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Existence of Exponential Attractors for Semigroup in Banach Space

Yongjun LI*, **Xiaoxian ZHANG**, **Zilong ZANG**
School of Mathematics, Lanzhou City University, Gansu 730070*, P. R. China*

Abstract We construct an exponential attractor for semigroup in Banach space by using ω-limit compactness method and provide a new method to prove the existence of an exponential attractor in uniformly convex Banach space. As a simple application, we prove the existence of an exponential attractor for reaction diffusion equations.

Keywords dynamical system; exponential attractors; fractal dimension; measure of Non-compactness

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1. Introduction

Global attractors is a suitable concept to describe the asymptotic behavior of infinite dimension dynamical system or semigroup generated by autonomous partial differential equations [1–3]. Many authors have paid much attention to this problem and have made much successful progress, see [1–5] and the references therein.

Global attractors is a compact invariant set attracting all bounded subsets of the phase space, but the attraction to it may be arbitrarily slow. To overcome this drawback creates the notion of the exponential attractors [6–12], a compact, positively invariant set of finite fractal dimension, which exponentially attracts each bounded subset. In 1994, Eden et al. [6] gave a method firstly, intending to construct an exponential attractor via the squeezing property in Hilbert space. Later in 2000, Efendiev et al. [7] proved the existence of an exponential attractor if the smooth property is replaced by the asymptotic smooth property. In 2002, Málek and Pražák proved the existence of an exponential attractor by the l-trajectory method [8]. These methods have been applied to a variety of concrete equations [9–12]. As we all know, for some problems it is difficult to verify exponential attractors by using these methods.

In this paper, we construct exponential attractor for semigroup in Banach space by using ω-limit compactness method. First, we give a sufficient condition for the existence of exponential attractor. Then we provide a new method to prove the existence of an exponential attractor in

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* Corresponding author
E-mail address: li_liyong120@163.com (Yongjun LI); zhangxx@lzcu.edu.cn (Xiaoxian ZHANG); zangzl@lzcu.edu.cn (Zilong ZANG)

uniformly convex Banach space. As a simple application, we will use our method to prove the existence of an exponential attractor for reaction diffusion equations.

2. Preliminaries

Let X be a complete metric space. A one parameter family of (nonlinear) mappings $S(t): X \to X$ $(t \geq 0)$ is called a semigroup provided that

(1) S(0)=I;

(2) $S(t+s) = S(t)S(s)$ for all $t, s \geq 0$.

The pair $(S(t), X)$ is usually referred to as a dynamical system, and $(S(n), X)$ $(n \in \mathbb{N})$ is called the discrete dynamical system generated by $(S(t), X)$. We say $\{S(t)\}_{t\geq 0}$ is a norm-to-weak continuous semigroup on X, if $t_n \to t$ and $x_n \to x$, then $S(t_n)x_n \rightharpoonup S(t)x$.

A set $\mathcal{A} \subset X$ is called the global attractor for $(S(t), X)$ or $(S(n), X)$ if (i) $\mathcal{A}$ is compact in X, (ii) $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$ or $S(n)\mathcal{A} = \mathcal{A}$ for all $n \in \mathbb{N}$ and (iii) for any $B \subset X$ that is bounded, $\mathrm{dist}(S(t)B, \mathcal{A}) \to 0$ as $t \to \infty$ or $\mathrm{dist}(S(n)B, \mathcal{A}) \to 0$ as $n \to \infty$, where $\mathrm{dist}(B, \mathcal{A}) = \sup_{b\in B} \inf_{a\in\mathcal{A}} \| b - a \|_X$.

A set B is called a bounded absorbing set to $(S(t), X)$, if for any bounded set $B_0 \subset X$, there exists $t_0 = t_0(B_0)$ such that $S(t)B_0 \subset B$ for all $t \geq t_0$. A set E is called positively invariant w.r.t $S(t)$ if for all $t \geq 0$, $S(t)E \subset E$.

Theorem 2.1([1–4]) *Continuous semigroup $S(t)$ has a global attractor $\mathcal{A}$ if and only if $S(t)$ has a bounded absorbing set B and for an arbitrary sequence of points $x_n \in B$, the sequence $S(t_n)x_n$ has a convergence subsequence in B.*

In fact, we know that

$$\mathcal{A} = \bigcap_{t\geq 0} \overline{\bigcup_{s\geq t} S(s)B}.$$

For discrete dynamical system $(S(n), X)$, the same conclusion holds true.

Now, we briefly review the basic concept about the Kuratowski measure of non-compactness and restate its basic property, which will be used to characterize the existence of exponential attractors for dynamical system $(S(t), X)$.

Let X be a Banach space and B be a bounded subset of X. The Kuratowski measure of non-compactness $\alpha(B)$ of B is defined as

$$\alpha(B) = \inf\{\delta > 0 \mid B \text{ admits a finite cover by sets of diameter} \leq \delta\}.$$

The following summarizes some of the basic properties of this measure of non-compactness [13].

Lemma 2.2 *Let $B,\ B_1,\ B_2 \subset X$. Then*

(1) $\alpha(B) = 0$ if, and only if, $\overline{B}$ is compact;

(2) $\alpha(B_1 + B_2) \leq \alpha(B_1) + \alpha(B_2)$;

(3) $\alpha(B_1) \leq \alpha(B_2)$ for $B_1 \subset B_2$;

(4) $\alpha(B_1 \bigcup B_2) \leq \max\{\alpha(B_1), \alpha(B_2)\}$;

(5) If $F_1 \supset F_2 \ldots$ are non-empty closed sets in X such that $\alpha(F_n) \to 0$ as $n \to \infty$, then $F = \bigcap_{n=1}^{\infty} F_n$ is nonempty and compact.

In addition, let X be an infinite dimensional Banach space with a decomposition $X = X_1 \oplus X_2$ and let $P : X \to X_1$, $Q : X \to X_2$ be projectors with $\dim X_1 < \infty$. Then

(6) $\alpha(B(\varepsilon))) = 2\varepsilon$, where $B(\varepsilon)$ is a ball of radius ε;

(7) $\alpha(B) < \varepsilon$ for any bounded subset B of X for which the diameter of QB is less than ε.

Definition 2.3 ([4,5]) *A semigroup $S(t)$ is called ω-limit compact if for every bounded subset B of X and for any $\varepsilon > 0$, there exists a $t_0 > 0$ such that $\alpha(\bigcup_{t \geq t_0} S(t)B) \leq \varepsilon$.*

Lemma 2.4 ([4,5]) *Assume semigroup $S(t)$ is ω-limit compactness, then for any sequence $t_n \in R^+, t_n \to \infty$, as $n \to \infty$, and any sequence $x_n \in B$, there exists a convergent subsequence of $\{S(t_n)x_n\}$ whose limit lies in $\omega(B)$, where B is bounded set in X and $\omega(B)$ is ω-limit set of B defined by*

$$\omega(B) = \bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} S(s)B}.$$

Definition 2.5 ([1–3]) *Let $n(\mathcal{M}, \varepsilon)$, $\varepsilon > 0$, denote the minimum number of the balls in X of radius ε which is necessary to cover $\mathcal{M}$. The fractal dimension of $\mathcal{M}$, which is also called the capacity of $\mathcal{M}$, is the number*

$$\dim_f \mathcal{M} = \overline{\lim_{\varepsilon \to 0^+}} \frac{\ln n(\mathcal{M}, \varepsilon)}{\ln \frac{1}{\varepsilon}}.$$

Lemma 2.6 ([1–3]) *Let g be a Lipschitzian mapping of one metric space into another. Then*

$$\dim_f g(M) \leq \dim_f M.$$

Lemma 2.7 ([1–3]) *Assume that $M_1 \times M_2$ is a direct product of two sets. Then*

$$\dim_f(M_1 \times M_2) \leq \dim_f M_1 + \dim_f M_2.$$

Lemma 2.8 ([1–3]) *Let B_R be a ball of the radius R in R^d equipped with Euclidean norm $|\cdot|$. Then for any $\varepsilon > 0$ there exists a finite set $\{x_k : k = 1, 2, \ldots, n_\varepsilon\} \subset B_R$ such that $B_R \subset \bigcup_{k=1}^{n_\varepsilon} \{x \in R^d : |x - x_k| < \varepsilon\}$ and $n_\varepsilon \leq (1 + \frac{2R}{\varepsilon})^d$.*

Definition 2.9 *Let $S(t)$ or $S(n) : X \to X$ be a semigroup on a complete metric space X. A set $\mathcal{M} \subset X$ is called an exponential attractor for $S(t)$ or $S(n)$ if the following properties hold:*

(1) The set $\mathcal{M}$ is compact in X and has finite fractal dimension.

(2) The set $\mathcal{M}$ is positively invariant, i.e., $S(t)\mathcal{M} \subset \mathcal{M}$ or $S(n)\mathcal{M} \subset \mathcal{M}$.

(3) The set $\mathcal{M}$ is an exponentially attracting set for the semigroup $S(t)$ or $S(n)$, i.e., there exists a constant $l > 0$ such that, for every bounded subset $B \subset X$, there exists a constant $k = k(B) > 0$ such that

$$\mathrm{dist}(S(t)B, \mathcal{M}) \leq ke^{-lt} \ \text{ or } \ \mathrm{dist}(S(n)B, \mathcal{M}) \leq ke^{-ln}.$$

3. Exponential attractor

Let X be a complete metric space, B be a bounded absorbing set for $S(t)$ or $S(n)$ on X, and $S(t)$ or $S(n)$ be a continuous semigroup in B. Without loss of generality, we can suppose that B is a positively invariant bounded absorbing set, as, if not, we could consider $\bigcup_{t\geq t_B} S(t)B$ as a new bounded absorbing set for some T_B and $S(t)B\subset B$ for all $t\geq t_B$. Next we will restrict the semigroup $S(t)$ or $S(n)$ in B and the same conclusion holds true.

Theorem 3.1 *Let $S(n)$ be a discrete dynamical system in X, and B be a positively invariant bounded absorbing set for $S(n)$. Then the following conclusions hold.*

(1) For some $0<r<1$, $K,N\in\mathbb{N}_+$, $\forall n\in\mathbb{N}$, there exist $x_{n1},x_{n2},\dots,x_{nN_n}\in S(n)B$ such that $S(n)B\subset\bigcup_{i=1}^{N_n}B(x_{ni},r^n)$ and $N_n\leq KN^n$. Then the semigroup $S(n)$ has exponential attractor.

(2) $S(n)$ has exponential attractor, then there exist $0<r<1$, $N\in\mathbb{N}$, $\forall n>N$, and $x_{n1},x_{n2},\dots,x_{nN_n}\in S(n)B$ such that $S(n)B\subset\bigcup_{i=1}^{N_n}B(x_{ni},r^n)$ and $N_n\leq KN^n$.

Proof (1) By Lemma 2.2 we know that the measure of non-compactness $\alpha(S(n)B)\leq 2r^n\to 0$. From the definition 2.3, we find that $S(n)$ is ω-limit compact. By Lemma 2.4 and Theorem 2.1, we get

$$\mathcal{A}=\bigcap_{n=1}^{\infty}\overline{\bigcup_{m\geq n}S(m)B}, \tag{3.1}$$

is the global attractor of $S(n)$.

By the condition of (1), we know that $S(n)B\subset\bigcup_{i=1}^{N_n}B(x_{n_i},r^n)$. Let $E_n=\{x_{n_i}\}$, setting $E_0=\bigcup_{m,k\geq 0}S(m)E_k$, and $\mathcal{M}=\mathcal{A}\bigcup E_0$. We claim that $\mathcal{M}$ is an exponential attractor of $S(n)$.

(Positively invariant) Since $S(n)\mathcal{A}=\mathcal{A}, S(n)E_0\subset E_0$, we get $S(n)\mathcal{M}\subset\mathcal{M}$.

(Exponential attraction) It is evident that $\operatorname{dist}(S(n)B,\mathcal{M})\leq\operatorname{dist}(S(n)B,E_n)$, $S(n)B\subset\bigcup_{i=1}^{N_n}B(x_{n_i},r^n)$, so we get $\operatorname{dist}(S(n)B,\mathcal{M})\leq r^n=e^{-(\ln\frac{1}{r})n}$, i.e., $\mathcal{M}$ exponentially attracting bounded set.

(Compactness) For any sequence $y_n\in\mathcal{M}$, there exist $x_n\in B$, $m_n\in\mathbb{N}$ such that $y_n=S(m_n)x_n$. By the condition of (1), $S(n)$ is ω-limit compactness, and by Lemma 2.4, y_n has a convergent subsequence in $\mathcal{A}$. Then we get $\mathcal{M}$ is compact.

(Boundedness of fractal dimension) Since B is a positively invariant bounded absorbing set, and $\mathcal{A}$ is the global attractor, we have

$$\mathcal{A}\subset\overline{S(n)B},\ \ S(n+1)B\subset S(n)B,\ \ \forall n\in\mathbb{N}.$$

For any $\varepsilon>0$, there exists $n_0=[\frac{\ln\varepsilon}{\ln r}]+1$ and $r^{n_0}<\varepsilon$. $\forall n>n_0$,

$$E_{n+1}\subset S(n+1)B\subset S(n)B\subset S(n_0)B\subset\bigcup_{i=1}^{N_{n_0}}B(x_{n_0i},r^{n_0})\subset\bigcup_{i=1}^{N_{n_0}}B(x_{n_0i},\varepsilon).$$

$$\mathcal{M}=\mathcal{A}\bigcup E_0\subset\mathcal{A}\bigcup\{\bigcup_{m+k\leq n_0-1}S(m)E_k\}\bigcup\{\bigcup_{i=1}^{N_{n_0}}B(x_{n_0i},r^{n_0})\}$$

$$\subset \bigcup_{m+k\le n_0-1} S(m)E_k \bigcup \{\bigcup_{i=1}^{N_{n_0}} B(x_{n_0 i}, r^{n_0})\}.$$

Without loss of generality, we can assume that $N > 1$. Let $E = \bigcup_{m+k\le n_0} S(m)E_k$. Hence the number of the points in E is $\sum_{m=0}^{n_0}\sum_{k=0}^{n_0-m} KN^k \le \frac{KN^{n_0+2}}{(N-1)^2}$ at most, and $\dim_f^X \mathcal{M} \le \lim_{\varepsilon\to 0^+}\sup \frac{\ln n_\varepsilon^X(\mathcal{M})}{\ln\frac{1}{\varepsilon}} \le \lim_{\varepsilon\to 0^+}\sup \frac{\ln KN^{n_0+2}}{\ln\frac{1}{\varepsilon}} \le \lim_{\varepsilon\to 0^+}\sup \frac{(n_0+2)\ln N}{\ln\frac{1}{\varepsilon}} \le \lim_{\varepsilon\to 0^+} \frac{(\frac{\ln\varepsilon}{\ln r}+3)\ln N}{\ln\frac{1}{\varepsilon}} = \frac{\ln N}{\ln\frac{1}{r}}$.

(2) Let $\mathcal{M}$ be the exponential attractor for $S(n)$, i.e., there exist $k, l > 0$, such that $\mathrm{dist}(S(n)B, \mathcal{M}) \le ke^{-ln}$. Since the $\lim_{n\to+\infty} ke^{-ln} = 0$, we get there exists $N_0 \in \mathbb{N}$, $\forall n > N_0$, $\mathrm{dist}(S(n)B, \mathcal{M}) \le r_0^n$ and $r_0 = k^{\frac{1}{N_0}} e^{-l} < 1$, which implies that $S(n)B \subset B(\mathcal{M}, r_0^n)$. The fractal dimension is bounded for $\mathcal{M}$. By Definition 2.5, for some $0 < r_1 < 1 - r_0$, $\dim_f^X \mathcal{M} = \overline{\lim}_{n\to+\infty} \frac{\ln n(\mathcal{M}, r_1^n)}{\ln\frac{1}{r_1^n}}$, there exists $P > 0$ such that $\frac{\ln n(\mathcal{M}, r_1^n)}{\ln\frac{1}{r_1^n}} < P$ for all $n \ge N_0$. Therefore, $n(\mathcal{M}, r_1^n) < \frac{1}{{r_1}^{np}}$. Let $r_1 < r_2 < 1 - r_0$. By Definition 2.5, there exist $x'_{n1}, x'_{n2}, \ldots, x'_{nN_n} \in X$ such that $\mathcal{M} \subset \bigcup_{i=1}^{N_n} B(x'_{ni}, r_2^n)$ and $N_n \le \frac{1}{{r_1}^{np}}$, so we we get $S(n)B \subset B(\mathcal{M}, r_0^n) \subset \bigcup_{i=1}^{N_n} B(x'_{ni}, r_3^n), r_3 = r_0 + r_2 < 1$. This gives us that there exist $x_{n1}, x_{n2}, \ldots, x_{nN_n} \in S(n)B$ such that $S(n)B \subset \bigcup_{i=1}^{N_n} B(x_{ni}, r_3^n)$, which implies there exist $0 < r < 1$, $N \in \mathbb{N}$, $\forall n > N$, and $N_n \le KN^n$ such that $S(n)B \subset \bigcup_{i=1}^{N_n} B(x_{ni}, r^n)$.

Theorem 3.2 *Assume that B is a positively invariant bounded absorbing set of the semigroup $S(t)$ in X, for some $T > 0$ and for all $(t, x) \in [0, T]\times B$, $F(t, x) = S(t)x$ is a Lipschitzian mapping. Let $T(n) = S(nT)$ and assume that there exist $0 < r < 1$, $K, N \in \mathbb{N}_+$, and $\forall n \in \mathbb{N}$, there exist $x_{n1}, x_{n2}, \ldots, x_{nN_n} \in T(n)B$ such that $T(n)B \subset \bigcup_{i=1}^{N_n} B(x_{ni}, r^n)$ and $N_n \le KN^n$. Then the semigroup $S(t)$ has an exponential attractor.*

Proof According to Theorem 3.1, $T(n)$ has an exponential attractor $\mathcal{M}$. Let $\mathcal{M}_0 = \bigcup_{0\le s\le T} S(s)\mathcal{M}$. We will prove that $\mathcal{M}_0$ is an exponential attractor of $S(t)$.

(Positively invariant) $\forall x \in \mathcal{M}_0$, there exist $\tau \in [0, T], x_0 \in \mathcal{M}$, such that $x = S(\tau)x_0$. $\forall t \ge 0$, $S(t)x = S(t+\tau)x_0$. Set $t + \tau = n_0T + \tau_0$, $n_0 \in \mathbb{N}, \tau_0 \in [0, T)$, $S(t)x = S(t+\tau)x_0 = S(\tau_0 + n_0T)x_0 = S(\tau_0)S(n_0T)x_0 = S(\tau_0)T(n_0)x_0$. Since $\mathcal{M}$ is the exponential attractor of $T(n)$, it follows that $\forall n \in \mathbb{N}$, $T(n)\mathcal{M} \subset \mathcal{M}$, and we get $T(n_0)x_0 \in \mathcal{M}$, $S(\tau_0)T(n_0)x_0 \in \bigcup_{0\le s\le T} S(s)\mathcal{M}$, i.e., $S(t)x \in \bigcup_{0\le s\le T} S(s)\mathcal{M}$, which implies $S(t)\mathcal{M}_0 \subset \mathcal{M}_0$.

(Exponential attraction) $\mathcal{M}$ is the exponential attractor of $T(t)$, which implies there exist $k, l > 0$ such that $\mathrm{dist}(T(n)B, \mathcal{M}) \le ke^{-ln}$. For all $t \ge 0$, there exist $n \in \mathbb{N}$, $\tau \in [0, T)$ such that $t = nT + \tau$. $S(t)B = S(nT+\tau)B = S(nT)S(\tau)B \subset S(nT)B = T(n)B$, $\mathrm{dist}(S(t)B, \mathcal{M}_0) \le \mathrm{dist}(S(t)B, \mathcal{M}) = \mathrm{dist}(S(nT+\tau)B, \mathcal{M}) = \mathrm{dist}(S(nT)S(\tau)B, \mathcal{M}) \le \mathrm{dist}(T(n)B, \mathcal{M}) \le ke^{-ln} = ke^{-l\frac{t-\tau}{T}} \le ke^{l}e^{-\frac{l}{T}t}$.

(Compactness) By the positively invariant property of $\mathcal{M}_0$, $\forall t \ge 0, S(t)\mathcal{M}_0 \subset \mathcal{M}_0$. For any sequence $y_n \in \mathcal{M}_0$, there exist $x_n \in \mathcal{M}_0 \subset B$, $t_n \in \mathbb{R}^+$ such that $y_n = S(t_n)x_n$. Let $t_n = k_nT + \tau_n$, $\tau_n \in [0, T)$. Then $y_n = S(k_nT)S(\tau_n)x_n = T(k_n)z_n$, where $z_n = S(\tau_n)x_n \in B$. By Theorem 3.1, we know that $T(n)$ is ω-limit compact, which implies that $\mathcal{M}_0$ is compact.

(Boundedness of fractal dimension) $F(t, x)$ is also a Lipschitzian mapping on $[0, T] \times \mathcal{M}$.

By Lemmas 2.6 and 2.7, we know that

$$\dim_f \mathcal{M}_0 = \dim_f F([0,T]\times\mathcal{M}) \le \dim_f([0,T]\times\mathcal{M}) \le 1+\dim_f \mathcal{M}.$$

$\mathcal{M}$ is the exponential attractor of $T(n)$, we get that the $\dim_f \mathcal{M}$ is bounded, which implies the fractal dimension of $\mathcal{M}_0$ is bounded.

We now present a method to verify the existence of exponential attractor for semigroup.

Let X be a uniformly convex Banach space, i.e., for all $\varepsilon>0$, there exists $\delta>0$ such that, given $x,y\in X$, $||x||\le 1, ||y||\le 1, ||x-y||>\varepsilon$, then $\frac{||x+y||}{2}<1-\delta$. Requiring a space to be uniformly convex is not a severe restriction in application, since this property is satisfied by all Hilbert space, the L^p space with $1<p<\infty$, and most Sobolev space $W^{k,p}$ with $1<p<\infty$. $\square$

Theorem 3.3 *Let X be a uniformly convex Banach space, and B be a bounded absorbing set of $S(t)$. For some $0<\theta<1$, there exists $T>0$ such that*

(1) $\|S(t)v_1-S(t)v_2\|\le l\|v_1-v_2\|, t\in[0,T], v_1,v_2\in B$,

(2) $\|S(t_1)v-S(t_2)v\|\le k|t_1-t_2|$, $t_1,t_2\in[0,T]$, $v\in B$,

(3) $\|\, (I-P_m)(\bigcup_{t\ge T} S(t)x)\, \|\le\theta$, $\forall x\in B$,

(4) $\|(I-P_m)(S(T)v_1-S(T)v_2)\|\le\delta\|v_1-v_2\|, v_1,v_2\in B$,

where $\delta,k>0$ and $\theta+\delta<1$, $P_m: X\to X_1$ is a bounded projector, in which m is the dimension of X_1. Then the semigroup $S(t)$ possesses an exponential attractor.

Proof Let $D=\bigcup_{t\ge T} S(t)B$. It is clear that $S(t)D\subset D\subset B$, i.e., D is a positively invariant bounded absorbing set. $\|S(t_1)v_1-S(t_2)v_2\| = \|S(t_1)v_1-S(t_2)v_1+S(t_2)v_1-S(t_2)v_2\| \le \|S(t_1)v_1-S(t_2)v_1\|+\|S(t_2)v_1-S(t_2)v_2\|$, by the assumptions (1) and (2),

$$\|S(t_1)v_1-S(t_2)v_2\|\le l\|v_1-v_2\|+k|t_1-t_2|\le \max\{l,k\}(\|v_1-v_2\|+|t_1-t_2|),$$

which implies that $S(t)x$ is a Lipschitzian mapping in $[0,T]\times B$.

Next we only need to prove that the discrete semigroup $T(n)=S(nT)$ satisfies the condition (1) in Theorem 3.1.

From the assumption we know that there exists $R>0$ such that $\mathrm{diam}B\le 2R$, where $\mathrm{diam}B$ denotes the diameter of B.

$$D=\bigcup_{t\ge T} S(t)B=\bigcup_{t\ge T} P_mS(t)B+\bigcup_{t\ge T}(I-P_m)S(t)B.$$

$\bigcup_{t\ge T} P_mS(t)B$ is a bounded set in finite dimension space X_1. By Lemma 2.8, there exist balls of $\{B_i\}_{i=1}^K$ in X_1 with radius δ such that $\{B_i\}_{i=1}^K$ is a cover of $\bigcup_{t\ge T} P_mS(t)B$. Let $D_i=B_i+\bigcup_{t\ge T}(I-P_m)S(t)B, i=1,2,\ldots,K$, and by the assumption (3), we know that $\mathrm{diam}\, D_i\le \mathrm{diam}\, B_i+2\theta\le 2(\delta+\theta)$, which implies D has a cover $\{D_i\}_{i=1}^K$ with radius $\delta+\theta$.

Let $D_i^1=D_i\cap D$. From the assumption (1), $\|P_m(S(T)v_1-S(T)v_2)\|\le\|S(T)v_1-S(T)v_2\|\le l\|v_1-v_2\|\le 2l(\delta+\theta), \forall v_1,v_2\in D_i^1, i=1,2,\ldots,K$. By Lemma 2.8, $P_mS(T)D_i^1$ has a cover $\{B_{ij}^1\}_{j=1}^{N_1}$ with radius $\theta(\delta+\theta)$ and the number of cover $k_1\le(1+\frac{2l(\delta+\theta)}{\theta(\delta+\theta)})^m=(1+\frac{2l}{\theta})^m$. Let $D_{ij}^1=B_{ij}^1+(I-P_m)D_i^1$. By the assumption (4), $\{D_{ij}^1\}_{i=1,2,\ldots,K}^{j=1,2,\ldots,k_1}$ is a cover $S(T)D$ with radius $(\delta+\theta)^2$ and the number of cover $N_1\le K(1+\frac{2l}{\theta})^m$. After iterations, we get there exist $K(1+\frac{2l}{\theta})^{nm}$

balls at most in X covering $T(n)D = S(nT)D$ with radius $(\delta+\theta)^{n+1}$. From the assumption we know that $(\delta+\theta) < 1$, and consequently that the discrete semigroup $T(n)$ satisfies the condition (1) in Theorem 3.1. □

4. Existence of exponential attractor for reaction diffusion equation

We consider the following nonlinear reaction diffusion equation:

$$\begin{cases} u_t - \triangle u + f(u) = g(x), \\ u|_{\partial\Omega} = 0, \\ u(x,0) = u_0, \end{cases} \tag{4.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$, $g(x) \in L^2(\Omega)$, f is a $\mathcal{C}^1$ function and there exist $p \geq 2$, $c_i > 0$, $i = 1, \ldots, 5$ such that

$$c_1|u|^p - c_2 \leq f(u)u \leq c_3|u|^p + c_4, \tag{4.2}$$

$$f_u(u) \geq -c_5, \tag{4.3}$$

for all $u \in R$.

The problem has been studied in [11]; here we will use our method to prove the existence of exponential attractor.

For convenience, hereafter let $|\cdot|_p$ be the norm of $L^p(\Omega)$ $(p \geq 1)$. If $p = 2$, we denote $|\cdot|_2 = |\cdot|$, and c the arbitrary positive constants, which may be different from line to line and even in the same line. Since $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, we take the equivalent norm in $H_0^1(\Omega)$ and $H^2(\Omega) \cap H_0^1(\Omega)$, respectively, by

$$|\nabla u| = \|u\| = \sum_{i=1}^{n} \Big(\int_\Omega |\frac{\partial u}{\partial x_i}|^2 \mathrm{d}x \Big)^{\frac{1}{2}} \quad \text{for all } u \in H_0^1(\Omega),$$

$$|\Delta u| = \sum_{i=1}^{n} \Big(\int_\Omega |\frac{\partial^2 u}{\partial x_i^2}|^2 \mathrm{d}x \Big)^{\frac{1}{2}} \quad \text{for all } u \in H^2(\Omega) \cap H_0^1(\Omega).$$

We state the general existence and uniqueness of solutions which can be obtained by the Faedo-Galerkin methods and the interested readers are referred to [2] for detail.

Lemma 4.1 *Assume that Ω is a bounded smooth domain in $\mathbb{R}^n$, and $g(x) \in L^2(\Omega)$, then for any $u_0 \in L^2(\Omega)$ and any $T > 0$, there exists a unique solution for Eq.(4.1) which satisfies*

$$u(t) \in \mathcal{C}([0,T); L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)) \cap L^p(0,T; L^p(\Omega)).$$

If furthermore, $u_0 \in H_0^1(\Omega)$, then

$$u \in \mathcal{C}([0,T]; H_0^1(\Omega)) \cap L^2(0,T; H^2(\Omega))).$$

By Lemma 4.1, we can define the semigroup $S(t)$ as follows

$$S(t)u_0 : L^2(\Omega) \times \mathbb{R}^+ \to L^2(\Omega), \quad \text{and} \quad S(t)u_0 : L^2(\Omega) \times \mathbb{R}^+ \to H_0^1(\Omega).$$

Lemma 4.2 ([5]) *Assume that Ω is a bounded smooth domain in $\mathbb{R}^n$, and $g(x) \in L^2(\Omega)$, then the semigroup $S(t)$ has a bounded absorbing set in $L^2(\Omega)$, $H_0^1(\Omega)$, $L^p(\Omega)$, $H^2(\Omega)$ and $L^{2p-2}(\Omega)$, respectively; that is, for any bounded subset B in $L^2(\Omega)$, there exists a positive constant T, such that*

$$|u(t)|^2 \le M,\ \|u(t)\|^2 \le M,\ |u(t)|_p^p \le M,\ |\Delta u(t)|^2 \le M,\ |u(t)|_{2p-2}^{2p-2} \le M$$

for any $u_0 \in B$ and $t \ge T$, where M is a positive constant independent of B, $u(t) = S(t)u_0$.

Theorem 4.3 *Assume that Ω is a bounded smooth domain in $\mathbb{R}^n$, $g(x) \in L^2(\Omega)$ and f satisfies (4.2) and (4.3), $2 \le p < \infty (n \le 2)$, $2 \le p \le \frac{n}{n-2}+1$ $(n \ge 3)$, and $S(t)$ is the semigroup associated with (4.1), then $S(t)$ has exponential attractor in $L^2(\Omega)$ and $H_0^1(\Omega)$.*

Next we will verify that $S(t)$ satisfies all the conditions in Theorem 3.3 in $L^2(\Omega)$ and $H_0^1(\Omega)$. We only prove the existence of exponential attractor in $H_0^1(\Omega)$, using the same method, and the same conclusion holds true in $L^2(\Omega)$.

We set $A = -\Delta$. Since A^{-1} is a continuous compact operator in $L^2(\Omega)$, by the classical spectral theorem, there exist a sequence $\{\lambda_j\}_{j=1}^{\infty}$,

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots,\quad \lambda_j \to +\infty,\ \text{as } j \to \infty,$$

and a family of elements $\{e_j\}_{j=1}^{\infty}$ of $H_0^1(\Omega)$ which are orthogonal in $L^2(\Omega)$ such that

$$Ae_j = \lambda_j e_j,\quad \forall j \in N.$$

Let $H_m = \text{span}\{e_1, e_2, \dots, e_m\}$ in $L^2(\Omega)$ and $P_m : L^2(\Omega) \to H_m$ be an orthogonal projector. For any $u \in L^2(\Omega)$ we write

$$u = P_m u + (I - P_m)u \triangleq u_1 + u_2.$$

Without loss of generality, we can suppose B is a positively invariant bounded absorbing set, i.e., $S(t)B \subset B, \forall t \in \mathbb{R}^+$.

Proof Taking inner product of (4.1) with $-\Delta u_2$ in $L^2(\Omega)$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_2\|^2 + |\Delta u_2|^2 \le |f(u)||\Delta u_2| + |g(x)||\Delta u_2|. \tag{4.4}$$

Since

$$|f(u)||\Delta u_2| \le \frac{|\Delta u_2|^2}{4} + |f(u)|^2,\quad |g(x)||\Delta u_2| \le \frac{|\Delta u_2|^2}{4} + |g(x)|^2,$$

(4.4) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_2\|^2 + |\Delta u_2|^2 \le 2(|f(u)|^2 + |g(x)|^2).$$

Using (4.2), we find

$$|f(u)|^2 = \int_\Omega |f(u)|^2 \mathrm{d}x \le c(|u|_{2p-2}^{2p-2} + 1).$$

By Poincaré inequality and Lemma 4.2, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_2\|^2 + \lambda_m \|u_2\|^2 \le c.$$

Applying the Gronwall's lemma, we have

$$\|u_2(t)\|^2 \le e^{-\lambda_m t}\|u_2(0)\|^2 + \frac{c}{\lambda_m}. \tag{4.5}$$

We set $u_1(t) = S(t)v_1$ and $u_2(t) = S(t)v_2$ to be solutions associated with equation (4.1) with initial data $v_1, v_2 \in B$. Since B is a positively invariant bounded absorbing set, we find that $u_1(t), u_2(t) \in B$ for all $t \in \mathbb{R}$.

Let $w(t) = u_1(t) - u_2(t)$. By (4.1), we get

$$w_t - \Delta w + f(u_1) - f(u_2) = 0. \tag{4.6}$$

Taking inner product of (4.6) with $-\triangle w$ in $L^2(\Omega)$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + |\triangle w|^2 + (f(u_1) - f(u_2), -\triangle w) = 0.$$

Taking into account (4.3)and Hölder inequality, it is immediate to see that

$$\begin{aligned}|(f(u_1) - f(u_2), -\triangle w)| &\le \int_\Omega |f(u_1) - f(u_2)||\triangle w|\mathrm{d}x \\ &\le \frac{1}{2}|\triangle w|^2 + \frac{1}{2}\int_\Omega |f(u_1) - f(u_2)|^2\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}\int_\Omega |f(u_1) - f(u_2)|^2\mathrm{d}x &= \int_\Omega |f'(u_1 + \theta(u_2 - u_1))|^2|u_1 - u_2|^2\mathrm{d}x \\ &\le c\int_\Omega (1 + |u_1|^{p-2} + |u_2|^{p-2})^2|u_1 - u_2|^2\mathrm{d}x \\ &\le c\Big(\int_\Omega (1 + |u_1|^{2(p-1)} + |u_2|^{2(p-1)}\mathrm{d}x\Big)^{\frac{p-2}{p-1}}\Big(\int_\Omega |u_1 - u_2|^{2(p-1)}\Big)^{\frac{1}{p-1}} \\ &\le c(1 + |u_1|^{2(p-2)}_{2(p-1)} + |u_2|^{2(p-2)}_{2(p-1)})|w|^2_{2(p-1)}.\end{aligned}$$

Since $2 \le p < \infty (n \le 2)$, $2 \le p \le \frac{n}{n-2} + 1$ $(n \ge 3)$, using Sobolev embedding theorem, and

$$\int_\Omega |f(u_1) - f(u_2)|^2\mathrm{d}x \le c(1 + \|u_1\|^{2(p-2)} + \|u_2\|^{2(p-2)})\|w\|^2 \le c\|w\|^2,$$

we get $\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 \le c\|w\|^2$, hence

$$\|w(t)\|^2 \le \|v_1 - v_2\|^2 e^{ct} \le c\|v_1 - v_2\|^2, \quad \forall t \in [0, T]. \tag{4.7}$$

Let $w = w_1 + w_2$, w_1 be the projector in PH_m. Taking inner product of (4.6) with $-\triangle w_2$ in $L^2(\Omega)$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_2\|^2 + |\triangle w_2|^2 + (f(u_1) - f(u_2), -\triangle w_2) = 0, \tag{4.8}$$

$$|(f(u_1) - f(u_2), -\triangle w_2)| \le \frac{1}{2}|\Delta w_2|^2 + \frac{1}{2}\int_\Omega |f(u_1) - f(u_2)|^2\mathrm{d}x \le \frac{1}{2}|\Delta w_2|^2 + c\|w\|^2.$$

Using the Poincaré inequality $\lambda_n\|w_2\|^2 \le |\Delta w_2|^2$, it is immediate that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w_2\|^2 + \lambda_n\|w_2\|^2 \le c\|w\|^2.$$

By Gronwall's lemma and (4.7), we have

$$\|w_2(t)\|^2 \le e^{-\lambda_n t}\|w_2(0)\|^2 + \frac{c}{\lambda_n}\|v_1 - v_2\|^2 \le (e^{-\lambda_n t} + \frac{c}{\lambda_n + c})\|v_1 - v_2\|^2. \tag{4.9}$$

By (4.1), we have

$$u(t_2) - u(t_1) = \int_{t_1}^{t_2} \triangle u(s)\mathrm{d}s - \int_{t_1}^{t_2} f(u(s))\mathrm{d}s + (t_2 - t_1)g(x). \tag{4.10}$$

Taking inner product of (4.10) with $-\triangle(u(t_2) - u(t_1))$ in $L^2(\Omega)$, we get

$$\|u(t_2) - u(t_1)\|^2 \le |\triangle(u(t_2) - u(t_1))|\Big(|\int_{t_1}^{t_2} |\triangle u(s)|\mathrm{d}s| + |\int_{t_1}^{t_2} |f(u(s))|\mathrm{d}s| + |t_2 - t_1||g|\Big),$$

(4.2) implies that

$$|f(u(s))|^2 \le c(1 + |u(s)|^{2p-2}).$$

And by Lemma 4.2, we obtain

$$\|u(t_2) - u(t_1)\|^2 \le c|t_2 - t_1|. \tag{4.11}$$

(4.5), (4.7), (4.9) and (4.11) imply that $S(t)$ satisfies all the conditions in Theorem 3.3, i.e., $S(t)$ has an exponential attractor in $H_0^1(\Omega)$. $\square$

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