative superrings. A (associative) ring R is called a $\mathbb{Z}_2$-graded ring or superring if $R = R_{\bar{0}} \oplus R_{\bar{1}}$ for some additive subgroups $R_{\bar{0}}$ and $R_{\bar{1}}$ of R such that $R_\alpha R_\beta \subseteq R_{\alpha+\beta}$ for $\alpha, \beta \in \mathbb{Z}_2$. The elements in $R_{\bar{0}}$ ($R_{\bar{1}}$) are called even (odd) and both even and odd elements are called homogenous. For a homogenous element $r \in R_\alpha, \alpha \in \mathbb{Z}_2$, its $\mathbb{Z}_2$-degree is denoted by $|r|$. We adopt the convention that if $|r|$ occurs in a formula, the corresponding element r is assumed to be homogeneous. A superring R

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is called supercommutative provided that

$$ab = (-1)^{|a||b|}ba \quad \text{for all} \quad a, b \in R.$$

Note that if $1 \in R$, then $1 \in R_{\bar{0}}$; if 2 is invertible, then $a_{\bar{1}}^2 = 0$ for all $a_{\bar{1}} \in R_{\bar{1}}$.

An ideal I of a superring R is called a $\mathbb{Z}_2$-graded ideal (or graded ideal for short) provided that

$$I = I \cap R_{\bar{0}} \oplus I \cap R_{\bar{1}}.$$

Obviously, $\{0\}$ and R are graded ideals of R, which are called trivial.

A graded ideal I of a superring R is called a maximal graded ideal, if there exists no nontrivial graded ideals containing I strictly.

A superring R is called a divisible superring, if $1 \in R$ and all nonzero elements of R are invertible. In view of the remarks above, we have

Proposition 2.1 *A supercommutative divisible superring is a field.*

From now on, assume that R is a unital supercommutative superring with 2 invertible.

A left R-module M is called a left R-supermodule, if $M = M_{\bar{0}} \oplus M_{\bar{1}}$ for some additive subgroups $M_{\bar{0}}$ and $M_{\bar{1}}$ such that

$$R_\alpha M_\beta \subseteq M_{\alpha+\beta} \quad \text{for all} \quad \alpha, \beta \in \mathbb{Z}_2.$$

If we define

$$xr = (-1)^{|r||x|} rx \quad \text{for all} \quad x \in M, r \in R,$$

then M is also a right R-supermodule.

A submodule N of R-supermodule $M = M_{\bar{0}} \oplus M_{\bar{1}}$ is called an R-subsupermodule ($\mathbb{Z}_2$-graded submodule) if

$$N = N \cap M_{\bar{0}} \oplus N \cap M_{\bar{1}}.$$

Let M, N, P be R-supermodules. A map $\varphi : M \times N \to P$ is called an R-bilinear map of degree γ if

(1) $\varphi(M_\alpha, N_\beta) \subseteq P_{\alpha+\beta+\gamma}$;

(2) $\varphi(x_1 + x_2, y) = \varphi(x_1, y) + \varphi(x_2, y)$;

(3) $\varphi(x, y_1 + y_2) = \varphi(x, y_1) + \varphi(x, y_2)$;

(4) $\varphi(xr, y) = \varphi(x, ry), \varphi(x, yr) = \varphi(x, y)r$,

for all $x, x_1, x_2 \in M, y, y_1, y_2 \in N, r \in R, \alpha, \beta, \gamma \in \mathbb{Z}_2$. From the definition, we have

$$\varphi(rx, y) = (-1)^{|r||\varphi|} r\varphi(x, y) \quad \text{for all} \quad r \in R, x, y \in M.$$

An R-superalgebra is an R-supermodule $\mathfrak{A} = \mathfrak{A}_{\bar{0}} + \mathfrak{A}_{\bar{1}}$ with an even R-bilinear map $\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$. A subalgebra $\mathfrak{B}$ of a superalgebra $\mathfrak{A}$ is called a $\mathbb{Z}_2$-graded subalgebra (graded subalgebra), if $\mathfrak{B}$ is also a $\mathbb{Z}_2$-graded submodule of $\mathfrak{A}$.

Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be an R-superalgebra with multiplication $[-, -]$. Then L is called a Lie superalgebra over R or R-Lie superalgebra if

(L1) $[x, y] = -(-1)^{|x||y|}[y, x]$,

(L2) $(-1)^{|z||x|}[x,[y,z]] + (-1)^{|x||y|}[y,[z,x]] + (-1)^{|y||z|}[z,[x,y]] = 0$,
for all $x, y, z \in L$.

3. Construction of maximal subalgebras

Suppose m and n are positive integers and let $\mathfrak{gl}_R(m,n)$ denote the R-supermodule consisting of all $(m+n)\times(m+n)$ matrices over R. Then

$$\mathfrak{gl}_R(m,n) = \mathfrak{gl}_R(m,n)_{\bar 0} \oplus \mathfrak{gl}_R(m,n)_{\bar 1},$$

where $\mathfrak{gl}_R(m,n)_{\bar 0}$ consists of all the even matrices

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$$

with

$$A_1 \in M_m(R_{\bar 0}), B_1 \in M_{m,n}(R_{\bar 1}), C_1 \in M_{n,m}(R_{\bar 1}), D_1 \in M_n(R_{\bar 0});$$

and $\mathfrak{gl}_R(m,n)_{\bar 1}$ consists of all the odd matrices

$$\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$$

with

$$A_2 \in M_m(R_{\bar 1}), B_2 \in M_{m,n}(R_{\bar 0}), C_2 \in M_{n,m}(R_{\bar 0}), D_2 \in M_n(R_{\bar 1}).$$

Then $\mathfrak{gl}_R(m,n)$ is an R-Lie superalgebra with respect to the following bracket:

$$[x,y] = xy - (-1)^{|x||y|}yx, \quad x, y \in \mathfrak{gl}_R(m,n).$$

Let e_{ij} be the $(m+n)\times(m+n)$ matrix having 1 in the (i,j) position and 0 elsewhere. Then

$$e_{ij}, e_{kl}] = \delta_{jk}e_{il} - (-1)^{|e_{ij}||e_{kl}|}\delta_{li}e_{kj}.$$

Obviously, all diagonal matrices of $\mathfrak{gl}_R(m,n)$ constitute a graded subalgebra, denoted by $\mathfrak{d}_R(m,n)$. All upper triangular matrices of $\mathfrak{gl}_R(m,n)$ also constitute a graded subalgebra, denoted by $\mathfrak{b}_R(m,n)$.

Suppose that X is a graded subalgebra of $\mathfrak{gl}_R(m,n)$ containing $\mathfrak{d}_R(m,n)$. Then

$$X = X_{\bar 0} \oplus X_{\bar 1} = X \cap \mathfrak{gl}_R(m,n)_{\bar 0} \oplus X \cap \mathfrak{gl}_R(m,n)_{\bar 1}.$$

For X and $1 \le i, j \le m+n$, define

$$A_{ij}^X = \{a \in R \mid ae_{ij} \in X\}.$$

Similarly to [7, Lemma 2.1], we have

Lemma 3.1 (1) *If $x = (x_{ij}) \in X$, then $x_{ij} \in A_{ij}^X$. In particular, $X = \sum_{1\le i,j\le m+n} A_{ij}^X e_{ij}$.*

(2) All A_{ij}^X are graded ideals of R, and

$$A_{ii}^X = R, \quad A_{ij}^X A_{jk}^X \subseteq A_{ik}^X,$$

for $i, j, k = 1, \ldots, m+n$.

Proof It is enough to show that all A_{ij}^X are graded ideals of R and one may finish the proof as in [7, Lemma 2.1]. When $i \neq j$, for $a, b \in A_{ij}^X$ and $r \in R$, we have $ae_{ij}, be_{ij} \in X$. It follows that $ae_{ij} - be_{ij} = (a-b)e_{ij} \in X$ and then $a - b \in A_{ij}^X$. Since $rae_{ij} = [re_{ii}, ae_{ij}] \in X$, we have $ra \in A_{ij}^X$ and $ar = (-1)^{|a||r|}ra \in A_{ij}^X$. So A_{ij}^X is an ideal of R. Let $a \in A_{ij}^X$ and write $a = a_{\bar{0}} + a_{\bar{1}}$, where $a_{\bar{0}} \in R_{\bar{0}}, a_{\bar{1}} \in R_{\bar{1}}$. Then

$$ae_{ij} = a_{\bar{0}}e_{ij} + a_{\bar{1}}e_{ij} \in X.$$

Since X is a graded subalgebra, we have $a_{\bar{0}}e_{ij}, a_{\bar{1}}e_{ij} \in X$ and then $a_{\bar{0}}, a_{\bar{1}} \in A_{ij}^X$. Hence, all A_{ij}^X are graded ideals of R. $\square$

Write

$$\Phi = \{1, 2, \ldots, m \mid m+1, \ldots, m+n\}, \Phi_1 = \{1, 2, \ldots, m\},$$

and

$$\Phi_2 = \{m+1, \ldots, m+n\}.$$

Write $\Phi_{ij} = \Phi_i \times \Phi_j$, where $i, j = 1, 2$. Let $\varnothing \neq \triangle \subsetneq \Phi$ and $\triangle^-$ be the complementary set of $\triangle$ in Φ, i.e., $\triangle \cap \triangle^- = \varnothing$, $\triangle \cup \triangle^- = \Phi$. Suppose that I is a maximal graded ideal of R and define

$$M(I, \triangle) = \sum_{(i,j)\in\triangle^-\times\triangle} Ie_{ij} + \sum_{(i,j)\in\Phi\times\Phi\setminus\triangle^-\times\triangle} Re_{ij}.$$

It is easy to show that $\mathfrak{d}_R(m,n) \subseteq M(I, \triangle)$ and that $M(I, \triangle)$ is an R-module.

Lemma 3.2 *$M(I, \triangle)$ is a maximal graded subalgebra of $\mathfrak{gl}_R(m,n)$ containing $\mathfrak{d}_R(m,n)$.*

Proof Firstly, we prove that it is an R-subsupermodule, i.e.,

$$M(I, \triangle) = M(I, \triangle) \cap \mathfrak{gl}_R(m,n)_{\bar{0}} \oplus M(I, \triangle) \cap \mathfrak{gl}_R(m,n)_{\bar{1}}.$$

The inclusion "$\supseteq$" is obvious. Conversely, for an arbitrary $x = \sum_{1\leq i,j\leq m+n} x_{ij}e_{ij} \in M(I, \triangle)$, write

$$x_{ij} = (x_{ij})_{\bar{0}} + (x_{ij})_{\bar{1}} \in R_{\bar{0}} \oplus R_{\bar{1}}.$$

Then

$$x = \sum_{(i,j)\in\Phi_{11}\cup\Phi_{22}} (x_{ij})_{\bar{0}}e_{ij} + \sum_{(k,l)\in\Phi_{12}\cup\Phi_{21}} (x_{kl})_{\bar{1}}e_{kl} + \sum_{(i,j)\in\Phi_{11}\cup\Phi_{22}} (x_{ij})_{\bar{1}}e_{ij} + \sum_{(k,l)\in\Phi_{12}\cup\Phi_{21}} (x_{kl})_{\bar{1}}e_{kl}.$$

Since I is a graded ideal, we get $x \in M(I, \triangle) \cap \mathfrak{gl}_R(m,n)_{\bar{0}} \oplus M(I, \triangle) \cap \mathfrak{gl}_R(m,n)_{\bar{1}}$.

Secondly, we show that it is a subalgebra. For $\sum_{1\leq i,j\leq m+n} x_{ij}e_{ij}$ and $\sum_{1\leq k,l\leq m+n} x_{kl}e_{kl}$ in $M(I, \triangle)$, we have to show that $[x_{ij}e_{ij}, x_{kl}e_{kl}] \in M(I, \triangle)$. Note that

$$[x_{ij}e_{ij}, x_{kl}e_{kl}] = \delta_{jk}x_{ij}x_{kl}e_{il} - (-1)^{(|x_{ij}|+|e_{ij}|)(|x_{kl}|+|e_{kl}|)}\delta_{li}x_{kl}x_{ij}e_{kj}.$$

When $(i,l) \in \triangle^- \times \triangle$, if $j = k \in \triangle$, then $(i,j) \in \triangle^- \times \triangle$ and $x_{ij} \in I$; if $j = k \in \triangle^-$, then $(k,l) \in \triangle^- \times \triangle$ and $x_{kl} \in I$. So, $x_{ij}x_{kl} \in I$ in the above two cases, furthermore,

$$[x_{ij}e_{ij}, x_{kl}e_{kl}] = x_{ij}x_{kl}e_{il} \in M(I, \triangle).$$

When $(k,j) \in \triangle^- \times \triangle$, if $l = i \in \triangle$, then $(k,l) \in \triangle^- \times \triangle$ and $x_{kl} \in I$; if $l = i \in \triangle^-$, then $(i,j) \in \triangle^- \times \triangle$ and $x_{ij} \in I$. So, $x_{kl}x_{ij} \in I$ in the above two cases, furthermore,

$$[x_{ij}e_{ij}, x_{kl}e_{kl}] = -(-1)^{(|x_{ij}|+|e_{ij}|)(|x_{kl}|+|e_{kl}|)}\delta_{li}x_{kl}x_{ij}e_{kj} \in M(I,\triangle).$$

Hence, $[M(I,\triangle), M(I,\triangle)] \subseteq M(I,\triangle)$ and $M(I,\triangle)$ is a graded subalgebra of $\mathfrak{gl}_R(m,n)$.

Finally, let us prove the maximality. Suppose that X is a graded subalgebra of $\mathfrak{gl}_R(m,n)$ satisfying $M(I,\triangle) \subsetneq X \subseteq \mathfrak{gl}_R(m,n)$. Write $x = (x_{ij}) \in X \setminus M(I,\triangle)$. There exists some $(p,q) \in \triangle^- \times \triangle$ such that $x_{pq} \notin I$. By Lemma 3.1, we know $x_{pq}e_{pq} \in X$. Since I is a maximal graded ideal, there exist some $r \in R$ and $r_0 \in I$ such that $rx_{pq} + r_0 = 1$. Furthermore,

$$e_{pq} = [re_{pp}, x_{pq}e_{pq}] + r_0e_{pq} \in X.$$

For every $i \in \triangle^-$, it follows that $(i,p) \notin \triangle^- \times \triangle$, then $e_{ip} \in M(I,\triangle) \subseteq X$ and $e_{iq} = [e_{ip}, e_{pq}] \in X$. For every $j \in \triangle$, it follows that $(q,j) \notin \triangle^- \times \triangle$, then $e_{qj} \in M(I,\triangle) \subseteq X$ and $e_{ij} = [e_{iq}, e_{qj}] \in X$. So X contains all e_{ij}, and this implies that $X = \mathfrak{gl}_R(m,n)$. Hence, $M(I,\triangle)$ is a maximal graded subalgebra of $\mathfrak{gl}_R(m,n)$ containing $\mathfrak{d}_R(m,n)$. □

For $1 \le s \le m+n-1$, put $\triangle_s = \{1,2,\ldots,s\}$. Then $M(I,\triangle_s)$ is called a standard maximal graded subalgebras of $\mathfrak{gl}_R(m,n)$ containing $\mathfrak{d}_R(m,n)$, where $s = 1,2,\ldots,m+n-1$.

Lemma 3.3 *$M(I,\triangle)$ is a standard maximal graded subalgebras if and only if $\mathfrak{b}_R(m,n) \subseteq M(I,\triangle)$.*

Proof Suppose that $M(I,\triangle)$ is standard with $\triangle = \triangle_s = \{1,2,\ldots,s\}$, where $1 \le s \le m+n-1$. For every $1 \le i \le j \le m+n$, we have that $(i,j) \notin \triangle^- \times \triangle$, then $e_{ij} \in M(I,\triangle)$. Hence, $\mathfrak{b}_R(m,n) \subseteq M(I,\triangle)$.

Now suppose that $M(I,\triangle)$ is not standard. Write

$$\triangle = \{j_1, j_2, \ldots, j_s\} \subseteq \Phi, \quad \text{where } j_1 < j_2 < \cdots < j_s.$$

Since $M(I,\triangle)$ is not standard, there exists some $1 \le k \le s$ such that $k < j_k$. Let p be the minimal one such that $k < j_k$. It follows that $j_1 = 1, j_2 = 2, \ldots, j_{p-1} = p-1$, $j_p > p$. Then $e_{p,j_p} \in \mathfrak{b}_R(m,n) \subseteq M(I,\triangle)$, which contradicts the fact that $e_{p,j_p} \notin M(I,\triangle)$ for $p \in \triangle^-$ and $j_p \in \triangle$. □

4. Main results

We begin with the following lemma.

Lemma 4.1 *Let $\mathbb{F}$ be a field of characteristic not 2, and X be a graded subalgebra of $\mathfrak{gl}_{\mathbb{F}}(m,n)$. Then X is a maximal graded subalgebra of $\mathfrak{gl}_{\mathbb{F}}(m,n)$ containing $\mathfrak{d}_{\mathbb{F}}(m,n)$ if and only if there exists $\varnothing \ne \triangle \subsetneq \Phi$ such that $X = M(0,\triangle)$.*

Proof According to Lemma 3.2, for any $\varnothing \ne \triangle \subseteq \Phi$, $X = M(0,\triangle)$ is a maximal graded subalgebra of $\mathfrak{gl}_{\mathbb{F}}(m,n)$ containing $\mathfrak{d}_{\mathbb{F}}(m,n)$.

Now suppose that X is a maximal graded subalgebra of $\mathfrak{gl}_{\mathbb{F}}(m,n)$ containing $\mathfrak{d}_{\mathbb{F}}(m,n)$.

According to Lemma 3.1, we have

$$X=\sum_{1\le i,j\le m+n} A^X_{ij}e_{ij},$$

where the A^X_{ij} are all graded ideals of $\mathbb{F}$. Since $\mathbb{F}$ is a field, $A^X_{ij}=0$ or $A^X_{ij}=\mathbb{F}$. Note that $X\subsetneq \mathfrak{gl}_{\mathbb{F}}(m,n)$. There must exist $p\neq q$ such that $A^X_{pq}=0$. Put

$$\triangle=\{j\in\Phi\mid A^X_{pj}=0\}.$$

Since $p\notin\triangle$ and $q\in\triangle$, it follows that $\varnothing\neq\triangle\subsetneq\Phi$. For above-mentioned $\triangle$, it is easy to show that $M(0,\triangle)$ is a maximal graded subalgebra of $\mathfrak{gl}_{\mathbb{F}}(m,n)$ containing $\mathfrak{d}_{\mathbb{F}}(m,n)$. Because both X and $M(0,\triangle)$ are of maximality, we have to show that $X\subseteq M(0,\triangle)$. By the maximality of X and $M(0,\triangle)$, in order to conclude the proof, we have to show that $A^X_{ij}=0$ for all $(i,j)\in\triangle^-\times\triangle$. If there exists some $(i_0,j_0)\in\triangle^-\times\triangle$ such that $A^X_{i_0j_0}\neq 0$, then we have $A^X_{i_0j_0}=R$. Since $(p,i_0)\notin\triangle^-\times\triangle$, it follows that $A^X_{pi_0}=R$. According to Lemma 3.1, we have

$$A^X_{pi_0}A^X_{i_0j_0}\subseteq A^X_{pj_0}=R,$$

which contradicts the fact that $A^X_{pj_0}=0$ for $(p,j_0)\in\triangle^-\times\triangle$. Hence $X=M(0,\triangle)$. □

Theorem 4.2 *Let X be a graded subalgebra of $\mathfrak{gl}_R(m,n)$. Then X is a maximal graded subalgebra of $\mathfrak{gl}_R(m,n)$ containing $\mathfrak{d}_R(m,n)$ if and only if there exist $\varnothing\neq\triangle\subsetneq\Phi$ and a maximal ideal I of R such that $X=M(I,\triangle)$.*

Proof According to Lemma 3.2, the sufficiency is obvious. Now, we show the necessity. Suppose that X is a maximal graded subalgebra of $\mathfrak{gl}_R(m,n)$ containing $\mathfrak{d}_R(m,n)$. According to Lemma 3.1, we have

$$X=\sum_{1\le i,j\le m+n} A^X_{ij}e_{ij},$$

where the A^X_{ij} are all graded ideals of R. Note that $X\subsetneq\mathfrak{gl}_R(m,n)$. There must exist $p\neq q$ such that $A^X_{pq}\neq R$. Let I be a maximal ideal I of R containing A^X_{pq} and π be the canonical homomorphism from R to R/I. Define σ to be the map from $\mathfrak{gl}_R(m,n)$ to $\mathfrak{gl}_{R/I}(m,n)$ such that

$$\sigma((x_{ij}))=(\bar{x}_{ij}),\quad\text{where}\quad \bar{x}_{ij}=\pi(x_{ij}).$$

It is easy to show that σ is a surjective homomorphism of Lie superalgebras with $\ker\sigma=\mathfrak{gl}_I(m,n)$ and $\sigma(X)\subsetneq\mathfrak{gl}_{R/I}(m,n)$. Let M be a maximal graded subalgebra of $\mathfrak{gl}_{R/I}(m,n)$ satisfying $\sigma(X)\subsetneq M\subseteq\mathfrak{gl}_{R/I}(m,n)$. Then

$$X+\mathfrak{gl}_I(m,n)\subsetneq\sigma^{-1}(M)\subseteq\mathfrak{gl}_R(m,n).$$

By the maximality of X, we know that $\sigma^{-1}(M)=\mathfrak{gl}_R(m,n)$. Then $M=\mathfrak{gl}_{R/I}(m,n)$. So $\sigma(X)$ is a maximal graded subalgebra of $\mathfrak{gl}_{R/I}(m,n)$. According to Lemma 4.1, there exists $\varnothing\neq\triangle\subsetneq\Phi$ such that $\sigma(X)=M(\bar{0},\triangle)$. Then

$$X+\mathfrak{gl}_I(m,n)=M(0,\triangle)+\mathfrak{gl}_I(m,n).$$

Furthermore,

$$X \subseteq X + \mathfrak{gl}_I(m,n) = M(0,\triangle) + \mathfrak{gl}_I(m,n) = M(I,\triangle).$$

By the maximality of X, we have $X = M(I,\triangle)$. □

Note that if the number of graded ideals of R is finite, then we can determine the number of maximal graded subalgebras of $\mathfrak{gl}_R(m,n)$.

Corollary 4.3 *$\mathfrak{gl}_R(m,n)$ has $z(R)(2^{m+n}-2)$ maximal graded subalgebras containing $\mathfrak{d}_R(m,n)$, where $z(R)$ is the number of graded ideals of R.*

Proof According to Theorem 4.2, we know that the number of maximal graded subalgebras of $\mathfrak{gl}_R(m,n)$ containing $\mathfrak{d}_R(m,n)$ only depends on the number of graded ideals of R and the number of choices of nontrivial subsets of $\triangle$. The number of choices of nontrivial subsets of $\triangle$ is $\sum_{k=1}^{m+n-1} C_n^k = 2^{m+n}-2$. Thus the assertion holds. □

Finally, we show that each maximal graded subalgebra is conjugate under a permutation matrix (the matrix having the only nonzero element 1 in each row and each column) to a standard one.

Theorem 4.4 *Let X be a maximal graded subalgebra of $\mathfrak{gl}_R(m,n)$ containing $\mathfrak{d}_R(m,n)$. Then there exists a permutation matrix P such that $X = PM(I,\triangle_s)P^{-1}$.*

Proof According to Theorem 4.2, we can suppose $X = M(I,\triangle)$, where $\varnothing \neq \triangle \subsetneq \Phi$ and I is a maximal graded ideal of R. Define

$$\Psi(X) = \{(i,j) \mid 1 \leq i < j \leq m+n, e_{ij} \in X\}.$$

The number of elements in $\Psi(X)$ is denoted by $N(X)$. We use induction on $N(X)$ to prove the result.

When $N(X) = \frac{1}{2}(m+n)(m+n-1)$, we have $e_{ij} \in X$ for any $1 \leq i < j \leq m+n$. Then $\mathfrak{b}_R(m,n) \subseteq M(I,\triangle)$. According to Lemma 3.3, $M(I,\triangle)$ is standard.

Now let us prove the assertion when $N(X) = u+1$ assuming that it holds for $0 \leq u < \frac{1}{2}(m+n)(m+n-1)$.

When $N(X) = u$, there must exist some (i_0,j_0) such that $e_{i_0j_0} \notin X$, where $1 \leq i_0 < j_0 \leq m+n$. Furthermore, there must exist some $1 \leq p \leq m+n-1$ such that $e_{p,p+1} \notin X$. In fact, if $e_{i,i+1} \in X$ for any $1 \leq i \leq m+n-1$, then we have $e_{ij} \in X$ for any $1 \leq i < j \leq m+n$, which contradicts our assumption that $e_{i_0j_0} \notin X$. From $e_{p,p+1} \notin X = M(I,\triangle)$ it follows that $e_{p+1,p} \in X$. We now consider the action of the inner automorphism $\mathrm{Int}E_{p,p+1}$:

$$\mathrm{Int}E_{p,p+1}(\mathfrak{d}_R(m,n)) = \mathfrak{d}_R(m,n);$$

$$\mathrm{Int}E_{p,p+1}(X) = E_{p,p+1}^{-1} X E_{p,p+1};$$

$$\mathrm{Int}E_{p,p+1}(e_{p,p+1}) = e_{p+1,p}.$$

It is easy to show that the number of elements in $\Psi(E_{p,p+1}^{-1} X E_{p,p+1})$ is exactly $u+1$. According to the induction assumption, there exists some permutation matrix P such that $P^{-1}E_{p,p+1}^{-1} X E_{p,p+1} P$

$= M(I, \triangle_s)$. Then

$$X = E_{p,p+1} P M(I, \triangle_s) P^{-1} E_{p,p+1}^{-1}.$$

Let $Q = E_{p,p+1}P$. Then Q is a permutation matrix and

$$X = Q M(I, \triangle_s) Q^{-1},$$

where $M(I, \triangle_s)$ is standard. □

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