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On the Complex Oscillation of Second Order Linear Differential Equations with Entire Coefficients of $[p,q]$-Order

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Abstract In this paper, the authors investigate the zeros and growth of solutions of second order linear differential equations with entire coefficients of $[p,q]$-order and obtain some results which improve and generalize some previous results.

Keywords linear differential equations; $[p,q]$-order; $[p,q]$ exponent of convergence of zero-sequence

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1. Introduction and notations

We assume that readers are familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions [8,12,16]. In addition, we use $\sigma(f)$ and $\lambda(f)$ to denote the order and the exponent of convergence of zero sequence of meromorphic function $f(z)$, respectively. For sufficiently large $r\in[1,\infty)$, we define $\log_{i+1} r=\log_i(\log r)$ $(i\in\mathbb{N})$ and $\exp_{i+1} r=\exp(\exp_i r)$ $(i\in\mathbb{N})$ and $\exp_0 r=r=\log_0 r$, $\exp_{-1} r=\log r$.

Firstly, we will recall some notations about the finite iterated order of entire functions.

Definition 1.1 ([5,11]) *The iterated p-order of an entire function $f(z)$ is defined by*

$$\sigma_p(f)=\overline{\lim_{r\to\infty}}\frac{\log_p T(r,f)}{\log r}=\overline{\lim_{r\to\infty}}\frac{\log_{p+1} M(r,f)}{\log r}.$$

Definition 1.2 ([11]) *The finiteness degree (growth index) of the iterated order of an entire function $f(z)$ is defined by*

$$i(f)=\begin{cases}0, & \text{for } f \text{ polynomial},\\ \min\{j\in\mathbb{N}:\sigma_j(f)<\infty\}, & \text{for } f \text{ transcendental for which some}\\ & j\in\mathbb{N} \text{ with } \sigma_j(f)<\infty \text{ exists},\\ \infty, & \text{for } f \text{ with } \sigma_j(f)=\infty \text{ for all } j\in\mathbb{N}.\end{cases}$$

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Remark 1.3 By Definition 1.2, we can similarly give the definition of the growth index of the iterated exponent of convergence of zero-sequence of a meromorphic function $f(z)$ by $i_\lambda(f,0)$.

Definition 1.4 ([11]) *The iterated exponent of convergence of the zero sequence and the iterated exponent of convergence of distinct zero sequence of an entire function $f(z)$ are defined by*

$$\lambda_p(f)=\overline{\lim_{r\to\infty}}\frac{\log_p n(r,\frac{1}{f})}{\log r}=\overline{\lim_{r\to\infty}}\frac{\log_p N(r,\frac{1}{f})}{\log r}$$

and

$$\overline{\lambda}_p(f)=\overline{\lim_{r\to\infty}}\frac{\log_p \overline{n}(r,\frac{1}{f})}{\log r}=\overline{\lim_{r\to\infty}}\frac{\log_p \overline{N}(r,\frac{1}{f})}{\log r}.$$

For second order linear differential equation

$$f''+A(z)f=0 \tag{1.1}$$

where $A(z)$ is an entire function or meromorphic function of finite order, many authors have investigated the growth and zeros of non-trivial solutions of (1.1), and obtain many classical results [1–4,6].

In 1998, Kinnunen investigated equation (1.1) and obtained the following theorems, where $A(z)$ is an entire function of finite iterated order.

Theorem 1.5 ([11]) *Let $A(z)$ be an entire function with $i(A)=p$ $(p\in\mathbb{N})$. Let f_1,f_2 be two linearly independent solutions of (1.1) and denote $E=f_1f_2$. Then $i_\lambda(E)\le p+1$ and*

$$\max\{\lambda_{p+1}(f_1),\lambda_{p+1}(f_2)\}=\lambda_{p+1}(E)=\sigma_{p+1}(E)\le\sigma_p(A).$$

If $i_\lambda(E)\le p+1$, then $i_\lambda(f_1)=p+1$ holds for all solutions of type $f=c_1f_1+c_2f_2$, where c_1,c_2 are complex numbers and $c_1c_2\neq 0$.

Theorem 1.6 ([11]) *Let $A(z)$ be an entire function satisfying $i(A)=p$ $(p\in\mathbb{N})$, and $\overline{\lambda}_p(A)<\sigma_p(A)$. Then $\lambda_{p+1}(f)\le\sigma_p(A)\le\lambda_p(f)$ holds for any non-trivial solution of (1.1).*

Theorem 1.7 ([11]) *Let $A(z)$ be an entire function with $i(A)=p$ and $\sigma_p(A)=\sigma$ $(p\in\mathbb{N})$. Let f_1,f_2 be two linearly independent solutions of (1.1), such that $\max\{\lambda_p(f_1),\lambda_p(f_2)\}<\sigma$. Let $\Pi(z)\not\equiv 0$ be any entire function satisfying either $i(\Pi)<p$ or $i(\Pi)=p$ and $\sigma_p(\Pi)<\sigma$. Then any two linearly independent solutions g_1 and g_2 of the differential equation*

$$f''+(A(z)+\Pi(z))f=0 \tag{1.2}$$

satisfy $\max\{\lambda_p(g_1),\lambda_p(g_2)\}\ge\sigma$.

In recent years, some authors investigated the higher order linear differential equation with entire coefficients of $[p,q]$-order in the complex plane [13,14]. In this paper, our aim is to investigate the zeros and growth of solutions of (1.1) with entire coefficients of $[p,q]$-order and improve Theorems 1.5–1.7.

First, we introduce the definitions of $[p,q]$-order of meromorphic functions, where p,q are positive integers satisfying $p\ge q\ge 1$.

Definition 1.8 ([9,10,13–15]) *If $f(z)$ is a meromorphic function, the $[p,q]$-order of $f(z)$ is defined by*

$$\sigma_{[p,q]}(f)=\overline{\lim_{r\to\infty}}\frac{\log_p T(r,f)}{\log_q r}.$$

Especially if $f(z)$ is an entire function, the $[p,q]$-order of $f(z)$ is defined by

$$\sigma_{[p,q]}(f)=\overline{\lim_{r\to\infty}}\frac{\log_p T(r,f)}{\log_q r}=\overline{\lim_{r\to\infty}}\frac{\log_{p+1} M(r,f)}{\log_q r}.$$

If $f(z)$ is a rational function, then $\sigma_{[p,q]}(f)=0$ for any $p\geq q\geq 1$. By Definition 1.8, we have that $\sigma_{[1,1]}=\sigma(f)$, $\sigma_{[2,1]}=\sigma_2(f)$ and $\sigma_{[p+1,1]}=\sigma_{p+1}(f)$.

Remark 1.9 ([9,10]) If a meromorphic function $f(z)$ satisfies $0<\sigma_{[p,q]}(f)<\infty$, then we have

(i) $\sigma_{[p-n,q](f)}=\infty$ $(n<p)$, $\sigma_{[p,q-n](f)}=0$ $(n<q)$, $\sigma_{[p+n,q+n](f)}=1$ $(n<p)$ for $n=1,2,\ldots$.

(ii) If $[p',q']$ is any pair of integers satisfying $q'=p'+q-p$ and $p'<p$, then $\sigma_{[p',q']}(f)=0$ if $0<\sigma_{[p,q]}(f)<1$ and $\sigma_{[p',q']}(f)=\infty$ if $1<\sigma_{[p,q]}(f)<\infty$.

(iii) $\sigma_{[p',q']}(f)=\infty$ for $q'-p'>q-p$ and $\sigma_{[p',q']}(f)=0$ for $q'-p'<q-p$.

Definition 1.10 ([9,10]) *A meromorphic function $f(z)$ is said to have index-pair $[p,q]$, if $0<\sigma_{[p,q]}(f)<\infty$ and $\sigma_{[p-1,q-1]}(f)$ is not a nonzero finite number.*

Remark 1.11 ([9,10]) If $\sigma_{[p,p]}(f)$ is never greater than 1 and $\sigma_{[p',p']}(f)=1$ for some integer $p'\geq 1$, then the index-pair of $f(z)$ is defined as $[m,m]$ where $m=\inf\{p':\sigma_{[p',p']}(f)=1\}$. If $\sigma_{[p,q]}(f)$ is never nonzero finite and $\sigma_{[p'',1]}(f)=0$ for some integer $p''\geq 1$, then the index-pair of $f(z)$ is defined as $[n,1]$ where $n=\inf\{p'':\sigma_{[p'',1]}(f)=0\}$. If $\sigma_{[p,q]}(f)$ is always infinite, then the index-pair of $f(z)$ is defined to be $[\infty,\infty]$.

Remark 1.12 ([9,10]) If a meromorphic function $f(z)$ has the index-pair $[p,q]$, then $\sigma=\sigma_{[p,q]}(f)$ is called its $[p,q]$-order. For example, set $f_1(z)=e^z$, $f_2(z)=e^{e^z}$, by Remark 1.11, we have that the index-pair of $f_1(z)$ is $[1,1]$ and the index-pair of $f_2(z)$ is $[2,1]$.

Definition 1.13 ([13,14]) *The $[p,q]$ exponent of convergence of the zero-sequence and the $[p,q]$ exponent of convergence of the distinct zero-sequence of a meromorphic function $f(z)$ are defined respectively by*

$$\lambda_{[p,q]}(f)=\overline{\lim_{r\to\infty}}\frac{\log_p n(r,\frac{1}{f})}{\log_q r}=\overline{\lim_{r\to\infty}}\frac{\log_p N(r,\frac{1}{f})}{\log_q r}$$

and

$$\overline{\lambda}_{[p,q]}(f)=\overline{\lim_{r\to\infty}}\frac{\log_p \overline{n}(r,\frac{1}{f})}{\log_q r}=\overline{\lim_{r\to\infty}}\frac{\log_p \overline{N}(r,\frac{1}{f})}{\log_q r}.$$

Remark 1.14 It is easy to know $\overline{\lambda}_{[p,q]}(f)\leq\lambda_{[p,q]}(f)\leq\sigma_{[p,q]}(f)$.

2. Main results

In this section, we give our results of this paper.

Theorem 2.1 *Let $A(z)$ be a transcendental entire function with $\sigma_{[p,q]}(A) \geq 0$. Let f_1, f_2 be two linearly independent solutions of (1.1) and denote $E = f_1 f_2$. Then*

$$\max\{\lambda_{[p+1,q]}(f_1), \lambda_{[p+1,q]}(f_2)\} = \lambda_{[p+1,q]}(E) = \sigma_{[p+1,q]}(E) \leq \sigma_{[p,q]}(A).$$

If $\sigma_{[p+1,q]}(E) < \sigma_{[p,q]}(A)$, then $\lambda_{[p+1,q]}(f) = \sigma_{[p,q]}(A)$ holds for all solutions of type $f = c_1 f_1 + c_2 f_2$, where c_1, c_2 are complex numbers and $c_1 c_2 \neq 0$.

Theorem 2.2 *Let $A(z)$ be an entire function with $\overline{\lambda}_{[p,q]}(A) < \sigma_{[p,q]}(A)$. Then any non-trivial solution of (1.1) satisfies $\lambda_{[p+1,q]}(f) \leq \sigma_{[p,q]}(A) \leq \lambda_{[p,q]}(f)$.*

Theorem 2.3 *Let $A(z)$ be a transcendental entire function with $\sigma_{[p,q]}(A) = \sigma > 0$. Let f_1 and f_2 be two linearly independent solutions of (1.1) such that $\max\{\lambda_{[p,q]}(f_1), \lambda_{[p,q]}(f_2)\} < \sigma$. Let $\Pi(z) \not\equiv 0$ be an entire function with $\sigma_{[p,q]}(\Pi) < \sigma$. Then any two linearly independent solutions g_1 and g_2 of (1.2) satisfy $\max\{\lambda_{[p,q]}(g_1), \lambda_{[p,q]}(g_2)\} \geq \sigma$.*

3. Preliminary lemmas

Lemma 3.1 ([14]) *Let $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ be entire functions satisfying*

$$\max\{\sigma_{[p,q]}(A_j) | j \neq 0\} < \sigma_{[p,q]}(A_0) < \infty.$$

Then every non-trivial solution $f(z)$ of the differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_0(z) f = 0 \qquad (3.1)$$

satisfies $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$.

Lemma 3.2 *Let $f_1(z), f_2(z)$ be two entire function of $[p,q]$-order, and denote $E = f_1 f_2$. Then*

$$\lambda_{[p,q]}(E) = \max\{\lambda_{[p,q]}(f_1), \lambda_{[p,q]}(f_2)\}.$$

Proof Let $n(r, E)$ denote the number of the zeros of $E(z)$ in disk $= \{z : |z| \leq r\}$, and so on for f_1 and f_2. Since for any given $r > 0$ we have $n(r, E) \geq n(r, f_1)$ and $n(r, E) \geq n(r, f_2)$, by Definition 1.13 we have

$$\lambda_{[p,q]}(E) \geq \max\{\lambda_{[p,q]}(f_1), \lambda_{[p,q]}(f_2)\}.$$

On the other hand, since the zero of $E(z)$ must be the zero of f_1 or f_2, for any given $r > 0$, we have

$$n(r, E) = n(r, f_1) + n(r, f_2) \leq 2 \max\{n(r, f_1), n(r, f_2)\}. \qquad (3.2)$$

Therefore, by Definition 1.13, we have

$$\lambda_{[p,q]}(E) \leq \max\{\lambda_{[p,q]}(f_1), \lambda_{[p,q]}(f_2)\}.$$

Thus we complete the proof of Lemma 3.2. □

Lemma 3.3 *Let $f(z)$ be a meromorphic function with $[p,q]$-order and $\sigma_{[p,q]}(f) = \sigma$, and let $k \geq 1$ be an integer. Then for any $\varepsilon > 0$,*

$$m(r, \frac{f^{(k)}}{f}) = O\{\exp_{p-1}\{(\sigma + \varepsilon) \log_q r\}\} \qquad (3.3)$$

holds outside of an exceptional set E_1 of finite linear measure.

Proof Let $k\geq 1$. Since $\sigma=\sigma_{[p,q]}(f)<\infty$, we have for all sufficiently large r,

$$T(r,f)<\exp_p\{(\sigma+\varepsilon)\log_q r\}. \tag{3.4}$$

By the lemma of the logarithmic derivative, we have

$$m(r,\frac{f^{(k)}}{f})=O\{\log T(r,f)+\log r\},\quad r\notin E_1$$

where $E_1\subset(1,\infty)$ is a set of finite linear measure, not necessarily the same at each occurrence. Hence we have

$$m(r,\frac{f'}{f})=O\{\exp_{p-1}\{(\sigma+\varepsilon)\log_q r\}\},\quad r\notin E_1. \tag{3.5}$$

Next, assume that we have

$$m(r,\frac{f^{(k)}}{f})=O\{\exp_{p-1}\{(\sigma+\varepsilon)\log_q r\}\},\quad r\notin E_1 \tag{3.6}$$

for some $k\in\mathbb{N}$. Since $N(r,f^{(k)})\leq(k+1)N(r,f)$, there holds

$$\begin{aligned}T(r,f^{(k)})&\leq m(r,f^{(k)})+N(r,f^{(k)})\leq m(r,\frac{f^{(k)}}{f})+m(r,f)+(k+1)N(r,f)\\&\leq(k+1)T(r,f)+O\{\exp_{p-1}\{(\sigma+\varepsilon)\log_q r\}\},\quad r\notin E_1.\end{aligned} \tag{3.7}$$

By (3.5), we again obtain

$$m(r,\frac{f^{(k+1)}}{f^{(k)}})=O\{\exp_{p-1}\{(\sigma+\varepsilon)\log_q r\}\},\quad r\notin E_1, \tag{3.8}$$

and hence,

$$m(r,\frac{f^{(k+1)}}{f})\leq m(r,\frac{f^{(k+1)}}{f^{(k)}})+m(r,\frac{f^{(k)}}{f})=O\{\exp_{p-1}\{(\sigma+\varepsilon)\log_q r\}\},\quad r\notin E_1.\ \square \tag{3.9}$$

Lemma 3.4 ([12]) *Let $g:[0,\infty)\longrightarrow R$ and $h:[0,\infty)\longrightarrow R$ be monotone increasing functions such that $g(r)\leq h(r)$ outside of an exceptional set E of finite linear measure. Then for any $\alpha>1$, there exists $r_0>0$, such that $g(r)\leq h(\alpha r)$ for all $r>r_0$.*

Lemma 3.5 ([7]) *Let $f(z)$ be a transcendental meromorphic function not of the form $e^{\alpha z+\beta}$. Then*

$$T(r,\frac{f}{f'})\leq 3\overline{N}(r,f)+7\overline{N}(r,\frac{1}{f})+4\overline{N}(r,\frac{1}{f''})+S(r,\frac{f}{f'}). \tag{3.10}$$

Similarly to the Hadamard theorem for entire functions and Lemma 1.8 in [11, p.390], we have the following results.

Lemma 3.6 *An entire function $f(z)$ with $[p,q]$ index can be represented by the form $f(z)=U(z)e^{V(z)}$, where $U(z)$ and $V(z)$ are entire functions such that*

$$\lambda_{[p,q]}(f)=\lambda_{[p,q]}(U)=\sigma_{[p,q]}(U),\quad \sigma_{[p,q]}(f)=\max\{\sigma_{[p,q]}(U),\sigma_{[p,q]}(e^V)\}. \tag{3.11}$$

4. Proofs of Theorems

Proof of Theorem 2.1 We denote $\sigma_{[p,q]}(A) = \sigma$. By Lemma 3.1 we have

$$\sigma_{[p+1,q]}(f_1) = \sigma_{[p+1,q]}(f_2) = \sigma.$$

Therefore,

$$\sigma_{[p+1,q]}(E) \leq \max\{\sigma_{[p+1,q]}(f_1), \sigma_{[p+1,q]}(f_2)\} = \sigma.$$

By Lemma 3.2, we know

$$\max\{\lambda_{[p+1,q]}(f_1), \lambda_{[p+1,q]}(f_2)\} = \lambda_{[p+1,q]}(E) \leq \sigma_{[p+1,q]}(E). \tag{4.1}$$

It remains to show that $\lambda_{[p+1,q]}(E) = \sigma_{[p+1,q]}(E)$. Assume that $\lambda_{[p+1,q]}(E) < \sigma_{[p+1,q]}(E)$. We obtain that all zeros of E are simple and that [12, pp.76-77]

$$E^2 = C^2\big((\frac{E'}{E})^2 - 2\frac{E''}{E} - 4A\big)^{-1}. \tag{4.2}$$

Hence,

$$\begin{aligned} 2T(r, E) &= T\big(r, (\frac{E'}{E})^2 - 2\frac{E''}{E} - 4A\big) + O(1) \\ &\leq O\big(\overline{N}(r, \frac{1}{E}) + m(r, \frac{E'}{E}) + m(r, \frac{E''}{E}) + m(r, A)\big). \end{aligned} \tag{4.3}$$

By Lemma 3.3, we have

$$m(r, \frac{E'}{E}) = O\{\exp_p\{(\sigma + \varepsilon)\log_q r\}\}, \quad m(r, \frac{E''}{E}) = O\{\exp_p\{(\sigma + \varepsilon)\log_q r\}\}, \quad r \notin E.$$

Since $\overline{N}(r, \frac{1}{E}) = N(r, \frac{1}{E}) = O\{\exp_{p+1}\{\beta \log_q r\}\}$ holds for some $\beta < \sigma_{[p+1,q]}(E)$, we obtain

$$T(r, E) = O\big(\overline{N}(r, \frac{1}{E}) + \exp_p\{(\sigma + \varepsilon)\log_q r\}\big), \quad r \notin E_1. \tag{4.4}$$

By (4.4), we have $T(r, E) = O\{\exp_{p+1}\{\beta \log_q r\}\}$ $(r \notin E)$ and by Lemma 3.4, we obtain $\sigma_{[p+1,q]}(E) \leq \beta < \sigma_{[p+1,q]}(E)$, this is a contradiction. Hence, $\lambda_{[p+1,q]}(E) = \sigma_{[p+1,q]}(E)$.

If $\sigma_{[p+1,q]}(E) < \sigma_{[p,q]}(A)$, let us assume $\lambda_{[p+1,q]}(f) < \sigma_{[p,q]}(A)$ for any solution of type $f = c_1 f_1 + c_2 f_2$ $(c_1 c_2 \neq 0)$. We denote $E = f_1 f_2$ and $F = f f_1$, then

$$\lambda_{[p+1,q]}(E) < \sigma_{[p,q]}(A), \quad \lambda_{[p+1,q]}(F) < \sigma_{[p,q]}(A).$$

Since $F = (c_1 f_1 + c_2 f_2) f_1 = c_1 f_1^2 + c_2 E$, by (4.4), we have

$$T(r, f_1) = O(T(r, F) + T(r, E)) = O\big(\overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{E}) + \exp_p\{(\sigma + \epsilon)\log_q r\}\big).$$

Since $\lambda_{[p+1,q]}(E) < \sigma_{[p,q]}(A), \lambda_{[p+1,q]}(F) < \sigma_{[p,q]}(A)$, we have

$$\overline{N}(r, \frac{1}{F}) < \exp_{p+1}\{\beta \log_q r\}, \quad \overline{N}(r, \frac{1}{E}) < \exp_{p+1}\{\beta \log_q r\}, \quad r \to \infty,$$

for some $\beta < \sigma_{[p,q]}(A)$. Thus we obtain $\sigma_{[p+1,q]}(f_1) \leq \beta < \sigma_{[p,q]}(A)$, this is a contradiction by Lemma 3.1. Hence we have that $\lambda_{[p+1,q]}(f) = \sigma_{[p,q]}(A)$ holds for all solutions of type $f = c_1 f_1 + c_2 f_2$, where $c_1 c_2 \neq 0$. □

Proof of Theorem 2.2 By Lemma 3.1 we have $\lambda_{[p+1,q]}(f) \leq \sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A)$. It remains to show that $\sigma_{[p,q]}(A) \leq \lambda_{[p,q]}(f)$. We assume that $\sigma_{[p,q]}(A) > \lambda_{[p,q]}(f)$. Since $A(z)$ is transcendental, the non-trivial solution of (1.1) is transcendental entire function of infinite order. Hence, by Lemma 3.5, we have for sufficiently large r

$$T(r,\frac{f}{f'})=O\big(\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{f''})\big),\quad r\notin E_1. \tag{4.5}$$

By $\overline{\lambda}_{[p,q]}(A)<\sigma_{[p,q]}(A)$ and the assumption $\lambda_{[p,q]}(f)<\sigma_{[p,q]}(A)$, form (4.5), we have for sufficiently large r

$$T(r,\frac{f}{f'})=O\{\exp_p\{\beta\log_q r\}\},\quad r\notin E_1 \tag{4.6}$$

for some $\beta<\sigma_{[p,q]}(A)$. Hence,

$$\sigma_{[p,q]}(\frac{f}{f'})=\sigma_{[p,q]}(\frac{f'}{f})\le\beta<\sigma_{[p,q]}(A).$$

Since

$$-A(z)=(\frac{f'}{f})'+(\frac{f'}{f})^2, \tag{4.7}$$

we obtain $\sigma_{[p,q]}(A)\le\sigma_{[p,q]}(\frac{f'}{f})<\sigma_{[p,q]}(A)$, this is a contradiction. Thus $\sigma_{[p,q]}(A)\le\lambda_{[p,q]}(f)$. □

Proof of Theorem 2.3 Similarly to the proof of Theorem 3.1 in [4], we denote $E=f_1f_2$ and $F=g_1g_2$. Let us assume

$$\lambda_{[p,q]}(F)=\max\{\lambda_{[p,q]}(g_1),\lambda_{[p,q]}(g_2)\}<\sigma.$$

By Lemma 3.1, we have $\sigma_{[p+1,q]}(E)\le\max\{\sigma_{[p+1,q]}(f_1),\sigma_{[p+1,q]}(f_2)\}=\sigma$, and hence, by Lemma 3.3, for any integer $k\ge1$ and for any $\varepsilon>0$, we have

$$m(r,\frac{E^{(k)}}{E})=O\{\exp_p\{(\sigma+\varepsilon)\log_q r\}\},\quad r\notin E_1.$$

Furthermore, by the assumption $\lambda_{[p,q]}(E)<\sigma$, we have $\overline{N}(r,\frac{1}{E})=O\{\exp_p\{\beta\log_q r\}\}$ for some $\beta<\sigma$, and the $[p,q]$-order of the function $A(z)$ implies that

$$T(r,A)=O\{\exp_p\{(\sigma+\varepsilon)\log_q r\}\},\quad r\to\infty.$$

By (4.4), we obtain

$$T(r,E)=O\{\exp_p\{(\sigma+\varepsilon)\log_q r\}\}$$

and hence, $\sigma_{[p,q]}(E)\le\sigma$. On the other hand, by

$$4A=(\frac{E'}{E})^2-2\frac{E''}{E}-\frac{1}{E^2}, \tag{4.8}$$

we have that $\sigma_{[p,q]}(A)=\sigma\le\sigma_{[p,q]}(E)$, hence $\sigma_{[p,q]}(E)=\sigma$. By the same reasoning for the function F, we have

$$4(A+\Pi)=(\frac{F'}{F})^2-2\frac{F''}{F}-\frac{1}{F^2} \tag{4.9}$$

and $\sigma_{[p,q]}(F)=\sigma$. Since $\lambda_{[p,q]}(E)<\sigma,\lambda_{[p,q]}(F)<\sigma$, by Lemma 3.6, we may write

$$E=Qe^P,\quad F=Re^S, \tag{4.10}$$

where P, Q, R, S are entire functions satisfying $\sigma_{[p,q]}(Q) = \lambda_{[p,q]}(E) < \sigma$, $\sigma_{[p,q]}(R) = \lambda_{[p,q]}(F) < \sigma$ and $\sigma_{[p,q]}(e^P) = \sigma_{[p,q]}(e^S) = \sigma$. Substituting (4.10) into (4.8) and (4.9), we have

$$4A = -\frac{1}{Q^2e^{2P}} + G_1(z), \tag{4.11}$$

$$4(A+\Pi) = -\frac{1}{R^2e^{2S}} + G_2(z), \tag{4.12}$$

where $G_1(z)$ and $G_2(z)$ are meromophc functions satisfying $\sigma_{[p,q]}(G_j) < \sigma$ $(j = 1, 2)$. Subtracting (4.12) from (4.11) gives

$$\frac{1}{R^2e^{2S}} - \frac{1}{Q^2e^{2P}} = G_3(z), \tag{4.13}$$

where $G_3(z)$ is a meromophic function satisfying $\sigma_{[p,q]}(G_3) < \sigma$. From (4.13), we have

$$e^{-2S} + H_1e^{-2P} = H_2, \tag{4.14}$$

where H_1, H_2 are meromorphic functions satisfying $\sigma_{[p,q]}(H_j) < \sigma$ $(j = 1, 2)$, and $H_1 = -\frac{R^2}{Q^2}$. Derivating (4.14), we have

$$-2S'e^{-2S} + (H_1' - 2PH_1)e^{-2P} = H_3, \tag{4.15}$$

where H_3 is a meromophic function with $\sigma_{[p,q]}(H_3) < \sigma$. Eliminating e^{-2S} by (4.14) and (4.15), we have

$$(H_1' - 2(P' - S')H_1)e^{-2P} = H_4, \tag{4.16}$$

where H_4 is a meromorphic function satisfying $\sigma_{[p,q]}(H_4) < \sigma$. Since $\sigma_{[p,q]}(e^S) = \sigma$, by (4.16) we have $H_1' - 2(P' - S')H_1 \equiv 0$, thus we have $H_1 = ce^{2(P-S)}$, $c \neq 0$. Hence,

$$\frac{E^2}{F^2} = \frac{Q^2}{R^2}e^{2(P-S)} = -\frac{1}{c}. \tag{4.17}$$

From (4.8), (4.9), (4.17), we have

$$4(A + \Pi + \frac{1}{c}A) = (\frac{F'}{F})^2 - 2\frac{F''}{F} + \frac{1}{c}(\frac{E'}{E})^2 - \frac{2}{c}\frac{E''}{E}.$$

By Lemma 3.3, we obtain

$$T(r, A(1 + \frac{1}{c}) + \Pi) = m(r, A(1 + \frac{1}{c}) + \Pi) = O\{\exp_{p-1}\{(\sigma + \varepsilon)\log_q r\}\}, \quad r \to \infty.$$

This implies $\sigma_{[p,q]}(A(1 + \frac{1}{c}) + \Pi) = 0$. Hence $c = -1$. Since $E^2 = F^2$, we have

$$\frac{E'}{E} = \frac{F'}{F}, \quad \frac{E''}{E} = \frac{F''}{F}.$$

From (4.8) and (4.9), we see that $\Pi(z) \equiv 0$, this is a contradiction. The proof of the theorem is completed. □

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