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C^*-Algebra $B_H(I)$ Consisting of Bessel Sequences in a Hilbert Space

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Abstract Let H be a separable Hilbert space, $B_H(I)$, $B(H)$ and $K(H)$ the sets of all Bessel sequences $\{f_i\}_{i\in I}$ in H, bounded linear operators on H and compact operators on H, respectively. Two kinds of multiplications and involutions are introduced in light of two isometric linear isomorphisms $\alpha_H : B_H(I) \to B(\ell^2), \beta : B_H(I) \to B(H)$, respectively, so that $B_H(I)$ becomes a unital C^*-algebra under each kind of multiplication and involution. It is proved that the two C^*-algebras $(B_H(I), \circ, \sharp)$ and $(B_H(I), \cdot, *)$ are $*$-isomorphic. It is also proved that the set $F_H(I)$ of all frames for H is a unital multiplicative semi-group and the set $R_H(I)$ of all Riesz bases for H is a self-adjoint multiplicative group, as well as the set $K_H(I) := \beta^{-1}(K(H))$ is the unique proper closed self-adjoint ideal of the C^*-algebra $B_H(I)$.

Keywords C^*-algebra; Bessel sequence; Hilbert space; frame; Riesz basis

MR(2010) Subject Classification 39B52

1. Introduction

Frames in Hilbert spaces were firstly introduced by Duffin and Schaeffer [1] in the study of nonharmonic Fourier series in 1952. Recently, frame theory plays an important role in mathematics, science, and engineering [2–4], and various generalizations of frames have been obtained. For example, Sun in [5,6] introduced and discussed the concept of a G-frame for a Hilbert space, which generalizes the concepts of frames [7], pseudoframes [8], oblique frames [9,10], outer frames [11], bounded quasi-projectors [12,13], frames of subspaces [14,15], operator frames for $B(H)$ (see [16]). In [17], (p, Y)-operator frames for a Banach space X were introduced, which makes a G-frame $\{T_j\}_{j\in\Lambda}$ for a Hilbert space H with respect to a sequence $\{K_j\}_{j\in\Lambda}$ of closed subspaces of a Hilbert space K be a $(2, K)$-operator frame for H. Hence, the concept of a (p, Y)-operator frame for a Banach space generalizes all of the concepts of frames. In Banach space setting, X_d frames, X_d Bessel sequences, tight X_d frames, independent X_d frames, and X_d Riesz basis for a Banach

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space were introduced and discussed, a necessary and sufficient condition for a Banach space X to have an X_d frame or to have an X_d Riesz basis as well as a necessary and sufficient condition for an X_d frame to have a dual frame were obtained in paper [18], some relations among basis, X_d frame and X_d Riesz basis in a Banach space were also established there. In paper [19], the properties of frames and atomic decompositions for a Banach space were explored in terms of the theory of frames for a Hilbert space, a sufficient condition for a complete Bessel sequence in a Banach space X to be a Banach frame was given and a sufficient condition for a complete Bessel sequence $\{y_n\}$ to have a sequence $\{x_n\}$ such that $(\{y_n\}, \{x_n\})$ becomes an atomic decomposition for X was also established and a relation between Banach frames and atomic decompositions was discussed there.

Multivariate Riesz multiwavelet bases with short support in $(L^2(\mathbb{R}^s))^{r\times 1}$ have applications in many areas, such as image processing, computer graphics and numerical algorithms. Pan in [20] characterized an algorithm to derive Riesz bases from refinable function vectors and several other important results about Riesz wavelet bases in $(L^2(\mathbb{R}^s))^{r\times 1}$ were also given there.

Traditionally, Gabor and wavelet analysis were studied by using classical Fourier analysis methods. But in recent years, more and more abstract tools have been introduced such as operator theory, operator algebra, abstract harmonic analysis and group-representations, etc. Especially, the author in [7] proved that the set $B_H(I)$ of all Bessel sequences $\{f_i\}_{i\in I}$ in a Hilbert space H is a Banach space and established an isometric isomorphism α from $B_H(I)$ onto the operator space $B(H, \ell^2)$ defined by $\alpha(\{f_i\}_{i\in I})(x) = \{\langle x, f_i\rangle\}_{i\in I}$, which suggests a relationship between wavelet analysis and operator theory. The aim of this paper is to introduce multiplication and involution on the Banach space $B_H(I)$ so that it becomes a unital C^*-algebra. This enables us to establish a corresponding connection between wavelet analysis and operator algebra.

Now, let us recall some definitions. Throughout this paper, H denotes a complex Hilbert space, I is an index set so that $\dim(H) = |I|$ (the cardinality of I), and $B(H)$ is the C^*-algebra of all bounded linear operators on H, $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$, $\ell^2 \equiv \ell^2(I)$ is the Hilbert space of all square-summable complex sequences with the inner product defined by

$$\langle \{c_n\}_{n\in I}, \{d_n\}_{n\in I}\rangle = \sum_{n\in I} c_n \overline{d_n},$$

which has the canonical orthonormal basis $e = \{e_n\}_{n\in I}$, where $e_n = \{\delta_{n,k}\}_{k\in I}$. Moreover,

$$S_H(I) = \big\{\{f_n\}_{n\in I} : f_n \in H (\forall n \in I)\big\},$$

and denote by $O_H(I)$ the set of all orthonormal bases for H.

Definition 1.1 ([7]) *Let $f = \{f_n\}_{n\in I} \in S_H(I)$. If there exists a positive constant B such that*

$$\sum_{n\in I} |\langle x, f_n\rangle|^2 \le B\|x\|^2, \quad \forall x \in H, \tag{1.1}$$

then we call f is a Bessel sequence in H. Denote by $B_H(I)$ the set of all Bessel sequences in H.

For every $f, g \in B_H(I)$, define

$$\alpha f + \beta g = \{\overline{\alpha} f_n + \overline{\beta} g_n\}_{n\in I}, \quad \forall \alpha, \beta \in \mathbb{F};$$
$$\|f\|^2 = \sup_{\|x\|\le 1} \Big(\sum_{n\in I} |\langle x, f_n\rangle|^2 \Big).$$

Then $(B_H(I), \|\cdot\|)$ becomes a normed linear space over $\mathbb{F}$ and

$$\|f\| = \inf\{B : B \text{ satisfies } (1.1)\}, \quad \forall f \in B_H(I).$$

For every $f \in B_H(I)$, put

$$T_f : H \to \ell^2, T_f x = \{\langle x, f_n\rangle\}_{n\in I}, \quad \forall x \in H.$$

It is evident that $T_f \in B(H, \ell^2)$ and $\|T_f\| = \|f\|$ for all f in $B_H(I)$. Thus, the map $f \mapsto T_f$ induces an isometric homomorphism α from $B_H(I)$ into $B(H, \ell^2)$.

In the sequel, we shall need the following.

Theorem 1.2 *(1)* $T_f^*\{c_n\}_{n\in I} = \sum_{n\in I} c_n f_n, \forall f \in B_H(I), \{c_n\}_{n\in I} \in \ell^2(I)$*;*

(2) $\alpha : f \mapsto T_f$ *is an isometrically linear isomorphism;*

(3) $(B_H(I), \|\cdot\|)$ *is a Banach space;*

(4) If $A \in B(H)$ *and* $f = \{f_n\}_{n\in I} \in B_H(I)$*, then* $Af := \{Af_n\}_{n\in I} \in B_H(I)$ *and* $T_{Af} = T_f A^*$.

Proof Similar to the proof of Corollary 2 in [7]. □

Definition 1.3 ([7]) *A Bessel sequence* $f = \{f_n\}_{n\in I}$ *in* H *is called a frame if there exists a positive constant* A *such that for every* $x \in H$*,*

$$A\|x\|^2 \le \sum_{n\in I} |\langle x, f_n\rangle|^2. \tag{1.2}$$

Denote by $F_H(I)$ the set of all frames for H. If conditions (1.1) and (1.2) hold for every $x \in H$ and the bounds A, B coincide, then the frame is called tight, f is called a Parseval frame if $A = B = 1$.

Definition 1.4 ([7]) *Let* $f = \{f_n\}_{n\in I} \in S_H(I)$*. If there exist two positive constants* C, D *such that*

$$C\|\{c_n\}\|^2 \le \sum_{n\in I} \|c_n f_n\|^2 \le D\|\{c_n\}\|^2, \quad \forall \{c_n\}_{n\in I} \in \ell^2, \tag{1.3}$$

and $\overline{\text{span}}\{f_n | n \in I\} = H$*, then* f *is said to be a Riesz basis for* H*.*

Denote by $R_H(I)$ the set of all Riesz bases for H. It is well-known that a frame has the unique dual if and only if it is a Riesz basis.

Theorem 1.5 *Let* $f = \{f_n\}_{n\in I} \in S_H(I)$*. Then*

(1) $f \in B_H(I) \Leftrightarrow T_f \in B(H, \ell^2)$*;*

(2) $f \in F_H(I) \Leftrightarrow T_f$ *is below-bounded* $\Leftrightarrow T_f^* T_f$ *is invertible;*

(3) $f \in R_H(I) \Leftrightarrow T_f$ *is invertible;*

(4) $f \in O_H(I) \Leftrightarrow T_f$ *is unitary.*

Proof Similar to the proof of Theorem 2 in [7]. $\square$

Definition 1.6 *If* $f = \{f_n\}_{n\in I}, g = \{g_n\}_{n\in I}$ *are two frames for* H *and*

$$x = \sum_{n\in I}\langle x, f_n\rangle g_n = \sum_{n\in I}\langle x, g_n\rangle f_n, \quad \forall x \in H,$$

then we say that g *is a dual frame of* f*. In this case,* f *is also a dual frame of* g*, so* $\{f, g\}$ *is called a dual pair of frames.*

For every frame $f = \{f_i\}_{i\in I}$ for H, Theorem 1.2(2) implies that the operator $T_f^*T_f$ is invertible. Thus, $\widetilde{f} := \{(T_f^*T_f)^{-1}f_i\}_{i\in I}$ is a dual frame of f, called the canonical dual of f.

2. Main results

In this section, we will define two kinds of multiplications and involutions of Bessel sequences in a Hilbert space H by using two maps

$$\alpha_H : B_H(I) \to B(\ell^2) \text{ and } \beta : B_H(I) \to B(H),$$

respectively, and then obtain C^*-algebra structures on the Banach space $B_H(I)$.

To define a multiplication on the Banach space $B_H(I)$, we first assume that $H = \ell^2(I)$.

Let $e = \{e_n\}_{n\in I}$ be the canonical orthonormal basis for $\ell^2(I)$. Then every element c of $\ell^2(I)$ can be written as $c = \{\langle c, e_n\rangle\}_{n\in I}$.

Definition 2.1 *For every* $f = \{f_n\}_{n\in I}, g = \{g_n\}_{n\in I} \in B_{\ell^2}(I)$*, we define*

$$f \circ g = \{(f \circ g)_n\}_{n\in I} = T_g^* f = \{T_g^* f_n\}_{n\in I}, \tag{2.1}$$

$$f^\sharp = \{(f^\sharp)_n\}_{n\in I} = T_f e = \{T_f e_n\}_{n\in I}. \tag{2.2}$$

Clearly, $f \circ g$ and $f^\sharp$ are both elements of $B_{\ell^2}(I)$. By Theorem 1.2(4), we get that $T_{f\circ g} = T_{T_g^* f} = T_f T_g$ and $T_{f^\sharp} = T_{T_f e} = T_e T_f^* = T_f^*$.

Theorem 2.2 $B_{\ell^2}(I)$ *is a unital* C^**-algebra with identity* $e = \{e_n\}_{n\in I}$ *and the mapping* α_{ℓ^2} : $B_{\ell^2}(I) \to B(\ell^2)$ *is an isometrically* $*$*-algebraic isomorphism.*

Proof (i) For all $f, g, h \in B_{\ell^2}(I)$, we compute that

$$f \circ (g \circ h) = T_{g\circ h}^* f = (T_g T_h)^* f = T_h^* T_g^* f = (f \circ g) \circ h,$$

$$(\lambda f) \circ g = T_g^*(\lambda f) = T_{\bar{\lambda} g}^* f = f \circ (\bar{\lambda} g)$$

for all $\lambda \in \mathbb{F}$ and

$$e \circ f = T_{f^*} e = T_f^* e = f = T_{e^*} f = f \circ e;$$

$$f \circ (g + h) = T_{g+h}^* f = (T_g + T_h)^* f = (T_g^* + T_h^*) f = f \circ g + f \circ h.$$

Similarly, $(f + g) \circ h = f \circ h + g \circ h$. For all $f, g \in B_{\ell^2}(I)$, we have

$$\|f \circ g\| = \|T_{f\circ g}\| = \|T_f T_g\| \le \|T_f\|\|T_g\| = \|f\|\|g\|.$$

So, $B_{\ell^2}(I)$ is a Banach algebra.

(ii) For all $f, g \in B_{\ell^2}(I)$ and $a, b \in \mathbb{F}$, we have

$$(af + bg)^\sharp = \{T_{af+bg}e_n\}_{n\in I} = \{aT_f e_n\}_{n\in I} + \{bT_g e_n\}_{n\in I} = \overline{a}f^\sharp + \overline{b}g^\sharp,$$

$$(f \circ g)^\sharp = \{T_{f\circ g}e_n\}_{n\in I} = \{T_f T_g e_n\}_{n\in I} = T_f g^\sharp = g^\sharp \circ f^\sharp,$$

$$(f^\sharp)^\sharp = \{T_{f^\sharp}e_n\}_{n\in I} = \{T_f^\sharp e_n\}_{n\in I} = f.$$

(iii) $\forall f \in B_H(I)$, $\|f^\sharp \circ f\| = \|T_{f^\sharp}T_f\| = \|T_f\|^2 = \|f\|^2$. Hence, $B_{\ell^2}(I)$ is a unital C^*-algebra with identity $e = \{e_n\}_{n\in I}$. Since

$$\alpha_{\ell^2}(f^\sharp) = T_{f^\sharp} = T_f^* = (\alpha_{\ell^2}(f))^*,$$

$$\alpha_{\ell^2}(f \circ g) = T_{f\circ g} = T_f T_g = \alpha_{\ell^2}(f)\alpha_{\ell^2}(g),$$

and $\alpha_{\ell^2} : B_{\ell^2}(I) \to B(\ell^2)$ is an isometrically linear isomorphism, $\alpha_{\ell^2} : B_{\ell^2}(I) \to B(\ell^2)$ is an isometrically $*$-algebraic isomorphism. $\Box$

It is well-known that every separable infinite-dimensional Hilbert space H with a basis $\{\varepsilon_i\}_{i\in I}$ is isomorphic to $\ell^2(I)$ in light of the unitary $U : H \to \ell^2$ defined by $U(\sum_{i\in I} c_i\varepsilon_i) = \{c_i\}_{i\in I}$. Thus, $\forall f = \{f_n\}_{n\in I}, g = \{g_n\}_{n\in I} \in B_H(I)$, we have

$$Uf := \{Uf_n\}_{n\in I} \in B_{\ell^2}(I), \quad Ug := \{Ug_n\}_{n\in I} \in B_{\ell^2}(I).$$

By Definition 2.1, we know that $Uf \circ Ug \in B_{\ell^2}(I)$. Put $\pi_U(f) = Uf = \{Uf_n\}_{n\in I}$ for every $f \in B_{\ell^2}(I)$. Then

$$\|\pi_U(f)\| = \|T_{\pi_U(f)}\| = \|T_f U^*\| = \|T_f U^* U T_f^*\|^{\frac{1}{2}} = \|T_f T_f^*\|^{\frac{1}{2}} = \|T_f\| = \|f\|,$$

and so we obtain an isometrically linear isomorphism $\pi_U : B_H(I) \to B_{\ell^2}(I)$ since U is a unitary.

Definition 2.3 *For all $f, g \in B_H(I)$, we define*

$$f \circ g = \pi_U^{-1}((Uf) \circ (Ug)), \quad f^\sharp = \pi_U^{-1}((Uf)^\sharp). \tag{2.3}$$

It is clear that $f \circ g \in B_H(I)$ and $f^\sharp \in B_H(I)$.

Theorem 2.4 *$B_H(I)$ is a unital C^*-algebra with identity $e_H = \pi_U^{-1}(e)$ and the mapping $\alpha_H \triangleq \alpha_{\ell^2}\pi_U : B_H(I) \to B(\ell^2)$ is an isometrically $*$-algebraic isomorphism.*

Proof Clearly, the Banach space $B_H(I)$ becomes a Banach algebra with the multiplication defined by (2.3). For all $f, g \in B_H(I)$ and $a, b \in \mathbb{F}$, we compute that

(i) $(af + bg)^\sharp = \pi_U^{-1}((U(af + bg))^\sharp) = \pi_U^{-1}(\overline{a}(Uf)^\sharp + \overline{b}(Ug)^\sharp) = \overline{a}f^\sharp + \overline{b}g^\sharp$;

(ii) $(f \circ g)^\sharp = \pi_U^{-1}((U(f \circ g))^\sharp) = \pi_U^{-1}((Ug)^\sharp \circ (Uf)^\sharp)) = g^\sharp \circ f^\sharp$;

(iii) $(f^\sharp)^\sharp = \pi_U^{-1}((Uf^\sharp)^\sharp) = \pi_U^{-1}(\{T_{Uf^\sharp}e_n\}) = \pi_U^{-1}(\{T_{f^\sharp}U^*e_n\}) = f$.

Thus, $B_H(I)$ becomes a Banach $*$-algebra with identity e_H. Moreover, $\forall f \in B_H(I)$, we have $\|f^\sharp \circ f\| = \|T_{f^\sharp}T_f\| = \|T_f\|^2 = \|f\|^2$. This shows that $B_H(I)$ is a unital C^*-algebra with identity e_H. Since

$$\pi_U(f \circ g) = (Uf) \circ (Ug) = \pi_U(f) \circ \pi_U(g), \pi_U(f^\sharp) = (Uf)^\sharp = (\pi_U(f))^\sharp,$$

we get that π_U is an isometrically $*$-algebraic isomorphism. It follows from Theorem 2.2 that α_H is an isometrically $*$-algebraic isomorphism. □

Remark 2.5 For $f, g \in B_H(I)$, the equation $f \circ g = g \circ f$ does not necessarily hold, so C^*-algebra $B_H(I)$ is not abelian in general.

For every $f \in B_H(I)$, define

$$S_f = \alpha_H(f). \tag{2.4}$$

By the definition of α_H, it is evident that $S_f = T_{Uf} = T_f U^* \in B(\ell^2)$ and $\|S_f\| = \|f\|$. For $f, g \in B_H(I)$, we have

$$S_{f\circ g} = T_{U(f\circ g)} = T_{(Uf)\circ(Ug)} = T_{Uf}T_{Ug} = T_f U^* T_g U^* = S_f S_g;$$

$$S_{f^\sharp} = T_{Uf^\sharp} = T_{(Uf)^\sharp} = T_{Uf}^* = (T_f U^*)^* = S_f^*.$$

From Theorem 1.5, we can get following results:

Corollary 2.6 *Let $f \in B_H(I)$. Then*

(1) $f \in F_H(I)$ if and only if S_f is below-bounded;

(2) $f \in R_H(I)$ if and only if S_f is invertible;

(3) $f \in O_H(I)$ if and only if S_f is unitary.

Corollary 2.7 *Let $f, g \in B_H(I)$. Then $A(f \circ g) = f \circ (Ag)$ whenever $A \in B(H)$.*

Proof Use the fact that $A(f \circ g) = AT_g^* f = (T_g A^*)^* f = T_{Ag}^* f = f \circ (Ag)$. □

The map α_H defined in (2.4) depends on the isomorphism U from H onto ℓ^2 and builds an isomorphism between $B_H(I)$ and $B(\ell^2)$. There is another way to obtain a new kind of multiplication of two Bessel sequences and a new involution of a Bessel sequence in light of a fixed orthonormal basis for H. Take an orthonormal basis $\delta = \{\delta_n\}_{n\in I}$ for H, then for arbitrary $f \in B_H(I)$ and $x \in H$, put

$$R_f x = \sum_{n\in I} \langle x, f_n\rangle \delta_n. \tag{2.5}$$

Clearly, $\sum_{n\in I} \langle x, f_n\rangle \delta_n = T_\delta^* T_f x \in H$ and so $R_f = T_\delta^* T_f \in B(H)$ and $\|R_f\| = \|T_f\| = \|f\|$. Now, we define a map

$$\beta : B_H(I) \to B(H), f \mapsto R_f.$$

It is clear that β is injective. For every $A \in B(H)$, take $f = \alpha^{-1}(T_\delta A)$, then $\beta(f) = R_f = T_\delta^* T_f = A$, so β is surjective. Thus, β is an isometrically linear bijection.

Definition 2.8 *$\forall f, g \in B_H(I)$, define $f \cdot g = \beta^{-1}(R_f R_g), f^* = \beta^{-1}(R_f^*)$. Clearly, $f \cdot g, f^* \in B_H(I)$ and*

$$\beta(f \cdot g) = R_f R_g = \beta(f)\beta(g), \beta(f^*) = R_f^* = \beta(f)^*.$$

Theorem 2.9 *$B_H(I)$ is a unital C^*-algebra with identity δ and the mapping $\beta : B_H(I) \to B(H)$ is an isometrically $*$-algebraic isomorphism.*

Proof Similar to the proof of Theorem 2.4. □

From $R_f = T_\delta^* T_f$ and Theorem 1.5, we can get following result:

Corollary 2.10 *Let $f \in B_H(I)$. Then*

(1) $f \in F_H(I)$ if and only if R_f is below-bounded;

(2) $f \in R_H(I)$ if and only if R_f is invertible;

(3) $f \in O_H(I)$ if and only if R_f is unitary.

Even though the maps from $B_H(I)$ into $B(\ell^2)$ and $B(H)$, respectively, are different, we obtain the same conclusion, which says that $(B_H(I), \circ, \sharp)$ and $(B_H(I), \cdot, *)$ are unital C^*-algebras with two different kinds of multiplications and involutions. The following theorem shows that these two C^*-algebras are isomorphic.

Theorem 2.11 *Set $\psi_U(A) = UAU^*$, then the mapping $\phi = \beta^{-1}\psi_U^{-1}\alpha_H$ is an isometric *-isomorphism from $(B_H(I), \circ, \sharp)$ onto $(B_H(I), \cdot, *)$, and C^*-algebras $(B_H(I), \circ, \sharp)$ and $(B_H(I), \cdot, *)$ are isomorphic. Moreover, $\phi(F_H(I)) = F_H(I), \phi(R_H(I)) = R_H(I)$.*

Proof Clearly, the mapping $\psi_U : B(H) \to B(\ell^2)$ is an isometric $*$-isomorphism and then $\phi = \beta^{-1}\psi_U^{-1}\alpha_H$ is an isometric $*$-isomorphism from $(B_H(I), \circ, \sharp)$ onto $(B_H(I), \cdot, *)$. By the fact that

$$\phi(f) = \beta^{-1}\psi_U^{-1}\alpha_H(f) = \beta^{-1}(U^* S_f U) = \beta^{-1}(U^* T_f U^* U) = \beta^{-1}(U^* T_\delta R_f),$$

and Corollary 2.10(1), we get that

$$\phi(f) \in F_H(I) \Leftrightarrow \beta(\phi(f)) \text{ is below bounded} \Leftrightarrow f \in F_H(I).$$

So, $\phi(F_H(I)) = F_H(I)$. Similarly, $\phi(R_H(I)) = R_H(I)$. $\Box$

3. Some applications

In this section, we will study some properties of some subsets of $B_H(I)$.

Theorem 3.1 *(1) $F_H(I)$ is a semi-group with identity δ with respect to multiplication $\cdot$;*

(2) $R_H(I)$ is a self-adjoint multiplicative group.

Proof (1) Clearly $\delta \in F_H(I)$. If $f, g \in F_H(I)$, then $R_f R_g$ is still a bounded-below operator, by Corollary 2.10, $f \cdot g \in F_H(I)$. This implies $F_H(I)$ is a multiplicative semi-group with identity δ.

(2) It is clear that $R_H(I)$ is a semi-group with identity δ. The following discussion will prove that every $f \in R_H(I)$ has a unique inverse in $R_H(I)$. Let $f \in R_H(I)$. Then R_f is invertible. Thus, there is a unique $g \in B_H(I)$ such that $\beta(g) = R_f^{-1}$, that is, $R_g = R_f^{-1}$. Since $R_{f\cdot g} = R_f R_g = I = R_\delta$ and $R_{g\cdot f} = R_g R_f = R_\delta$, $f \cdot g = g \cdot f = \delta$. Hence, g is the inverse of f. Clearly, $g \in R_H(I)$. $\Box$

For arbitrary frame f for H, define $\widehat{f} = \{g | g \text{ is a dual frame of } f\}$. Clearly, $\widetilde{f} \in \widehat{f}$ for every frame f for H.

Theorem 3.2 *(1) If $f \in F_H(I)$, then $g^* \cdot f = \delta$ for all $g \in \widehat{f}$.*

(2) Let $f \in F_H(I)$, $g \in B_H(I)$. If $g \cdot f = \delta$, then $g^* \in F_H(I)$ and $g^* \in \widehat{f}$.

(3) If $f \in F_H(I)$, then $f^* \in F_H(I)$ if and only if $f \in R_H(I)$;

(4) If $f \in F_H(I)$, then f is idempotent if and only if f is the identity δ of $B_H(I)$.

Proof (1) If $f \in F_H(I)$ and $g \in \widehat{f}$, then

$$x = \sum_{n \in I} \langle x, f_n \rangle g_n = \sum_{n \in I} \langle x, g_n \rangle f_n, \quad \forall x \in H,$$

by the definitions of T_f and T_g, we have $x = T_g^* T_f x = T_f^* T_g x$. So, $T_g^* T_f = T_f^* T_g = I$ and thus we have $T_g^* T_f = T_{g^* \cdot f} = I$, so $g^* \cdot f = \delta$.

(2) If $g \cdot f = \delta$, then $T_f^* T_{g^*} = R_f^* R_{g^*} = R_g R_f = \beta(g)\beta(f) = \beta(g \cdot f) = I$. So we can see that T_{g^*} is left-invertible. This implies that $g^* \in F_H(I)$ and $g^* \in \widehat{f}$.

(3) Let $f \in F_H(I)$. Suppose that $f^* \in F_H(I)$. By Corollary 2.10, we have R_f, R_f^* are both bounded-below. So R_f is invertible and thus $f \in R_H(I)$. Conversely, we suppose that $f \in R_H(I)$. Then R_f is invertible, so R_f^* is also invertible, therefore $f^* \in R_H(I) \subset F_H(I)$.

(4) f is idempotent if and only if R_f is idempotent. Since $f \in F_H(I)$ implies that $R_f : \ell^2 \to \ell^2$ is below-bounded, now we only need prove that an idempotent injective operator R_f is the identity I. Suppose that $\operatorname{ran}(R_f)^\perp \neq \{0\}$, where $\operatorname{ran}(R_f)$ denotes the range of R_f, then there exists $x \in \operatorname{ran}(R_f)^\perp \backslash \{0\}$ such that $R_f x \neq x$. However, $R_f(R_f x - x) = 0$. This contradicts the fact that R_f is injective. So, $R_f = I$. Hence, f is idempotent if and only if f is the identity δ of $B_H(I)$. $\Box$

Remark 3.3 In $F_H(I)$, the dual frame of f is not unique, so its left-inverse element is also not unique. But the convex combination of left-inverse elements of f is still a left-inverse element of f.

In the following, we consider ideals of $B_H(I)$ as a C^*-algebra and obtain an important result. Denote by $K(H)$ the set of all compact operators on H,

$$c_0(H) = \big\{ \{f_n\}_{n \in I} : \|f_n\| \to 0 \ (n \to \infty) \big\}$$

and denote $K_H(I) := \beta^{-1}(K(H))$.

Theorem 3.4 *$K_H(I)$ is the unique proper self-adjoint closed ideal of $B_H(I)$ and $K_H(I) \subset c_0(H) \cap B_H(I)$.*

Proof Since $K(H)$ is the unique non-trivial closed ideal of C^*-algebra $B(H)$ and β is an isometrically $*$-isomorphism, $K_H(I)$ is the unique proper closed ideal of $B_H(I)$. Since $T \in K(H)$ if and only if $T^* \in K(H)$, we see that $K_H(I)$ is self-adjoint. Also, $\forall f \in K_H(I)$, we have $R_f \in K(H)$. Take the identity δ of $B_H(I)$, then $f_n = R_f^* \delta_n$, so we compute that

$$\|f_n\| = \|R_f^* \delta_n\| \to 0 (n \to \infty).$$

So, $K_H(I) \subset c_0(H)$. Clearly, $K_H(I) \subset B_H(I)$. This shows that $K_H(I) \subset c_0(H) \cap B_H(I)$. $\Box$

Remark 3.5 The " $\subset$ " of Theorem 3.4 may be a proper inclusion in general. For instance, let

$\{e_n\}_{n\in I}$ be an orthonormal basis for H and define the sequence $\{f_n\}_{n\in I}$ by

$$\{e_1, \frac{e_2}{\sqrt{2}}, \frac{e_2}{\sqrt{2}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \dots\}.$$

It is easily seen that $\{f_n\}_{n\in I}$ is a Parseval frame for H and $\{f_n\}_{n\in I} \in c_0(H)$. But R_f is a co-isometry, i.e., $R_f R_f^* = I$. Hence, R_f is not a compact operator. Applying properties of an ideal, we also derive following results.

Corollary 3.6 *(1)* $K_H(I) \cap F_H(I) = \emptyset$;

(2) If $f \in B_H(I)$ *and* $A \in K(H)$, *then* $Af \in K_H(I)$.

Proof (1) Suppose that $K_H(I) \cap F_H(I) \neq \emptyset$, that is, at least there exists $f \in K_H(I) \cap F_H(I)$, then we have $R_f \in K(H)$ and furthermore $R_f^* \in K(H)$. On the other hand, for $f \in F_H(I)$, take $g \in \widehat{f}$, then by Definition 1.6 and Theorem 1.5, we have $R_f^* R_g = T_f^* T_\delta T_\delta^* T_g = I$. Since $K(H)$ is the unique ideal of $B(H)$, $I \in K(H)$. This is a contradiction. Therefore, $K_H(I) \cap F_H(I) = \emptyset$.

(2) Since $R_{Af} = T_{Af}^* T_\delta = AT_f^* T_\delta \in K(H)$, $Af \in K_H(I)$. $\square$

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