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Corepresentation and Categorical Realization of Dual Hom-Quasi-Hopf Algebras

Yongsheng CHENG[1,*], Guang-ai SONG[2]

1. *School of Mathematics and Statistics* & *Institute of Contemporary Mathematics, Henan University, Henan* 475004*, P. R. China;*
2. *College of Mathematics and Information Science, Shandong Institute of Business and Technology, Shandong* 264005*, P. R. China*

Abstract In this paper, we introduce the dual Hom-quasi-Hopf algebra and prove that the comodules category of a (braided) dual Hom-quasi-bialgebra is a monoidal category. Finally, we give a categorical realization of dual Hom-quasi-Hopf algebras.

Keywords monoidal category; dual Hom-quasi-Hopf algebra; Hom-comodules

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1. Introduction

Drinfeld introduced the concept of quasi-Hopf algebra in connection with the Knizhnik-Zamolodchikov system of partial differential equations in [1]. The quasi-Hopf algebra H is required to be associative as an algebra but coassociative only up to conjugation by an invertible element $\Phi \in H^{\otimes 3}$ which obeys a certain pentagon cocycle condition. The modules of a quasi-Hopf algebra form a monoidal category ${}_H\mathcal{M}$ by pulling back along the coproduct Δ as usual. If we draw our attention to the category of their corepresentations, we get the concepts of dual quasi-bialgebra and of dual quasi-Hopf algebra. Dual quasi-Hopf algebra was studied by Majid in order to prove a Tannaka-Krein type theorem, and he proved that every monoidal category with a functor to vector spaces that respects tensor products leads to a dual quasi-Hopf algebra [2].

Categorification was firstly introduced by Louis Crane and Igor Frenkel for constructing four-dimensional quantum field theory [3], which is the process of finding category theoretic analogs of set theoretic concepts by replacing sets with categories, functions with functors, and equations between functions by natural isomorphisms between functors satisfying certain coherence laws. Categorification can make many intricate-looking things much simpler than they usually appear, thus it begins to appear more and more in the physics literature [4–6].

In this paper, following the idea of Drinfeld and Majid [1,2], we introduce dual Hom-quasi-bialgebra and dual Hom-quasi-Hopf algebra, which are generalization of dual quasi-bialgebra and

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* Corresponding author
E-mail address: yscheng@ustc.edu.cn (Yongsheng CHENG); gasong@ccec.edu.cn (Guang-ai SONG)

dual quasi-Hopf algebras. The key difference compared to an ordinary bialgebra or Hopf algebra of Hom-type is that multiplication is no longer Hom-associative, but only so up to conjugation by ϕ, we called quasi-Hom-associative [7–9]. Using the monoidal category language, we study the corepresentation theory of dual Hom-quasi-bialgebra and prove that the comodules category of a dual Hom-quasi-bialgebra is monoidal category. Finally, we give a categorical realization of dual Hom-quasi-bialgebra, and show that any dual Hom-quasi-Hopf algebra can be viewed as a 2-category.

2. Hom-bialgebras in monoidal categories

Recall that a monoidal category is a six-tuple $(\mathcal{C}, \otimes, I, f, l, r)$ where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor, called tensor product bifunctor, and functorial isomorphisms $f : \otimes(\otimes \times \mathrm{id}) \to \otimes(\mathrm{id} \times \otimes)$, $l : \otimes(I \times \mathrm{id}) \to \mathrm{id}$ (resp., $r : \otimes(\mathrm{id} \times I) \to \mathrm{id}$). The functorial morphism f is called the associativity constraint for $\otimes$ and satisfies Pentagon axiom, that is, the following relation

$$(\mathrm{id}_U \otimes f_{V,W,X}) \circ f_{U,V\otimes W,X} \circ (f_{U,V,W} \otimes \mathrm{id}_X) = f_{U,V,W\otimes X} \circ f_{U\otimes V,W,X} \tag{1}$$

holds, for any $U, V, W, X \in \mathcal{C}$. The morphisms l and r are called the left (resp., right) unit constraint with respect to the unital object I and they obey Triangle axiom, that is, for any $V, W \in \mathcal{C}$,

$$(\mathrm{id}_V \otimes l_W) f_{U,I,W} = r_V \otimes \mathrm{id}_W. \tag{2}$$

Let $(\mathcal{C}, \otimes, I, f, l, r)$ be a tensor category. A braiding in $\mathcal{C}$ is a commutativity constraint $c : \otimes \to \otimes\tau$ where τ is the flip functor defined by $\tau_{V,W}(V \otimes W) = W \otimes V$ on any pair of objects of the category, satisfying

$$f^{-1}_{W,U,V} \circ c_{U\otimes V,W} \circ f^{-1}_{W,U,V} = (c_{U,W} \otimes \mathrm{id}_V) \circ f^{-1}_{U,W,V} \circ (\mathrm{id}_U \otimes c_{V,W}), \tag{3}$$

$$f_{V,W,U} \circ c_{U,V\otimes W} \circ f_{U,V,W} = (\mathrm{id}_V \otimes c_{U,W}) \circ f_{V,U,W} \circ (c_{U,V} \otimes \mathrm{id}_W). \tag{4}$$

A tensor category with a braiding is called a braided tensor category.

Let $\mathcal{C} = (\mathcal{C}, \otimes, I, f, l, r)$ and $\mathcal{D} = (\mathcal{D}, \otimes, J, g, m, t)$ be two monoidal categories. A tensor functor from $\mathcal{C}$ to $\mathcal{D}$ is a triple (F, F_0, F_2) where $F : \mathcal{C} \to \mathcal{D}$ is a functor, F_0 is an isomorphism from J to $F(I)$, and

$$F_2(U, V) : F(U) \otimes F(V) \to F(U \otimes V)$$

such that the followings hold:

$$\begin{aligned} &F(f_{U,V,W}) \circ F_2(U \otimes V, W) \circ (F_2(U, V) \otimes \mathrm{id}_{F(W)}) \\ &\quad = F_2(U, V \otimes W) \circ (\mathrm{id}_{F(U)} \otimes F_2(V, W)) \circ g_{F(U),F(V),F(W)}; \end{aligned}$$

$$F(l_U) \circ F_2(I, U) \circ (F_0 \otimes \mathrm{id}_{F(U)}) = m_{F(U)}; \quad F(r_U) \circ F_2(U, I) \circ (\mathrm{id}_{F(U)} \otimes F_0) = t_{F(U)}.$$

The monoidal functor is called strict if the isomorphisms F_0, F_2 are identities of $\mathcal{D}$.

A natural transformation $\eta : (F, F_0, F_2) \to (F', F'_0, F'_2)$ between functors from $\mathcal{C}$ to $\mathcal{D}$ is a natural transformation $\eta : F \to F'$, for each couple objects (U, V) of $\mathcal{C}$, satisfying the following condition:

$$F'_2(U,V)(\eta(U) \otimes \eta(V)) = \eta(U \otimes V)F_2(U,V), \quad \eta(I)F_0 = F'_0.$$

Thus we can say that in category theory, monoidal functors are functors between monoidal categories which preserve the monoidal structure. More specifically, a monoidal functor between two monoidal categories consists of a functor between the categories, along with two coherence maps, a natural transformation and a morphism that preserve monoidal multiplication and unit, respectively.

A 2-category $\mathcal{C}$ is a category equipped with a class Obj$\mathcal{C}$ of objects, a family $(\mathrm{Hom}_{\mathcal{C}}(X,Y)|X, Y \in \mathrm{Obj}\,\mathcal{C})$ of categories, and a family $(M_{X,Y,Z} : \mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)|X, Y, Z \in \mathrm{Obj}\,\mathcal{C})$ of functors such that

$$M_{W,Y,Z}(\mathrm{id} \times M_{W,X,Y}) = M_{W,X,Z}(M_{X,Y,Z} \times \mathrm{id})$$

holds and there is an object id_X of $\mathrm{Hom}_{\mathcal{C}}(X,X)$ such that $M_{X,X,Z}(-, \mathrm{id}_X)$ and $M_{W,X,X}(\mathrm{id}_X, -)$ are identity functors. The objects of the $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ of categories are called 1-morphisms. The morphisms of these categories are called 2-morphisms.

That is to say that a 2-category is a data with categories as objects, functors as 1-morphisms, natural transformations as 2-morphisms satisfying some compositions. We denote the set of 2-morphisms between two 1-morphisms f and g as $\mathrm{Mor}_{\mathcal{C}}(f,g)$.

Let $(\mathcal{C}, \otimes, I, f, l, r)$ be a monoidal category with tensor product $\otimes$ and object I. In the following, using the techniques in [7,10], we give the notions of Hom-associative algebra, Hom-coassociative coalgebras and Hom-bialgebras in monoidal category.

Definition 2.1 *A Hom-associative algebra is a quadruple (A, μ, η, α), where A is an object in $\mathcal{C}$, $\mu : A \otimes A \to A$, $\eta : I \to A$ and $\alpha : A \to A$ are morphisms in $\mathcal{C}$ such that*

$$\mu \circ (\alpha \otimes \mu) = \mu \circ (\mu \otimes \alpha) \tag{5}$$

and

$$\mu \circ (\mathrm{id} \otimes \eta) = \mathrm{id}_A = \mu \circ (\eta \otimes \mathrm{id}). \tag{6}$$

If

$$\alpha \circ \mu = \mu \circ \alpha^{\otimes 2}, \tag{7}$$

then we call (A, μ, α) a multiplicative Hom-associative algebra; if α is an automorphism of A in $\mathcal{C}$, then we call it a regular Hom-associative algebra.

Let (A, μ, η, α) and $(A', \mu', \eta', \alpha')$ be two Hom-associative algebras. $f : A \to A'$ is said to be a morphism of Hom-associative algebras if

$$\mu' \circ (f \otimes f) = f \circ \mu, \quad f \circ \alpha = \alpha' \circ f \quad \text{and} \quad \eta' = f \circ \eta.$$

By the Hom-associative property, we have the following proposition.

Proposition 2.2 *Let (A, μ, α) be a Hom-associative algebra. Then*

$$\mu \circ ((\mu \circ \alpha^{\otimes 2}) \otimes (\alpha \circ \mu)) = \mu \circ ((\alpha \circ \mu) \otimes (\mu \circ \alpha^{\otimes 2})). \tag{8}$$

Proof

$$\begin{aligned}\mu \circ ((\mu \circ \alpha^{\otimes 2}) \otimes (\alpha \circ \mu)) &= \mu \circ (\alpha^2 \otimes (\mu \circ (\alpha \otimes \mu))) = \mu \circ (\alpha^2 \otimes \mu \circ (\mu \otimes (\mu \circ \alpha)))\\ &= \mu \circ (\mu \circ (\alpha \otimes \mu) \otimes \alpha^2) = \mu \circ ((\mu \circ (\mu \otimes \alpha)) \otimes \alpha^2)\\ &= \mu \circ ((\alpha \circ \mu) \otimes (\mu \circ \alpha^{\otimes 2})). \ \square \end{aligned} \tag{9}$$

In particular, for a multiplicative Hom-associative algebra (V, μ, α), we have

$$\mu \circ ((\mu \circ \alpha^{\otimes 2}) \otimes (\alpha \circ \mu)) = \mu \circ ((\mu \circ \alpha^{\otimes 2}) \otimes (\mu \circ \alpha^{\otimes})) = \mu \circ ((\alpha \circ \mu) \otimes (\mu \circ \alpha^{\otimes})). \tag{10}$$

Definition 2.3 *A left module of a Hom-associative algebra (A, μ, η, α) is a triple (V, μ_V, α_V), where V is an object in $\mathcal{C}$, $\mu_V: A \otimes V \to V$ and $\alpha_V: V \to V$ are morphisms in $\mathcal{C}$ such that*

$$\mu_V \circ (\mu \otimes \alpha_V) = \mu_V \circ (\alpha \otimes \mu_V), \tag{11}$$

for any $x \in A$, $v \in V$.

If (U, μ_U, α_U) and (V, μ_V, α_V) are two A-modules, then a morphism of A-modules

$$f: (U, \mu_U, \alpha_U) \to (V, \mu_V, \alpha_V)$$

is a morphism of the underlying A-modules such that

$$f \circ \mu_U = \mu_V \circ (\mathrm{id}_A \otimes f). \tag{12}$$

If we take the module category as the monoidal category, then we can write $\mu(x \otimes v) = xv$ for $x \in A$ and $v \in V$, where V is an A-module, then (11) can be rewritten as $(xy)\alpha_V(v) = \alpha(x)(yv)$, and (12) can be rewritten as $f(xv) = xf(v)$.

Definition 2.4 *A Hom-coassociative coalgebra is a quadruple $(A, \Delta, \varepsilon, \alpha)$, where A is an object in $\mathcal{C}$, $\Delta: A \to A \otimes A$ (called the comultiplication), $\varepsilon: A \to I$ (called the counit), and $\alpha: A \to A$ are morphisms in $\mathcal{C}$ satisfying the following conditions:*

$$(\alpha \otimes \Delta) \circ \Delta = (\Delta \otimes \alpha) \circ \Delta \ \textit{(Hom-coassociative axiom)}; \tag{13}$$

$$(\mathrm{id} \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} \ \ \textit{(Hom-counit axiom)}. \tag{14}$$

A Hom-coassociative coalgebra is called comultiplicative if $\Delta \circ \alpha = (\alpha \otimes \alpha) \circ \Delta$. In particular, if α is an isomorphism, we call $(A, \Delta, \varepsilon, \alpha)$ a regular Hom-coassociative coalgebra.

Let $(A, \Delta, \alpha, \varepsilon)$ and $(B, \delta, \beta, \epsilon)$ be two Hom-coassociative coalgebras. A morphism $f: A \to B$ is a homomorphism of Hom-coassociative coalgebras if

$$(f \otimes f) \circ \Delta = \delta \circ f, \quad \beta f = f\alpha, \ \text{and} \ \ \varepsilon = \epsilon \circ f. \tag{15}$$

Definition 2.5 *Let $(A, \Delta, \varepsilon, \alpha)$ be a Hom-coassociative coalgebra. A right A-comodule is a triple (V, α_V, δ_V), where $\delta_V: V \to V \otimes A$ is a morphism satisfying*

$$(\delta_V \otimes \alpha) \circ \delta_V = (\alpha_V \otimes \Delta) \circ \delta_V, \tag{16}$$

and

$$(\mathrm{id}_V \otimes \varepsilon) \circ \delta_V = \mathrm{id}_V. \tag{17}$$

Similarly, for the module category, we can denote $\delta_V(v) = \sum v_{(0)} \otimes v_{(1)} \in V \otimes A$ for $v \in V$. Then (16) can be rewritten as

$$\sum \alpha(v_{(0)}) \otimes \big(v_{(11)} \otimes v_{(12)}\big) = \sum \big(v_{(00)} \otimes v_{(01)}\big) \otimes \alpha(v_{(1)}) \tag{18}$$

and (17) can be rewritten as

$$\sum v_{(0)} \varepsilon(v_{(1)}) = v, \tag{19}$$

where we denote $v_{(i)(j)} = v_{(ij)}$.

If (U, α_U, δ_U) and (V, α_V, δ_V) are two right A-comodules, then a morphism of right A-comodules $f : U \to V$ is a morphism of the Hom-modules such that $f \circ \alpha_V = \alpha_V \circ f$ and

$$\delta_V \circ f = (f \otimes \mathrm{id}_A) \circ \delta_U,$$

which can be rewritten as

$$\sum (f(u))_{(0)} \otimes (f(u))_{(1)} = \sum f(u_{(0)}) \otimes u_{(1)}. \tag{20}$$

Actually, the comodules we have just defined are right comodules. A left A-comodule can be defined similarly.

Definition 2.6 *Let $(A, \Delta, \varepsilon, \alpha)$ be a Hom-coassociative coalgebra. A right A-comodule is a triple (V, α_V, δ_V), where $\delta_V : V \to A \otimes V$ is a morphism satisfying*

$$(\Delta \otimes \alpha_V) \circ \delta_V = (\alpha \otimes \delta) \circ \delta_V. \tag{21}$$

For module category, denote $\delta_V(v) = \sum v_{(-1)} \otimes v_{(0)} \in A \otimes V$ for $v \in V$. Then (21) can be rewritten as

$$\sum \alpha(v_{(-1)}) \otimes v_{(0-1)} \otimes v_{(00)} = \sum \big(v_{(-11)} \otimes v_{(-12)}\big) \otimes \alpha_V(v_{(0)}).$$

If U and V are two A-comodules, then a morphism of A-comodules $f : U \to V$ is a morphism satisfying

$$\delta_V \circ f = (\mathrm{id}_A \otimes f) \circ \delta_U, \tag{22}$$

which can be rewritten as

$$\sum (f(u))_{(-1)} \otimes (f(u))_{(0)} = \sum u_{(-1)} \otimes f(u_{(0)}).$$

A right A-comodule is the same as a left comodule over the opposite Hom-coassociative coalgebra A^{cop}. Obviously, a Hom-coassociative coalgebra A is A-comodule. The unit is I, which is regarded as a right A-comodule via the trivial coaction i.e., $\delta(a) = a \otimes I$.

Definition 2.7 *A Hom-bialgebra is a tuple $(A, \mu, \eta, \Delta, \varepsilon, \alpha)$, where (A, μ, η, α) is a Hom-associative algebra, $(A, \Delta, \varepsilon, \alpha)$ is a Hom-coassociative coalgebra, satisfying the following compatible conditions:*

$$\Delta \circ \mu = (\mu \otimes \alpha)(\alpha \otimes \Delta) + (\alpha \otimes \mu)(\Delta \otimes \alpha) \tag{23}$$

$$\Delta(I)=I\otimes I,\ \varepsilon(I)=I,\ \varepsilon\circ\mu=\mu\circ\varepsilon^{\otimes 2},\ \varepsilon\circ\alpha=\varepsilon. \tag{24}$$

For the module category, if x is an element of A, the element $\Delta(x)$ in $A\otimes A$ is of the form $\Delta(x)=\sum_{(x)}x_{(1)}\otimes x_{(2)}$. Denote $\Delta(x_{(1)})=\sum_{(x_{(1)})}x_{(11)}\otimes x_{(12)}$ and $\Delta(x_{(2)})=\sum_{(x_{(2)})}x_{(21)}\otimes x_{(22)}$. Thus, the coassociativity of Δ (13) can be expressed as follows

$$\sum_{(x)}\sum_{(x_{(2)})}\alpha(x_{(1)})\otimes\big(x_{(21)}\otimes x_{(22)}\big)=\sum_{(x)}\sum_{(x_{(1)})}\big(x_{(11)}\otimes x_{(12)}\big)\otimes\alpha(x_{(2)}). \tag{25}$$

The compatible condition (23) in Hom-bialgebra can be written as

$$\Delta(\mu(x\otimes y))=\sum_{(x)}\alpha(x_{(1)})\otimes\mu(x_{(2)}\otimes\alpha(y))+\sum_{(y)}\mu(\alpha(x)\otimes y_{(1)})\otimes\alpha(y_{(2)}). \tag{26}$$

Given a Hom-associative algebra $A\in\mathcal{C}$ one can define the categories ${}_A\mathcal{M}$, $\mathcal{M}_A$ and ${}_A\mathcal{M}_A$ of left, right and two-sided modules over A, respectively. Similarly, given a Hom-coassociative coalgebra $C\in\mathcal{C}$, one can define the categories of left, right and two-sided comodules over C, denote ${}^C\mathcal{M}$, $\mathcal{M}^C$, ${}^C\mathcal{M}^C$, respectively.

If H is a Hom-bialgebra, similarly, we have the notion of a Hopf H-module which is defined by an H-module (right or left) and H-comodule (right or left) such that the H-comodule structure is an H-module map, or, equivalently, the H-module structure is an H-comodule map. That is, a Hopf H-module is an H-module in the category of H-comodules, or, equivalently, an H-comodule in the category of H-modules.

3. Dual Hom-quasi-Hopf algebras and their corepresentation categories

Quasi-bialgebras, quasi-Hopf algebras and gauge transformations were invented by Drinfeld [7] relation with his treatment of the monodromy of the Knizhnik-Zamolodchikov equations. In [8], using Drinfeld's definition, the authors give the definition of Hom-quasi-bialgebra and its gauge transformation. Following the idea of Majid [13], Elhamdadi, Makhlouf [8], and Yau [18], we introduce the concept of dual Hom-quasi-Hopf algebra and braided dual Hom-quasi-Hopf algebra, which is a generalized dual quasi-Hopf algebra. In this section, we will mainly study their corepresentations category, here we take module category as the monoidal category.

Definition 3.1 *A dual Hom-quasi-bialgebra is a datum $(H,\mu,\eta,\Delta,\varepsilon,\alpha,\phi)$ in monoidal category, where $(H,\Delta,\varepsilon,\alpha)$ is a regular Hom-coassociative coalgebra; $\mu:\ H\otimes H\to H$ (we write $\mu(a\otimes b)=ab$) and $\eta:\ I\to H$ are Hom-coassociative coalgebra morphisms called multiplication and unit, respectively; a morphism $\phi:\ H\otimes H\otimes H\to I$ is convolution invertible, satisfying $\phi(\alpha^{\otimes 3})=\phi$ such that for all $a,b,c,d\in H$ the following relations hold:*

$$\sum\alpha(a_{(1)})(b_{(1)}c_{(1)})\phi(a_{(2)}\otimes b_{(2)}\otimes c_{(2)})=\sum\phi(a_{(1)}\otimes b_{(1)}\otimes c_{(1)})(a_{(2)}b_{(2)})\alpha(c_{(2)}), \tag{27}$$

$$\eta(I)a=a\eta(I)=\alpha(a), \tag{28}$$

$$\begin{aligned}&\sum\phi(\alpha(a_{(1)})\otimes\alpha(b_{(1)})\otimes c_{(1)}d_{(1)})\phi((a_{(2)}b_{(2)})\otimes\alpha(c_{(2)})\otimes\alpha(d_{(2)}))\\&\quad=\sum\phi(b_{(1)}\otimes c_{(1)}\otimes d_{(1)})\phi(\alpha(a_{(1)})\otimes(b_{(2)}c_{(2)})\otimes\alpha(d_{(2)}))\phi(a_{(2)}\otimes b_{(3)}\otimes c_{(3)}),\end{aligned} \tag{29}$$

$$\phi(a \otimes \eta(I) \otimes b) = \varepsilon(a)\varepsilon(b). \tag{30}$$

Note that from (13) and (27) we see that the main difference between a Hom-bialgebra and a Hom-quasi-bialgebra lies in the fact that the multiplication of a Hom-quasi-bialgebra is no longer Hom-associative. Nevertheless, Relation (27) shows that it is almost; At last, by (29), we also call Drinfeld reassociator ϕ a normalized 3-cocycle of Hom-bialgebra. If ϕ is trivial, a dual Hom-quasi-bialgebra is just a Hom-bialgebra.

Definition 3.2 *A morphism of dual Hom-quasi-bialgebras*

$$g : (H, \mu, \eta, \Delta, \varepsilon, \alpha, \phi) \to (H', \mu', \eta', \Delta', \varepsilon', \alpha', \phi')$$

is a Hom-coalgebra homomorphism $g : (H, \Delta, \varepsilon, \alpha) \to (H', \Delta', \varepsilon', \alpha')$ *such that*

$$\mu'(g \otimes g) = g\mu, g\eta = \eta', \phi'(g^{\otimes 3}) = \phi'.$$

It is an isomorphism of Hom-quasi-bialgebras if, in addition, it is invertible.

Definition 3.3 *A dual Hom-quasi-bialgebra* $(H, \mu, \eta, \Delta, \varepsilon, \alpha, \phi)$ *is called braided dual Hom-quasi-bialgebra if there exists an invertible element* $\varphi \in (H \otimes H)^*$ *satisfying* $\varphi(\alpha^{\otimes 2}) = \varphi$ *and*

$$\begin{aligned}\varphi((xy) \otimes \alpha(z)) = \sum \phi(z_{(1)}, x_{(1)}, y_{(1)})\varphi(\alpha(x_{(2)}), z_{(2)})\\ \phi^{-1}(x_{(3)}, z_{(3)}, y_{(2)})\varphi(\alpha(y_{(3)}), z_{(4)})\phi(x_{(4)}, y_{(4)}, z_{(5)}),\end{aligned} \tag{31}$$

$$\begin{aligned}\varphi(\alpha(x) \otimes (yz)) = \sum \phi^{-1}(y_{(1)}, z_{(1)}, x_{(1)})\varphi(x_{(2)}, \alpha(z_{(2)}))\\ \phi(y_{(2)}, x_{(3)}, z_{(2)})\varphi(x_{(4)}, \alpha(y_{(3)}))\phi^{-1}(x_{(5)}, y_{(4)}, z_{(4)}),\end{aligned} \tag{32}$$

$$\sum \varphi(x_{(1)} \otimes y_{(1)})x_{(2)}y_{(2)} = \sum y_{(1)}x_{(1)}\varphi(x_{(2)} \otimes y_{(2)}). \tag{33}$$

Definition 3.4 *A dual Hom-quasi-bialgebra* $(H, \mu, \eta, \Delta, \varepsilon, \alpha, \phi)$ *is called dual Hom-quasi-Hopf algebra if there is an antimorphism* $s : H \to H$ *of the Hom-coassociative coalgebra* $(H, \Delta, \varepsilon, \alpha)$ *and invertible elements* $f, g : H \to I$ *such that for all* $x \in H$, $s\alpha = \alpha s$ *and*

$$\sum s(x_1)f(x_2)x_3 = f(x)\eta(I), \quad \sum x_1 g(x_2)s(x_3) = g(x)\eta(I), \tag{34}$$

$$\sum \phi(x_1 \otimes g(x_2)s(x_3)f(x_4) \otimes x_5) = \sum \phi^{-1}(s(x_1) \otimes f(x_2)x_3 g(x_4) \otimes s(x_5)) = \varepsilon(x). \tag{35}$$

We shall write $(H, \mu, \varepsilon, \alpha, \phi, s, f, g)$ to express the complete data of a dual Hom-quasi-Hopf algebra.

Lemma 3.5 *If* $(H, \mu, \eta, \Delta, \varepsilon, \alpha, \phi, s, f, g)$ *is a dual Hom-quasi-Hopf algebra, then* $H^{\mathrm{op,\,cop}}$ *and* $H \otimes H^{\mathrm{op}}$ *are also dual Hom-quasi-Hopf algebras.*

Proof The structure maps are

$$\phi_{\mathrm{op,\,cop}}(x, y, z) = \phi(z, y, x), \ s_{\mathrm{op,\,cop}} = s, \ f_{\mathrm{op,\,cop}} = g, \ g_{\mathrm{op,\,cop}} = f.$$

$$\phi_{H \otimes H^{\mathrm{op}}} = \phi \otimes \phi_{\mathrm{op}}, \ s_{H \otimes H^{\mathrm{op}}} = s \otimes s, \ f_{H \otimes H^{\mathrm{op}}} = f \otimes g, \ g_{H \otimes H^{\mathrm{op}}} = g \otimes f. \ \Box$$

Let $(H, \Delta, \varepsilon, \alpha)$ be a comultiplicative Hom-coassociative coalgebra. Let (U, α_U) and (V, α_V) be two right H-comodules. Similarly to [10], the next lemma shows that two right H-comodules give rise to an H-comodule via diagonal coaction.

Lemma 3.6 *If $(U,\alpha_U,\delta_U),(V,\alpha_V,\delta_V)\in\mathcal{M}^H$, we define $(U,\alpha_U,\delta_U)\otimes(V,\alpha_V,\delta_V)=(U\otimes V,\alpha_U\otimes\alpha_V,\delta_{U\otimes V})$ according to*

$$\delta_{U\otimes V}=(\mathrm{id}_{U\otimes V}\otimes\mu)(\mathrm{id}_U\otimes\tau_{A,V}\otimes\mathrm{id}_A)(\delta_U\delta_V) \tag{36}$$

for $u\in U,v\in V$. Then the map $\delta_{U\otimes V}$ endows the tensor product $U\otimes V$ with a right H-comodule structure.

Proof Let $\delta_U(u)=\sum u_{(0)}\otimes u_{(1)}$, $\delta_V(v)=\sum v_{(0)}\otimes v_{(1)}$. Then (36) can be clearly expressed as follows:

$$\delta_{U\otimes V}(u\otimes v)=\sum u_{(0)}\otimes v_{(0)}\otimes u_{(1)}v_{(1)}. \tag{37}$$

For any $u\in(U,\alpha_U)$, $v\in(V,\alpha_V)$, we have

$$\begin{aligned}\delta_{U\otimes V}\circ(\alpha_U\otimes\alpha_V)(u\otimes v)&=\delta\big(\alpha_U(u)\otimes\alpha_V(v)\big)\\&=\sum(\alpha_U(u))_{(0)}\otimes(\alpha_V(v))_{(0)}\otimes(\alpha(u))_{(1)}(\alpha(v))_{(1)}\\&=\sum\alpha_U(u_{(0)})\otimes\alpha_V(v_{(0)})\otimes\alpha(u_{(1)})\alpha(v_{(1)})\\&=\sum\alpha_U(u_{(0)})\otimes\alpha_V(v_{(0)})\otimes\alpha(u_{(1)}v_{(1)})\\&=\alpha\otimes(\alpha_U\otimes\alpha_V)\circ\delta_{U\otimes V}(u\otimes v),\end{aligned}$$

which shows that $\delta_{U\otimes V}$ is a morphism of Hom-comodules. On the other hand,

$$\begin{aligned}(\delta_{U\otimes V}\otimes\alpha)\circ\delta_{U\otimes V}(u\otimes v)&=(\delta\otimes\alpha)\big(\sum u_{(0)}\otimes v_{(0)}\otimes u_{(1)}v_{(1)}\big)\\&=\sum\big(u_{(0)(0)}\otimes v_{(0)(0)}\otimes u_{(0)(1)}v_{(0)(1)}\big)\otimes\alpha(u_{(1)})\alpha(v_{(1)})\\&=\sum\alpha_U(u_{(0)})\alpha_V(v_{(0)})\otimes\big(u_{(1)(1)}v_{(1)(1)}\otimes u_{(1)(2)}v_{(1)(2)}\big)\\&=\sum\alpha_U(u_{(0)})\alpha_V(v_{(0)})\otimes\big((u_{(1)}v_{(1)})_{(1)}\otimes(u_{(1)}v_{(1)})_{(2)}\big)\\&=((\alpha_U\otimes\alpha_V)\otimes\Delta)\circ\delta_{U\otimes V},\end{aligned}$$

which shows that (36) satisfies (16). $\Box$

Similarly, we also have

Lemma 3.7 *If $(U,\alpha_U,\delta_U),(V,\alpha_V,\delta_V)\in\mathcal{M}_H$, we define $(U,\alpha_U,\delta_U)\otimes(V,\alpha_V,\delta_V)=(U\otimes V,\alpha_U\otimes\alpha_V,\delta_{U\otimes V})$ according to*

$$\delta_{U\otimes V}(u\otimes v)=\sum u_{(-1)}v_{(-1)}\otimes u_{(0)}\otimes v_{(0)} \tag{38}$$

for $u\in U,v\in V$. Then the map $\delta_{U\otimes V}$ endows the tensor product $U\otimes V$ with a left H-comodule structure.

By Definition 2.5 and formula (37), we can obtain the following lemma directly.

Lemma 3.8 The conditions are as above. Then we have

$$(u\otimes v)_{(0)}=u_{(0)}\otimes v_{(0)},\ (u\otimes v)_{(1)}=u_{(1)}v_{(1)},\ (u\otimes v)_{(-1)}=u_{(-1)}v_{(-1)}. \tag{39}$$

Lemma 3.9 *Let $(H, \Delta, \varepsilon, \alpha)$ be a comultiplicative Hom-coassociative coalgebra, (V, α_V, δ_V) be a right H-comodule. Then*

$$(\alpha_V(v))_{(0)} = \alpha_V(v_{(0)}),\ (\alpha_V(v))_{(1)} = \alpha_V(v_{(1)}),\ (\alpha_V(v))_{(-1)} = \alpha_V(v_{(-1)}). \tag{40}$$

By Lemma 3.6, we obtain that the tensor product comodule $(U \otimes V, \alpha_U \otimes \alpha_V, \delta_{U\otimes V})$ of two H-comodules (U, α_U, δ_U) and (U, α_U, δ_U) is defined by the coaction (37). The tensor product restricts to a functor

$$\otimes : \mathcal{M}^H \times \mathcal{M}^H \longrightarrow \mathcal{M}^H.$$

Similarly, for $(U, \alpha_U, \delta_U), (V, \alpha_V, \delta_V), (W, \alpha_W, \delta_W) \in \mathcal{M}^H$, we know that $\big((U\otimes V)\otimes W,\ \delta_{(U\otimes V)\otimes W}\big)$ and $\big(U \otimes (V \otimes W),\ \delta_{U\otimes(V\otimes W)}\big)$ are all right H-comodules. Their H-comodule structures are given by

$$\begin{aligned}
&\delta_{(U\otimes V)\otimes W}\big((u\otimes v)\otimes w\big) = \sum (u\otimes v)_{(0)} \otimes w_{(0)} \otimes (u\otimes v)_{(1)} w_{(1)}\\
&\quad = \sum u_{(0)} \otimes v_{(0)} \otimes w_{(0)} \otimes (u_{(1)}v_{(1)})w_{(1)}; \qquad (41)\\
&\delta_{U\otimes(V\otimes W)}\big(u\otimes(v\otimes w)\big) = \sum u_{(0)} \otimes (v\otimes w)_{(0)} \otimes u_{(1)}(v\otimes w)_{(1)}\\
&\quad = \sum u_{(0)} \otimes v_{(0)} \otimes w_{(0)} \otimes u_{(1)}(v_{(1)}w_{(1)}). \qquad (42)
\end{aligned}$$

By (27), we know that (41) and (42) are not equal in general.

Lemma 3.10 *Let $(H, \Delta, \varepsilon, \alpha)$ be a regular Hom-coassociative coalgebra with two morphisms of Hom-coalgebra $\mu : H \otimes H \to H$ and $\eta :\ I \to H$ satisfying (27)–(30). For all (U, α_U), (V, α_V), $(W, \alpha_W) \in \mathcal{M}^H$ and $u \in U, v \in V, w \in W$, define $f_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ by*

$$f_{U,V,W}\big((u\otimes v)\otimes w\big) = \sum \phi(u_{(1)}, v_{(1)}, w_{(1)})\alpha_U(u_{(0)}) \otimes (v_{(0)} \otimes \alpha_W^{-1}(w_{(0)})), \tag{43}$$

where $\delta(u) = \sum u_{(0)} \otimes v_{(1)}$ and the map ϕ is defined in Definition 3.1. Then $f_{U,V,W}$ is a right H-comodule homomorphism and satisfies the Pentagon axiom.

Proof Since α is an isomorphism and $\phi \in (H \otimes H \otimes H)^*$, we easily obtain that $f_{U,V,W}$ is an isomorphism of vector spaces $(U \otimes V) \otimes W$ and $U \otimes (V \otimes W)$. By the isomorphic property of α, and (27), we easily get that $f_{U,V,W}$ is an isomorphism with inverse

$$f^{-1}_{U,V,W}\big(u\otimes(v\otimes w)\big) = \sum \phi^{-1}(u_{(1)}, v_{(1)}, w_{(1)})\big(\alpha_U^{-1}(u_{(0)}) \otimes (v_{(0)})\big) \otimes \alpha_W(w_{(0)}).$$

We want to prove $f_{U,V,W}$ is a right H-comodule homomorphism, according to (22), we just need to prove that the following equation holds.

$$\delta_{U\otimes(V\otimes W)} \circ f_{U,V,W} = (f_{U,V,W} \otimes \mathrm{id}_H) \circ \delta_{(U\otimes V)\otimes W}. \tag{44}$$

On the one hand, using (42), (43) and Lemmas 3.8 and 3.9, we have

$$\begin{aligned}
&\delta_{U\otimes(V\otimes W)} \circ f_{U,V,W}\big((u\otimes v)\otimes w\big)\\
&= \delta_{U\otimes(V\otimes W)} \sum \phi(u_{(1)} \otimes v_{(1)} \otimes w_{(1)})\alpha_U(u_{(0)}) \otimes \big(v_{(0)} \otimes \alpha_W^{-1}(w_{(0)})\big)\\
&= \sum \phi(u_{(1)} \otimes v_{(1)} \otimes w_{(1)})(\alpha_U(u_{(0)}))_{(0)} \otimes (v_{(0)})_{(0)} \otimes (\alpha_W^{-1}(w_{(0)}))_{(0)} \otimes\\
&\quad (\alpha_U(u_{(0)}))_{(1)}\big((v_{(0)})_{(1)}(\alpha_W^{-1}(w_{(0)}))_{(1)}\big)
\end{aligned}$$

$$= \sum \phi(u_{(1)} \otimes v_{(1)} \otimes w_{(1)})\alpha_U(u_{(00)}) \otimes v_{(00)} \otimes \alpha_W^{-1}(w_{(00)}) \otimes \\ \alpha_U(u_{(01)})\big(v_{(01)}\alpha_W^{-1}(w_{(01)})\big). \tag{45}$$

On the other hand, by (41) and (43), we have

$$\begin{aligned}&(f_{U,V,W} \otimes \mathrm{id}_H) \circ \delta_{(U\otimes V)\otimes W}\big((u \otimes v) \otimes w\big)\\ &\quad = (f_{U,V,W} \otimes \mathrm{id}_H) \sum u_{(0)} \otimes v_{(0)} \otimes w_{(0)} \otimes (u_{(1)}v_{(1)})w_{(1)}\\ &\quad = \sum \phi(u_{(01)} \otimes v_{(01)} \otimes w_{(01)})\alpha_U(u_{(00)}) \otimes v_{(00)} \otimes \alpha_W^{-1}(w_{(00)}) \otimes (u_{(1)}v_{(1)})w_{(1)}.\end{aligned} \tag{46}$$

Comparing (27), (45) and (46), we conclude that (44) holds.

Let

$$g_{U,V,W}\big((u \otimes v) \otimes w\big) = \sum \phi(u_1 \otimes v_1 \otimes w_1)\big(u_0 \otimes (v_0 \otimes w_0)\big). \tag{47}$$

By (29) and Theorem 2.1 in [13], we know that $g_{U,V,W}$ satisfies the Pentagon axiom. Then

$$f_{U,V,W} = \big(\alpha_U \otimes (\mathrm{id}_V \otimes \alpha_W^{-1})\big)g_{U,V,W} = g_{U,V,W}\big((\alpha_U \otimes \mathrm{id}_V) \otimes \alpha_W^{-1}\big). \tag{48}$$

Thus, on the one hand

$$\begin{aligned}&f_{U,V,W\otimes P}f_{U\otimes V,W,P}\\ &g_{U,V,W\otimes P} \circ \Big(\alpha_U \otimes \big(\mathrm{id}_V \otimes (\alpha_W^{-1} \otimes \alpha_P^{-1})\big)\Big) \circ g_{U\otimes V,W,P} \circ \Big(\big((\alpha_U \otimes \alpha_V) \otimes \mathrm{id}_W\big) \otimes \alpha_P^{-1}\Big)\\ &\quad = g_{U,V,W\otimes P} \circ g_{U\otimes V,W,P} \circ \Big(\big((\alpha_U \otimes \mathrm{id}_V) \otimes \alpha_W^{-1}\big) \otimes \alpha_P^{-1}\Big) \circ \Big(\big((\alpha_U \otimes \alpha_V) \otimes \mathrm{id}_W\big) \otimes \alpha_P^{-1}\Big)\\ &\quad = g_{U,V,W\otimes P} \circ g_{U\otimes V,W,P} \circ \Big(\big((\alpha_U^2 \otimes \alpha_V) \otimes \alpha_W^{-1}\big) \otimes \alpha_P^{-2}\Big).\end{aligned}$$

On the other hand,

$$\begin{aligned}&(\mathrm{id}_U \otimes f_{V,W,P}) \circ f_{U,V\otimes W,P} \circ (f_{U,V,W} \otimes \mathrm{id}_P)\\ &\quad = \Big(\mathrm{id}_U \otimes g_{V,W,P} \circ \big((\alpha_V \otimes \mathrm{id}_W) \otimes \alpha_P^{-1}\big)\Big) \circ \Big(g_{U,V\otimes W,P} \circ \big((\alpha_U \otimes \mathrm{id}_V \otimes \mathrm{id}_W\big) \otimes \alpha_P^{-1}\Big) \circ\\ &\qquad \Big((g_{U,V,W} \circ (\alpha_U \otimes \mathrm{id}_V) \otimes \alpha_W^{-1}) \otimes \mathrm{id}_P\Big)\\ &\quad = (\mathrm{id}_U \otimes g_{V,W,P}) \circ g_{U,V\otimes W,P} \circ \big((g_{U,V,W} \otimes \mathrm{id}_P\big) \circ\\ &\qquad \big(\mathrm{id}_U \otimes (\alpha_V \otimes \mathrm{id}_W) \otimes \alpha_P^{-1}\big) \circ \big((\alpha_U \otimes \mathrm{id}_V \otimes \mathrm{id}_W\big) \otimes \alpha_P^{-1}\big)\big((\alpha_U \otimes \mathrm{id}_V) \otimes \alpha_W^{-1}) \otimes \mathrm{id}_P\big)\\ &\quad = (\mathrm{id}_U \otimes g_{V,W,P}) \circ g_{U,V\otimes W,P} \circ \big((g_{U,V,W} \otimes \mathrm{id}_P\big) \circ \Big(\big((\alpha_U^2 \otimes \alpha_V) \otimes \alpha_W^{-1}\big) \otimes \alpha_P^{-2}\Big).\end{aligned}$$

Using (1), we obtain that

$$f_{U,V,W\otimes P}f_{U\otimes V,W,P} = (\mathrm{id}_U \otimes f_{V,W,P}) \circ f_{U,V\otimes W,P} \circ (f_{U,V,W} \otimes \mathrm{id}_P),$$

that is to say that the Pentagon axiom holds. □

In general, we call $f_{U,V,W}$ defined by (43) the associativity constraint of the tensor product. Similarly, with the idea of Kassel [10], the following theorem gives a characterization of dual Hom-quasi-bialgebra and its corepresentation category.

Theorem 3.11 *Let $(H, \Delta, \varepsilon, \alpha)$ be a Hom-coassociative coalgebra equipped with Hom-coalgebra homomorphisms $\mu : H \otimes H \to H$ and $\eta :\ I \to H$. If H is a dual Hom-quasi-bialgebra, then $\mathcal{M}^H$*

is a monoidal category.

Proof By Lemma 3.10, we know that the associativity constraint $f_{U,V,W}$ is a right A-comodule homomorphism and an isomorphism satisfying the Pentagon axiom. Define the unit constraints

$$l_V(\eta(I) \otimes v) = \alpha_V(v), \quad r_V(v \otimes \eta(I)) = \alpha_V(v). \tag{49}$$

They are natural isomorphisms and H-linear thanks to Relations (27), (30) and $\alpha^3(\phi) = \phi$. By (28) and (30), we have

$$\begin{aligned}(\mathrm{id}_V \otimes l_W) \circ f_{V,I,W} &= (\mathrm{id}_V \otimes l_W)\big(\sum \phi(v_{(1)} \otimes I_{(1)} \otimes w_{(1)})\alpha_V(v_{(0)}) \otimes (I_{(0)} \otimes \alpha_W(w_{(0)}))\big)\\ &= (\mathrm{id}_V \otimes l_W)\big(\alpha_V(v) \otimes \alpha_W(\alpha_W^{-1}(w_{(0)}))\big)\\ &= \alpha_V(v) \otimes w = l_V \otimes \mathrm{id}_W.\end{aligned}$$

That implies the Triangle axioms. □

Similarly to Theorem 3.11, we have

Theorem 3.12 *Let $(H, \Delta, \varepsilon, \alpha)$ be a Hom-coassociative coalgebra equipped with Hom-coalgebra homomorphisms $\mu : H \otimes H \to H$ and $\eta : I \to H$. If H is a dual Hom-quasi-bialgebra, then ${}^H\mathcal{M}$ and ${}^H\mathcal{M}^H$ are monoidal categories.*

Proof Define the associative constraint

$${}^H f_{U,V,W}\big((u \otimes v) \otimes w\big) = \sum \phi^{-1}((u_{(-1)} \otimes v_{(-1)}) \otimes w_{(-1)})\alpha_U(u_{(0)}) \otimes (v_{(0)} \otimes \alpha_W^{-1}(w_{(0)})),$$

$$\begin{aligned}&{}^H f_{U,V,W}{}^H\big((u \otimes v) \otimes w\big)\\ &\quad= \sum \phi^{-1}(u_{(-1)} \otimes v_{(-1)} \otimes w_{(-1)})\alpha_U(u_{(0)}) \otimes (v_{(0)} \otimes \alpha_W^{-1}(w_{(0)}))\phi(u_{(1)} \otimes v_{(1)} \otimes w_{(1)}),\end{aligned}$$

respectively. We can complete the proof in a way similar to that of Theorem 3.11. □

Theorem 3.13 *Let $(H, \Delta, \varepsilon, \alpha)$ be a Hom-coassociative coalgebra equipped with Hom-coalgebra homomorphisms $\mu :\ H \otimes H \to H$ and $\eta :\ I \to H$. If H is a braided dual Hom-quasi-bialgebra, then $\mathcal{M}^H$ is a braided monoidal category.*

Proof By Theorem 3.11, we know that $(\mathcal{M}^H, \otimes, I, f_{U,V,W}, l, r)$ is a monoidal category. Define

$$c_{V,W}(u \otimes v) = \sum \mu(u_{(1)} \otimes v_{(1)})v_{(0)} \otimes u_{(0)}. \tag{50}$$

We just prove that $c_{V,W}$ is a braiding of $(\mathcal{M}^H, \otimes, I, f_{U,V,W}, l, r)$. First, we prove that $c_{V,W}$ is an isomorphism of H-modules, that is, for $U, V \in \mathcal{M}^H$, the following commutative diagram holds.

$$\begin{array}{ccc} U \otimes V & \xrightarrow{c_{U,V}} & V \otimes U \\ {\scriptstyle \delta_{U\otimes V}}\big\downarrow & & {\scriptstyle \delta_{V\otimes U}}\big\downarrow \\ (U \otimes V) \otimes H & \xrightarrow{c_{H,H} \otimes \mathrm{id}_H} & (V \otimes U) \otimes H \end{array}$$

where $\delta_{U\otimes V}$ is defined by (36). By (33), we have

$$\begin{aligned}\delta_{V\otimes U}c_{U,V}(u\otimes v)&=\delta_{V\otimes U}\Big(\sum\mu(u_{(1)}\otimes v_{(1)})v_{(0)}\otimes u_{(0)}\Big)\\&=\sum\mu(u_{(1)}\otimes v_{(1)})v_{(00)}\otimes u_{(00)}\otimes v_{(01)}u_{(01)}\\&=\sum\mu(u_{(01)}\otimes v_{(01)})v_{(00)}\otimes u_{(00)}\otimes u_{(1)}v_{(1)}\\&=(c_{U,V}\otimes\mathrm{id}_H)\Big(\sum u_{(0)}\otimes v_{(0)}\otimes u_{(1)}v_{(1)}\Big)\\&=(c_{U,V}\otimes\mathrm{id}_H)\delta_{U,V}(u\otimes v).\end{aligned}$$

Thus we prove that the above diagram commutes. Next we just check that (43) satisfies (4). On the one hand,

$$\begin{aligned}&f_{V,W,U}\circ c_{U,V\otimes W}\circ f_{U,V,W}\\&\quad=g_{V,W,U}\circ((\alpha_V\otimes\mathrm{id}_W)\otimes\alpha_U^{-1})\circ c_{U,V\otimes W}\circ f_{U,V,W}\circ((\alpha_U\otimes\mathrm{id}_V)\otimes\alpha_W^{-1})\\&\quad=g_{V,W,U}\circ c_{U,V\otimes W}\circ(\alpha_U^{-1}\otimes(\alpha_V\otimes\mathrm{id}_W))\circ g_{U,V,W}\circ((\alpha_U\otimes\mathrm{id}_V)\otimes\alpha_W^{-1})\\&\quad=g_{V,W,U}\circ c_{U,V\otimes W}\circ f_{U,V,W}\circ(\mathrm{id}_U\otimes(\alpha_V\otimes\alpha_W^{-1})).\end{aligned}$$

On the other hand,

$$\begin{aligned}&(\mathrm{id}_V\otimes c_{U,W})\circ f_{V,U,W}\circ(c_{U,V}\otimes\mathrm{id}_W)\\&\quad=(\mathrm{id}_V\otimes c_{U,W})\circ g_{V,W,U}\circ((\alpha_V\otimes\mathrm{id}_W)\otimes\alpha_U^{-1})\circ(c_{U,V}\otimes\mathrm{id}_W)\\&\quad=(\mathrm{id}_V\otimes c_{U,W})\circ g_{V,W,U}\circ(c_{U,V}\otimes\mathrm{id}_W)\circ(\mathrm{id}_U\otimes(\mathrm{id}_V\otimes\alpha_W^{-1})),\end{aligned}$$

where $g_{U,V,W}$ is defined in (47). By [12], we know that $g_{V,W,U}$ satisfies (4). Then, from the above equations, we obtain that (4) holds for $\mathcal{M}^H$. Similarly, we can show that (3) holds. $\square$

In fact, for a dual Hom-quasi-bialgebra $(H,\mu,\eta,\Delta,\varepsilon,\alpha)$, we do not have a notion of H-module, and we do not have a notion of H-module in the category of H-comodules either, since H is not a Hom-associative algebra in the category of H-comodules. On the other hand, following idea of [16], we want to define H-bicomodules. Since $\big((H,\delta_H^l,\delta_H^r),\mu,\eta,\alpha\big)$ is a Hom-associative algebra in the monoidal category $({}^H\mathcal{M}^H,\otimes,I,{}^Hf^H,l,r)$ with $\delta_H^l=\delta_H^r=\Delta$. Hence we set

$${}^H\mathcal{M}_H^H:=({}^H\mathcal{M}^H)_H$$

which is called the category of right dual Hom-quasi-Hopf H-bicomodules.

By Lemma 3.10, we can directly check that

Theorem 3.14 *Let $(H,\mu,\eta,\Delta,\varepsilon,\alpha)$ be a dual Hom-quasi-bialgebra. Let $U\otimes^H\mathcal{M}^H$, $V\in^H\mathcal{M}_H^H$. Then $(U\otimes V,\delta_{U\otimes V}^l,\delta_{U\otimes V}^r,\rho_{U\otimes V}^r)\in^H\mathcal{M}_H^H$, where $\delta_{U\otimes V}^r$ and $\delta_{U\otimes V}^l$ are defined as (37) and (38) respectively, and*

$$\rho_{U\otimes V}^r\big((u\otimes v)\otimes h\big)=\phi^{-1}(u_{(-1)}\otimes v_{(-1)}\otimes h_{(1)})u_{(0)}\otimes v_{(0)}h_{(2)}\phi(u_{(1)}\otimes v_{(1)}\otimes h_{(3)}).$$

4. Categorical realization of dual Hom-quasi-Hopf algebras

In this section, we will use the categorical technique of [6, 11], [15–17] to realize dual Hom-quasi-Hopf algebra. Recall a 2-vector space is an internal category in the category of all vector spaces over $\mathbb{F}$ as objects and linear transformations as morphisms. Let $\mathcal{V}$ and $\mathcal{W}$ be two 2-vector spaces. We have the obvious definition of the 2-vector space $\mathcal{V} \otimes \mathcal{W}$ by the tensor product of vector spaces and the tensor product of corresponding maps [14]. We also have a unique 2-vector space, also denoted $\mathbb{F}$, which is the categorified ground field $\mathbb{F}$, where the two objects are also itself and the morphisms are identity, the composition map is trivial.

Given a category $\mathcal{C}$, whose objects are 2-vector spaces. As we say in Section 2, firstly, we build a Hom-coassociative coalgebra structure on $\mathcal{C}$. Secondly, we define a multiplication map $\mu:\ \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a unit map $\eta:\ \mathbb{F} \to \mathcal{C}$, and a tensor functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ satisfying

$$\mu(\eta \otimes \mathrm{id}) = \alpha = \mu(\mathrm{id} \otimes \eta). \tag{51}$$

Next we want to build another Hom-associative algebra structure on $\mathcal{C}$. While, unlike ordinary regular Hom-associative algebra in a monoidal category, we will replace its functions with functors, and equations between functions by natural isomorphisms between functors satisfying certain coherence laws which are called the categorification process. For the tensor functor $\otimes$ of $\mathcal{C}$, we define an associative constraint isomorphism

$$f : \otimes(\alpha \times \otimes) \to \otimes(\otimes \times \alpha). \tag{52}$$

The associative constraint isomorphism f should satisfy the following coherence conditions:

$$\begin{tikzcd}
& \otimes(\otimes \times \otimes)(\alpha \times \alpha \times \alpha \times \alpha) \arrow[dr, "f(\otimes\times\alpha\times\alpha)"] & \\
\otimes(\alpha \times \otimes)(\alpha \times \alpha \times \otimes) \arrow[ur, "f(\alpha\times\alpha\times\otimes)"'] \arrow[d, "\mathrm{id}\otimes f"'] & & \otimes(\otimes \times \alpha)(\otimes \times \alpha \times \alpha) \\
\otimes(\alpha \times \otimes)(\alpha \times \otimes \times \alpha) \arrow[rr, "f(\alpha\times\otimes\times\alpha)"] & & \otimes(\otimes \times \alpha)(\alpha \times \otimes \times \alpha) \arrow[u, "f\otimes\mathrm{id}"']
\end{tikzcd} \tag{53}$$

that is the famous Pentagon axiom.

For the multiplication map μ of $\mathcal{C}$, we define

$$\phi : \mu(\alpha \otimes \mu) \to \mu(\mu \otimes \alpha)$$

is a natural isomorphism in the sense that there exists a $*$-invertible $\phi:\ H \otimes H \otimes H \to \mathbb{F}$ such that

$$\phi * \mu(\alpha \otimes \mu) = \mu(\mu \otimes \alpha) * \phi, \tag{54}$$

where $*$ is the convolution defined by

$$(\varphi * \psi)(x) = \sum_{(x)} \varphi(x_{(1)})\psi(x_{(2)}).$$

Then (54) can be written as (27). Furthermore, the natural isomorphism ϕ should satisfy the following coherence laws:

$$\begin{tikzcd}
& \mu(\mu\times\mu)(\alpha\times\alpha\times\alpha\times\alpha) \arrow[dr, "\phi(\mu\otimes\alpha\otimes\alpha)"] & \\
\mu(\alpha\times\mu)(\alpha\times\alpha\times\mu) \arrow[ur, "\phi(\alpha\otimes\alpha\otimes\mu)"'] \arrow[d, "\mathrm{id}\otimes\phi"'] & & \mu(\mu\times\alpha)(\mu\times\alpha\times\alpha) \\
\mu(\alpha\times\mu)(\alpha\times\mu\times\alpha) \arrow[rr, "\phi(\alpha\otimes\mu\otimes\alpha)"] & & \mu(\mu\times\alpha)(\alpha\times\mu\times\alpha) \arrow[u, "\phi\otimes\mathrm{id}"']
\end{tikzcd} \tag{55}$$

From the commutative diagram, we can conclude that

$$(\phi\otimes\mathrm{id})*\phi(\alpha\otimes\mu\otimes\alpha)*(\mathrm{id}\otimes\phi)=\phi(\mu\times\alpha\times\alpha)*\phi(\alpha\times\mu\times\alpha) \tag{56}$$

as an equation for morphisms $H^{\otimes 4}\to\mathbb{F}$. (56) can be rewritten as (29).

From the above construction, we can obtain the following theorem at once.

Theorem 4.1 *Let $(H,\Delta,\varepsilon,\alpha)$ be a regular Hom-coassociative coalgebra over a field $\mathbb{F}$ equipped with Hom-coalgebra morphisms $\mu:\ H\otimes H\to H$ and $\eta:\mathbb{F}\to H$ and a $*$-invertible $\phi:\ H^{\otimes 3}\to\mathbb{F}$ such that equations (51), (54) and (56) hold. Then $(H,\mu,\eta,\Delta,\varepsilon,\alpha,\phi)$ is a dual Hom-quasi-bialgebra.*

Here we remark that, firstly, for the category $\mathcal{C}$, its objects are categories, the functors μ and $\otimes$ of $\mathcal{C}$ are 1-morphisms, the natural transformations f and ϕ are called 2-morphisms, which satisfy the corresponding coherence laws (53) and (55). Thus the category which realize the dual Hom-quasi-bialgebra is in fact a 2-category. Secondly, for any $x_1,x_2,x_3,x_4\in H$, the coherence law (55) can be written as

$$\begin{tikzcd}
& \mu(\alpha^2(x_1)\otimes\mu(\alpha(x_2)\otimes\mu(x_3\otimes x_4))) & \\
\mu(\alpha^2(x_1)\otimes\mu(\mu(x_2\otimes x_3)\otimes\alpha(x_4))) \arrow[ur, "\mathrm{id}_{\alpha^2(x_1)}\otimes f_{x_2,x_3,x_4}"'] & & \mu(\mu(\alpha(x_1)\otimes\alpha(x_2))\otimes(\mu(\alpha(x_3)\otimes\alpha(x_4)))) \arrow[ul, "f_{\alpha(x_1),\alpha(x_2),\mu(x_3\otimes x_4)}"'] \\
\mu(\mu(\alpha(x_1)\otimes\mu(x_2\otimes x_3))\otimes\alpha^2(x_4)) \arrow[u, "f_{\alpha(x_1),\mu(x_2\otimes x_3),\alpha(x_4)}"'] & & \mu(\mu(\mu(x_1\otimes x_2)\otimes\alpha(x_3))\otimes\alpha^2(x_4)) \arrow[u, "f_{\mu(x_1\otimes x_2),\alpha(x_3),\alpha(x_4)}"'] \arrow[ll, "f_{x_1,x_2,x_3}\otimes\mathrm{id}_{\alpha^2(x_4)}"]
\end{tikzcd} \tag{57}$$

which comes from (9) and (10) of Proposition 2.2 of a multiplicative Hom-associative algebra. Thus, through categorifying a regular Hom-associative algebra, we can obtain a generalized dual Hom-quasi-Hopf algebra.

Finally, we give an interesting corollary as follows:

Corollary 4.2 *Any dual Hom-quasi-Hopf algebra can be viewed as a 2-category.*

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References

[1] J. BAEZ, J. DOLAN. *Higher-dimensional algebra and topological quantum field theory*. J. Math. Phys., 1995, **36**(11): 6073–6105.
[2] L. CRANE, I. B. FRENKEL. *Four-dimensional topological quantum field theory, Hopf categories, and the canonical bases*. J. Math. Phys., 1994, **35**(10): 5136–5154.
[3] Yongsheng CHENG, Yucai SU. *Quantum deformations of the Heisenberg-Virasoro algebra*. Algebra Colloq., 2013, **20**(2): 299–308.
[4] Yongsheng CHENG, Yucai SU. *(Co)homology and universal central extension of Hom-Leibniz algebras*. Acta Math. Sin. (Engl. Ser.), 2011, **27**(5): 813–830.
[5] Yongsheng CHENG, Hengyun YANG. *Low-dimensional cohomology of q-deformed Heisenberg-Virasoro algebra of Hom-type*. Front. Math. China, 2010, **5**(4): 607–622.
[6] W. CHANG, Z. WANG, K. WU, et al. *Categorification and quasi-Hopf algebras*. Commun. Theor. Phys., 2011, **56**: 207–210.
[7] V. G. DRINFELD. *Quasi-Hopf algebras*. Leningrad Math. J., 1990, **1**(6): 1419–1457.
[8] M. ELHAMDADI, A. MAKHLOUF. *Hom-Quasi-Bialgebra*. Amer. Math. Soc., Providence, RI, 2013.
[9] I. FRENKEL, M. KHOVANOV, C. STROPPEL. *A categorification of finite-dimensional irreducible representations of quantum sl_2 and their tensor products*. Selecta Math. (N.S.), 2006, **12**(3-4): 379–431.
[10] C. KASSEL. *Quantum Groups*. Springer-Verlag, New York, 1995.
[11] M. KHOVANOV. *A categorification of the Jones polynomial*. Duke Math. J., 2000, **101**(3): 359–426.
[12] Jinqi LI. *Dual quasi-Hopf algebras and antipodes*. Algebra Colloq., 2006, **13**(1): 111–118.
[13] S. MAJID. *Tannaka-Krein Theorem for Quasi-Hopf Algebras and Other Results*. Amer. Math. Soc., Providence, RI, 1992.
[14] S. MACLANE. *Categories for a Working Mathematician* (2nd Ed.). Graduate Texts in Mathematics 5, Springer, 1998.
[15] A. MAKHLOUF, S. D. SILVESTROV. *Hom-algebra and Hom-coalgebras*. J. Algebra Appl., 2010, **9**(4): 553–589.
[16] P. SCHAUENBURG. *Tannaka Duality for Arbitrary Hopf Algebra*. Verlag Reinhard Fischer, Munich, 1992.
[17] P. SCHAUENBURG. *Hopf modules and double of a quasi-Hopf algebra*. Trans. Amer. Math. Soc., 2002, **354**(8): 3349–3378.
[18] D. YAU. *The Hom-Yang-Baxter equation, Hom-Lie algebras, and quasi-triangular bialgebras*. J. Phys. A, 2009, **42**(16): 1–12.