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A New Family of Exponential Slash Distributions with Elliptical Contours

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Abstract A new family of univariate exponential slash distribution is introduced, which is based on elliptical distributions and defined by means of a stochastic representation as the scale mixture of an elliptically distributed random variable with respect to the power of an exponential random variable. The same idea is extended to the multivariate case. General properties of the resulting families, including their moments and kurtosis coefficient, are studied. And inferences based on methods of moment and maximum likelihood are discussed. A real data is presented to show this family is flexible and fits much better than other related families.

Keywords slash distribution; exponential slashed elliptical distribution; scale mixtures of elliptical distributions; kurtosis coefficient

MR(2010) Subject Classification 62E10; 62F10

1. Introduction

Symmetric distributions generalizing normality have been the subject of much research in statistical literatures. One such extension is the class of standard slash distributions, and a random variable S is said to have a slash distribution if it can be represented as

$$S = \frac{Z}{U^{1/q}} \tag{1.1}$$

where $Z \sim N(0,1)$, independent of $U \sim U(0,1)$ and $q > 0$. This class contains the standard normal distribution as its special case and has heavier tails than the standard normal while keeping symmetry. The simplicity of the stochastic representation (1.1) turns out to be quite useful for studying properties of the corresponding family. General properties of this family were studied in references [1–3]. Maximum likelihood estimators (MLEs) for the location-scale case were studied in reference [4]. Reference [5] developed a multivariate version and also an asymmetric

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multivariate version and studied its properties and inference. Then references [6,7] studied multivariate skew-slash distribution and its maximum likelihood parameter estimation. Reference [8] gave a generalization of the multivariate slash distribution. References [9,10] extended the slash distribution by introducing the slashed-elliptical family. In the past year, reference [11] introduced a modified slash distribution in which the uniform variable in (1.1) was replaced by an exponential variable with scale parameter 2 and reference [12] introduced an alpha half normal slash distribution with an alpha half normal variable substituting the normal variable in (1.1) for analyzing nonnegative data. Main properties and inferences from these new distributions were presented too.

For elliptical distributions, the early works afforded by references [13] and [14] contain many interesting properties and results. A general compilation of such theory was given in [15]. Specially, a random variable W has elliptical distribution with location μ and scale parameter σ, denoted as $W \sim \mathrm{EI}(\mu, \sigma; g)$, if W has density function given by

$$f_W(x) = \frac{1}{\sigma} g\big((\frac{x-\mu}{\sigma})^2\big)$$

for some non-negative function $g(u), u \geq 0$ (referred to as the density generator), satisfying $\int_0^\infty u^{-1/2} g(u) \mathrm{d}u = 1$. For example, Normal, Cauchy, Student-t, Laplace, Type II pearson and Kotz-type distributions are all its special cases (see Table 1).

Type	$g(u)$
Normal	$(2\pi)^{-1/2} \exp(-u/2)$
Cauchy	$\pi^{-1}(1+u)^{-1}$
Student-t with degree ν	$\frac{\Gamma((1+\nu)/2)}{\Gamma(\nu/2)\sqrt{\pi\nu}}(1+\frac{u}{\nu})^{-(1+\nu)/2}$
Laplace	$2^{-1}\exp(-\sqrt{u})$
Type II pearson with parameter $\alpha > 0$	$\frac{\Gamma(\alpha+1/2)}{\Gamma(\alpha)\sqrt{\pi}}(1+u)^{\alpha-1}$
Kotz-type with parameters $r, s > 0, N > 1/2$	$\frac{sr^{(2N-1)/2s}}{\Gamma((2N-1)/2s)} u^{N-1} \exp(-ru^s)$

Table 1 Some special cases of generator function g

In this paper we will consider another slashed distribution after references [11–15], that is, we replace the uniform random variable in denominator with an exponential one underlying the definition of the slashed-elliptical family presented in [9]. i.e., any member of the new class of distributions can be represented as $Y = W/U^{1/q}$ for some $q > 0$, where $W \sim \mathrm{EI}(0, 1; g)$ is independent of $U \sim \mathrm{Exp}(\lambda)$ with $\lambda > 0$, which has density function $f_U(x; \lambda) = \lambda e^{-\lambda x},\ x > 0$. We denote this as $\mathrm{ESEI}(0, 1, \lambda, q; g)$. Members of this family are called exponential-slashed-elliptical distribution. A key advantage of our construction is the simplicity by which standard and well-known symmetric distributions can be modified to support increased kurtosis coefficient. We study the main properties of the corresponding induced location–scale family, and discuss an appropriate multivariate extension.

The rest of the paper is organized as follows: In Section 2, we develop the general density function of the exponential-slashed-elliptical distribution and study some of its important properties such as moments and kurtosis coefficient. We also discuss inference based on methods of moment and maximum likelihood. In Section 3, a real data is presented to show the flexibility and better performance of the exponential-slashed-normal distribution, one special case of the exponential-slashed-elliptical distribution. In Section 4, the multivariate extension is given and some related properties are derived. Section 5 includes some concluding remarks.

2. Exponential-slashed-elliptical distribution

We now give the general stochastic representation and the density function of the exponential-slashed-elliptical distribution, and study some of its important properties such as moments and kurtosis coefficient. For illustrative reason, we also provide some plots for the density function and kurtosis coefficient ranges for some special cases.

2.1. Density function in the general case

Let $Y \sim \mathrm{ESEI}(\mu, \sigma, \lambda, q; g)$, i.e., Y is a random variable that can be represented as

$$Y = \mu + \sigma \frac{W}{U^{1/q}}, \tag{2.1}$$

where $W \sim \mathrm{EI}(0, 1; g)$, $U \sim \mathrm{Exp}(\lambda)$ and they are independent, $\mu \in \mathbb{R}$, $\sigma > 0$, $\lambda > 0$ and $q > 0$.

In what follows we state the results for the standard case ($\mu = 0$ and $\sigma = 1$) from which extensions to the location-scale family are immediate and will be presented in Remark 2.3 later. Now we start with the general form of the density function.

Proposition 2.1 *Let $Y \sim \mathrm{ESEI}(0, 1, \lambda, q; g)$. Then the density function of Y is given by*

$$f_Y(y; 0, 1, \lambda, q) = \begin{cases} (\lambda q/2) \cdot \mid y \mid^{-(q+1)} \int_0^\infty g(v) v^{(q-1)/2} \exp\{-\lambda v^{q/2} \mid y \mid^{-q}\} \mathrm{d}v, & \textit{if } y \neq 0, \\ \lambda^{-1/q} g(0) \Gamma(1 + 1/q), & \textit{if } y = 0, \end{cases} \tag{2.2}$$

here $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \mathrm{d}x$ is the Gamma function.

Proof From representation (2.1), and using the independence of Z and U, it is easy to see that the density function of Y can be written as

$$f_Y(y; 0, 1, \lambda, q) = \int_0^\infty f_W(yu^{1/q}) f_U(u) u^{1/q} \mathrm{d}u,$$

where $f_W(x) = g(x^2)$ is the density function of W and $f_U(u)$ is the density function of U. Hence,

$$f_Y(y; 0, 1, \lambda, q) = \lambda \int_0^\infty g(y^2 u^{2/q}) u^{1/q} e^{-\lambda u} \mathrm{d}u. \tag{2.3}$$

If $y = 0$, the result can be obtained by the definition of Gamma function. On the other hand, if $y \neq 0$, formula (2.2) follows by changing the integration variable to $v = y^2 u^{2/q}$ in formula (2.3). $\Box$

Remark 2.2 From formula (2.3), we can see that $f_Y(y; 0, 1, \lambda, q)$ is continuous at $y = 0$ provided that g is right-continuous at 0. Also, $\lim\limits_{q \to \infty} f_Y(y; 0, 1, \lambda, q) = g(y^2)$, which indicates that the exponential-slashed-elliptical distribution contains the elliptical distribution as its limit case when $q \to \infty$.

Figure 1 depicts plots of the density function for some canonical cases of the exponential-slashed-elliptical distribution with $\lambda = \sqrt{2}$ and $q = 1$. They are the exponential-slashed normal (ESN), exponential-slashed student-t with $\nu = 3$ (ESt), exponential-slashed type II pearson with $\alpha = 1/2$ (ESP) and exponential-slashed kotz with $r = 1,\ s = 1,\ N = 2$ (ESK).

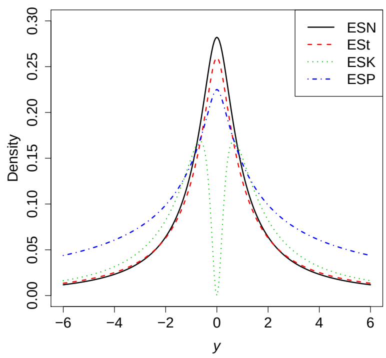

Figure 1 Density curve for some exponential-slashed-elliptical distributions with $\lambda = \sqrt{2}$ and $q = 1$.

Remark 2.3 From the representation (2.1), we know that if a random variable $X \sim \mathrm{ESEI}(\mu, \sigma, \lambda, q; g)$, then $Y = (X - \mu)/\sigma \sim \mathrm{ESEI}(0, 1, \lambda, q; g)$ and the density of X can be obtained by

$$f_X(x; \mu, \sigma, \lambda, q) = \frac{1}{\sigma} f_Y(\frac{x-\mu}{\sigma}; 0, 1, \lambda, q)$$
$$= \begin{cases} \frac{\lambda q}{2} \mid \frac{x-\mu}{\sigma} \mid^{-(q+1)} \int_0^\infty g(v) v^{(q-1)/2} \exp\{-\lambda v^{q/2} \mid \frac{x-\mu}{\sigma} \mid^{-q}\} \mathrm{d}v, & \text{if } x \neq \mu, \\ \sigma^{-1} \lambda^{-1/q} g(0) \Gamma(1 + 1/q), & \text{if } x = \mu. \end{cases}$$

2.2. Some properties

Consider a random variable $Y \sim \mathrm{ESEI}(0, 1, \lambda, q; g)$, and define $W = |Y|$ and $V = Y^2$. It is straightforward to see from formula (2.2) that the density functions of W and V are respectively given by

$$f_W(w; 0, 1, \lambda, q) = \lambda q w^{-(q+1)} \int_0^\infty g(t) t^{(q-1)/2} \exp\{-\lambda t^{q/2} w^{-q}\} \mathrm{d}t, \quad w > 0, \tag{2.4}$$

$$f_V(v; 0, 1, \lambda, q) = (\lambda q/2) \cdot v^{-(q/2+1)} \int_0^\infty g(t) t^{(q-1)/2} \exp\{-\lambda t^{q/2} v^{-q/2}\} \mathrm{d}t, \quad v > 0. \tag{2.5}$$

Formula (2.4) can be seen as a generalization of the half-symmetric family, while formula (2.5) gives an extension to results involving the square of symmetric random variables.

Example 2.1 Consider the normal case, i.e., $g(t) = (2\pi)^{-1/2} \exp(-t/2)$. For $q > 0$, formulas (2.4) and (2.5) then respectively become

$$f_W(w; 0, 1, \lambda, q) = 2\lambda q \int_0^\infty \phi(wt) t^q e^{-\lambda t^q} \mathrm{d}t, \quad w > 0,$$

$$f_V(v; 0, 1, \lambda, q) = \frac{\lambda q}{\sqrt{v}} \int_0^\infty \phi(\sqrt{v} t) t^q e^{-\lambda t^q} \mathrm{d}t, \quad v > 0,$$

here $\phi(t) = (2\pi)^{-1/2} \exp\{-t^2/2\}$ is the standard normal density function. In the canonical case $(q = 1)$, we get

$$f_W(w; 0, 1, \lambda, 1) = \sqrt{\frac{2}{\pi}} \frac{\lambda}{w^2} \exp\{\frac{\lambda^2}{2w^2}\}[1 - \frac{\lambda}{w}\sqrt{\frac{\pi}{2}}], \quad w > 0,$$

$$f_V(v; 0, 1, \lambda, 1) = \frac{\lambda}{\sqrt{2\pi} v^{3/2}} \exp\{\frac{\lambda^2}{2v}\}[1 - \frac{\lambda}{\sqrt{v}}\sqrt{\frac{\pi}{2}}], \quad v > 0.$$

We recognize $f_W(w; 0, 1, \lambda, 1)$ as the canonical exponential-slashed-half-normal (ESHN) density, while $f_V(v; 0, 1, \lambda, 1)$ is just the canonical exponential-slashed-$\chi^2(1)$ (ES-$\chi^2(1)$) density.

Figure 2 shows that density of the canonical ESHN (or ES-$\chi^2(1)$) is increasing first then decreasing with long right tail heavier than the one of half-normal (or $\chi^2(1)$). So they can be used in management engineering and survival analysis to model nonnegative data with outliers.

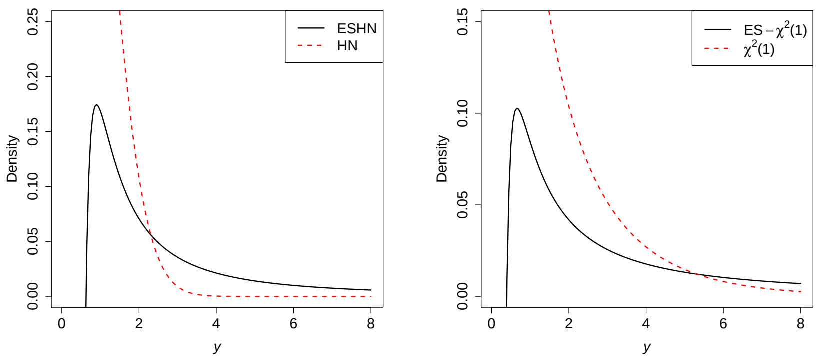

Figure 2 Density comparation for ESHN (or ES-$\chi^2(1)$) with HN (or $\chi^2(1)$) under $\lambda = \sqrt{2}$ and $q = 1$.

Proposition 2.4 *Let* $Y|U = u \sim \mathrm{EI}(0, u^{-1/q}; g)$ *and* $U \sim \mathrm{Exp}(\lambda)$. *Then random variable* $Y \sim \mathrm{ESEI}(0, 1, \lambda, q; g)$.

Proof Since the marginal distribution of Y can be written as

$$f_Y(y; \lambda, q) = \int_0^\infty f_{Y|U}(y|u) f_U(u) \mathrm{d}u = \int_0^\infty u^{1/q} g(u^{2/q} y^2) \lambda e^{-\lambda u} \mathrm{d}u,$$

the result follows from formula (2.3). □

Corollary 2.5 *Let* $X|U = u \sim \mathrm{EI}(\mu, u^{-1/q}\sigma; g)$ *and* $U \sim \mathrm{Exp}(\lambda)$. *Then random variable* $X \sim \mathrm{ESEI}(\mu, \sigma, \lambda, q; g)$.

Proof Let $Y = (X - \mu)/\sigma$. Then from Remark 2.3 $Y|U = u \sim \mathrm{EI}(0, u^{-1/q}; g)$. Hence the result follows from Proposition 2.4. □

Remark 2.6 Proposition 2.4 and Corollary 2.5 declare that the exponential-slashed-elliptical distribution can be represented as a mixture of a particular scale elliptical distribution and the exponential distribution. This is an important result in the sense that it provides a simple way for generating random numbers from the exponential-slashed-elliptical distribution.

2.3. Moments and kurtosis coefficient

In this section, the moments are derived, which can be used in deriving moments estimators and kurtosis coefficient evaluation.

Proposition 2.7 *Let $Y \sim \mathrm{ESEI}(0, 1, \lambda, q; g), X \sim \mathrm{ESEI}(\mu, \sigma, \lambda, q; g)$ and $r < q$. Then the rth moments of random variables Y and X are respectively given by*

$$\mu_r = \mathrm{E}(Y^r) = \begin{cases} \frac{r!}{i^r (r/2)!} a_{r/2} \lambda^{r/q} \Gamma(1 - \frac{r}{q}), & r \text{ even} \\ 0, & r \text{ odd} \end{cases} \quad \text{and} \quad \mu'_r = \mathrm{E}(X^r) = \sum_{k=0}^{r} \sigma^k \mu_k \mu^{r-k} \tag{2.6}$$

with $i = \sqrt{-1}$, $a_{r/2} = \mathrm{d}^{r/2} k(x)/\mathrm{d}x^{r/2}|_{x=0}$ for $r = 2, 4, \ldots,$ here $k(x)$ is the kernel of $g(x)$, i.e., the generator after removing all normalization constants.

Proof Notice Y symmetrically takes possible values in $(-\infty, \infty)$, so it is apparent that the odd moments of Y are zero. For even moments, by formula (2.1), it follows that

$$\mu_r = \mathrm{E}(Y^r) = \mathrm{E}(W^r U^{-r/q}) = \mathrm{E}\left(W^r\right) \mathrm{E}(U^{-r/q}).$$

After some simple calculations, we get $\mathrm{E}\left(U^{-r/q}\right) = \lambda^{r/q}\Gamma(1-r/q)$, $r < q$, hence using the results in reference [16] as below $\mathrm{E}\left(W^r\right) = \frac{r!}{i^r (r/2)!} a_{r/2}$ gives the first result. The second result follows from the fact that X can be represented as $X = \mu + \sigma Y$, where $Y \sim \mathrm{ESEI}(0, 1, \lambda, q; g)$. □

Corollary 2.8 *An immediate consequence of Proposition 2.7 is that the mean, variance and kurtosis coefficient of random variable $Y \sim \mathrm{ESEI}(\mu, \sigma, \lambda, q; g)$ are respectively given by*

$$\begin{aligned} &\mathrm{E}(Y) = \mu, \quad \mathrm{Var}(Y) = -2\sigma^2 a_1 \lambda^{2/q} \Gamma(1 - 2/q), \quad q > 2, \\ &\kappa = \mathrm{kurt}(Y) = \frac{3a_2}{a_1^2} \frac{\Gamma(1 - 4/q)}{\Gamma(1 - 2/q)^2}, \quad q > 4. \end{aligned} \tag{2.7}$$

Table 2 shows the value of the a_1 and a_2 functions and the supported range (over q) of kurtosis coefficient (2.7) for some usual cases.

Exponential-slash distribution	a_1	a_2	Kurtosis coefficient range
Normal	-0.5	0.25	$3 < \kappa < \infty$
Cauchy	-1	2	$6 < \kappa < \infty$
Student-t with degree $v > 0$	$-\frac{1+v}{2v}$	$\frac{(1+v)(3+v)}{4v^2}$	$\frac{3(3+v)}{1+v} < \kappa < \infty$
Type II pearson with parameter $\alpha > 0$	$1 - \alpha$	$(\alpha - 1)(\alpha - 2)$	$\frac{3(\alpha-2)}{\alpha-1} < \kappa < \infty$

Table 2 Kurtosis coefficient ranges for some special cases

Remark 2.9 From formula (2.7), we see that the kurtosis coefficient is not relative to scale parameters σ and λ and decreasing to the corresponding one of elliptical distribution (which is $3a_1^{-2}a_2$) with respect to parameter q. These indicate that the exponential-slashed-elliptical distribution presents heavier both tails than its normal parent and the heavier tails can be obtained by taking smaller q.

2.4. Inference

In this section, we will discuss moments and maximum likelihood estimation for parameters in the exponential-slashed-elliptical distribution. Notice that $\sigma^2\lambda^{1/q}$ is a combination of scale parameters, we may take $\sigma = 1$ below.

Proposition 2.10 *Let $Y_1, \ldots, Y_n$ be a random sample from the* $\mathrm{ESEI}(\mu, 1, \lambda, q; g)$ *distribution. Then the moment estimates of μ, λ and q ($q > 4$) are given by*

$$\hat{\mu} = \overline{Y}, \quad \hat{\lambda} = \Big(\frac{s^2}{-2a_1\Gamma(1-2/\hat{q})}\Big)^{\hat{q}/2},$$

where $\hat{q}$ is the solution of equation $(b_2a_1^2)/(3a_2) = \Gamma(1-4/q)/[\Gamma(1-2/q)^2]$, $\overline{Y}$, s^2 and b_2 respectively are the sample mean, sample variance and sample kurtosis coefficient.

Proof From Corollary 2.8 and the definition of the moment estimates, we make

$$\begin{cases} \overline{Y} = \mu, \\ s^2 = -2a_1\lambda^{2/q}\Gamma(1-2/q), \\ b_2 = 3a_2\Gamma(1-4/q)/[a_1^2\ \Gamma(1-2/q)^2]. \end{cases}$$

The results can be obtained immediately by solving above system of equations. $\square$

Proposition 2.11 *Let $y_1, \ldots, y_n$ be a group of sample observations from* $\mathrm{ESEI}(0, 1, \lambda, q; g)$ *distribution. Then the maximum likelihood estimates of λ and q can be found by solving following system of equations*

$$\frac{1}{\lambda} = \frac{1}{n}\sum_{i=1}^{n}\int_0^\infty g(y_i^2v^2)v^{2q}\exp\{-\lambda v^q\}\mathrm{d}v\Big/\int_0^\infty g(y_i^2v^2)v^q\exp\{-\lambda v^q\}\mathrm{d}v, \tag{2.8}$$

$$\frac{1}{q} = -\frac{1}{n}\sum_{i=1}^{n}\int_0^\infty g(y_i^2v^2)v^q(1-\lambda v^q)\log(v)\exp\{-\lambda v^q\}\mathrm{d}v\Big/\int_0^\infty g(y_i^2v^2)v^q\exp\{-\lambda v^q\}\mathrm{d}v. \tag{2.9}$$

Proof From formula (2.3), we can write the log-likelihood function as

$$l(\lambda, q) = n\log(\lambda) + n\log(q) + \sum_{i=1}^{n}\log\Big(\int_0^\infty g(y_i^2v^2)v^q\exp\{-\lambda v^q\}\mathrm{d}v\Big). \tag{2.10}$$

Hence we get the score functions as follows

$$\begin{cases} \partial l/\partial\lambda = n/\lambda - \sum_{i=1}^n\int_0^\infty g(y_i^2v^2)v^{2q}\exp\{-\lambda v^q\}\mathrm{d}v/\int_0^\infty g(y_i^2v^2)v^q\exp\{-\lambda v^q\}\mathrm{d}v, \\ \partial l/\partial q = n/q + \sum_{i=1}^n\int_0^\infty g(y_i^2v^2)v^q(1-\lambda v^q)\log(v)\exp\{-\lambda v^q\}\mathrm{d}v/\int_0^\infty g(y_i^2v^2)v^q\exp\{-\lambda v^q\}\mathrm{d}v. \end{cases}$$

Let the score functions be equal to zero. We can find λ and q satisfy the system of equations (2.8) and (2.9). $\square$

Remark 2.12 Usually, the explicit solution for the system of equations (2.8) and (2.9) in Proposition 2.11 is not solved easily, but it can be obtained by using numerical iteration such as the Newton–Raphson procedure. So the observed Fisher information matrix is needed, which is given by

$$I(\lambda, q) = -\begin{pmatrix} l_{\lambda\lambda} & l_{\lambda q} \\ l_{\lambda q} & l_{qq} \end{pmatrix}$$

with entries equal to minus the second partial derivatives of the log-likelihood function given in formula (2.10) with respect to the model parameters.

$$l_{\lambda,\lambda} = -\frac{n}{\lambda^2} + \sum_{i=1}^{n}\Big\{-\frac{\big[\int_0^\infty -\exp\{-v^q\lambda\}v^{2q}g(y_i^2v^2)\,\mathrm{d}v\big]^2}{\big[\int_0^\infty \exp\{-v^q\lambda\}v^q g(y_i^2v^2)\,\mathrm{d}v\big]^2} + \frac{\int_0^\infty \exp\{-v^q\lambda\}v^{3q}g(y_i^2v^2)\,\mathrm{d}v}{\int_0^\infty \exp\{-v^q\lambda\}v^q g(y_i^2v^2)\,\mathrm{d}v}\Big\},$$

$$l_{\lambda,q} = \sum_{i=1}^{n}\frac{\int_0^\infty \exp\{-v^q\lambda\}v^{2q}g(y_i^2v^2)\,\mathrm{d}v\cdot\int_0^\infty \exp\{-v^q\lambda\}\log(v)g(y_i^2v^2)v^q(1-v^q\lambda)\,\mathrm{d}v}{\big[\int_0^\infty \exp\{-v^q\lambda\}v^q g(y_i^2v^2)\,\mathrm{d}v\big]^2} + \sum_{i=1}^{n}\frac{\int_0^\infty \exp\{-v^q\lambda\}v^{2q}\log(v)g(y_i^2v^2)(-2+v^q\lambda)\,\mathrm{d}v}{\int_0^\infty \exp\{-v^q\lambda\}v^q g(y_i^2v^2)\,\mathrm{d}v},$$

$$l_{q,q} = -\frac{n}{q^2} - \sum_{i=1}^{n}\frac{\big[\int_0^\infty \exp\{-v^q\lambda\}\log(v)g(y_i^2v^2)v^q(1-v^q\lambda)\,\mathrm{d}v\big]^2}{\big[\int_0^\infty \exp\{-v^q\lambda\}v^q g(y_i^2v^2)\,\mathrm{d}v\big]^2} + \sum_{i=1}^{n}\frac{\int_0^\infty \exp\{-v^q\lambda\}\log(v)^2g(y_i^2v^2)v^q(1-3v^q\lambda+v^{2q}\lambda^2)\,\mathrm{d}v}{\int_0^\infty \exp\{-v^q\lambda\}v^q g(y_i^2v^2)\,\mathrm{d}v}.$$

3. Real data analysis

The dataset is obtained from RESSET Financial Research Database (http://www2.resset.cn/). We collect a total 127 observations about the monthly risk-free return reported by some bank of China from the period of March 2003 to September 2013. We consider the monthly changes (total 126 observations) in risk-free return (unit: 1/10000).

Table 3 summarizes the data set where b_1 is sample skewness coefficient. Figure 3 shows the histogram of the data, including estimated densities under a regular normal distribution, a slash distribution and an exponential-slashed-normal distribution where g is a normal generator using maximum likelihood method. Consider that $\sigma^2\lambda^{1/q}$ is a combination of scale parameters just as we mentioned in 2.4, there are multiple solutions for the system of log likelihood equations from ESN distribution. In fact, there are multiple solutions for the parameter combination (σ, λ), but unique solution for parameters μ and q, which is similar to the work given by Reyes et.al. (2013). And we can see this from Figure 7 in [11]. In our example, we take 1 as the starting point of λ and random values for other parameters. We find the algorithmic procedure works well and the MLE for (μ, σ, q) is unique. The results are reported in Table 4. The Akaike information criterion (AIC) is used to measure the goodness of fit of the models. AIC $= 2k - 2\log L$, where k is the number of parameters in the model and L is the maximized value of the likelihood function for the estimated model. The results show that the exponential-slashed-normal distribution fits much better.

sample size	mean	standard deviation	b_1	b_2
126	0.1581	2.5778	-0.2273	8.4287

Table 3 Summary for monthly changes in risk-free return data

Model	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\lambda}$	$\hat{q}$	loglik	AIC
Normal	0.1581	2.5676	–	–	−297.5988	599.1975
	(0.0523)	(0.0262)				
Slash	0.2096	0.3471	–	0.8414	−257.1507	520.3014
	(0.0052)	(0.0056)		(0.0192)		
ESN	0.1809	0.6292	1.0574	1.1019	−253.0093	514.0186
	(0.0044)	(0.0309)	(0.0756)	(0.0335)		

Table 4 MLEs (with (SD)) of the Normal, slash and ESN models for the monthly changes in risk-free return data

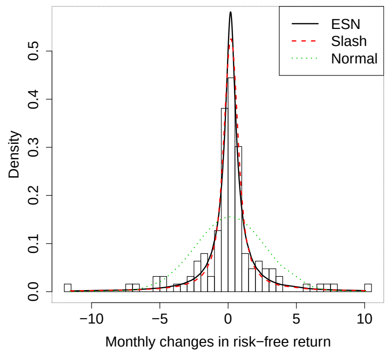

Figure 3 Histogram and fitted curves for the monthly changes in risk-free return data set

4. Multivariate exponential-slashed-elliptical distribution

Now we consider the multivariate extension of the exponential-slashed-elliptical family introduced earlier. Recall that $\mathbf{Y} = (Y_1, \ldots, Y_k)^T$ has elliptical distribution with location vector $\boldsymbol{\mu}$ and positive definite scale matrix $\boldsymbol{\Sigma}$ if its joint density is given by

$$f_{\mathbf{Y}}(\mathbf{y}) = |\boldsymbol{\Sigma}|^{-1/2} g((\mathbf{y}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu})), \quad \mathbf{y} \in \mathbb{R}^k,$$

here g represents the k-variate density generator function, assumed to satisfy the condition $\int_0^\infty u^{k-1} g(u^2)\mathrm{d}u < \infty$. We use the notation $\mathbf{Y} \sim \mathrm{EI}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma}; g)$. If the appropriate moments of $\mathbf{Y}$ are finite, then $\mathrm{E}(\mathbf{Y}) = \boldsymbol{\mu}$ and $\mathrm{Var}(\mathbf{Y}) = \alpha\boldsymbol{\Sigma}$, where α is a positive constant that depends on g (see [5]).

Motivated by the univariate representation (2.1), we say that a k-dimension random vector $\mathbf{Y}$ has exponential-slashed-elliptical distribution with location vector parameter $\boldsymbol{\mu}$, scale parameter λ, positive definite matrix scale parameter $\boldsymbol{\Sigma}$, and tail parameter $q > 0$, if it can be represented as

$$\mathbf{Y} = \boldsymbol{\mu} + \boldsymbol{\Sigma}^{1/2} \frac{\mathbf{W}}{U^{1/q}}, \tag{4.1}$$

where $\mathbf{W} \sim \mathrm{EI}_k(\mathbf{0}, \mathbf{I}_k; g)$ is assumed to be independent of $U \sim \mathrm{Exp}(\lambda)$ with $\lambda > 0$. We denote this as $\mathbf{Y} \sim \mathrm{ESEI}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \lambda, q; g)$. In formula (4.1) and in what follows $\boldsymbol{\Sigma}^{1/2}$ refers to the *square* *root* of $\mathbf{\Sigma}$.

Next we give the general expression for the corresponding density function.

Proposition 4.1 *Let* $\mathbf{Y} \sim \mathrm{ESEI}_k(\boldsymbol{\mu}, \mathbf{\Sigma}, \lambda, q; g)$. *Then the density function of* $\mathbf{Y}$ *is given by*

$$f_{\mathbf{Y}}(\mathbf{y}; \boldsymbol{\mu}, \mathbf{\Sigma}, \lambda, q) = \begin{cases} (\lambda q/2) \cdot |\mathbf{\Sigma}|^{-1/2} \gamma^{-(q+k)/2} \int_0^\infty g(v) v^{(q+k)/2-1} \exp\{-\lambda v^{q/2} \gamma^{-q/2}\} \mathrm{d}v, & \textit{if } \mathbf{y} \neq \boldsymbol{\mu}, \\ \lambda^{-k/q} |\mathbf{\Sigma}|^{-1/2} g(0) \Gamma(1 + k/q), & \textit{if } \mathbf{y} = \boldsymbol{\mu}, \end{cases}$$

here $\gamma = (\mathbf{y} - \boldsymbol{\mu})^T \mathbf{\Sigma}^{-1} (\mathbf{y} - \boldsymbol{\mu})$.

Proof From formula (4.1), we can find that the density function of $\mathbf{Y}$ is given by

$$\begin{aligned} f_{\mathbf{Y}}(\mathbf{y}; \boldsymbol{\mu}, \mathbf{\Sigma}, \lambda, q) &= \int_0^\infty |\, \mathbf{\Sigma} \,|^{-1/2} f_W(\mathbf{\Sigma}^{-1/2}(\mathbf{y} - \boldsymbol{\mu}) u^{1/q}) f_U(u) u^{k/q} \mathrm{d}u \\ &= \lambda q \,|\, \mathbf{\Sigma} \,|^{-1/2} \int_0^\infty t^{k+q-1} g\left(\gamma t^2\right) e^{-\lambda t^q} \mathrm{d}t, \end{aligned}$$

from which the result for the case $\mathbf{y} = \boldsymbol{\mu}$ is obtained by the definition of Gamma function. The other case follows by considering the transformation $v = \gamma t^2$. $\square$

The mean and covariance matrix of the multivariate exponential-slashed-elliptical distribution are given as follows.

Proposition 4.2 *Let* $\mathbf{Y} \sim \mathrm{ESEI}_k(\boldsymbol{\mu}, \mathbf{\Sigma}, \lambda, q; g)$. *Then the mean vector and covariance matrix of* $\mathbf{Y}$ *respectively are* $\mathrm{E}(\mathbf{Y}) = \boldsymbol{\mu}$, $\mathrm{COV}(\mathbf{Y}) = \alpha \lambda^{2/q} \Gamma(1 - 2/q) \mathbf{\Sigma}$, $q > 2$.

Proof The first result is easy to get from the expression (4.1). For the second result, it is easy to see from formula (4.1) that

$$\mathrm{COV}(\mathbf{Y}) = \mathbf{\Sigma}^{1/2} \mathrm{COV}(\mathbf{W}) \mathbf{\Sigma}^{1/2} \cdot \mathrm{E}(U^{-2/q}).$$

Hence the result follows by $\mathrm{COV}(\mathbf{W}) = \alpha \mathbf{I}_k$ and $E(U^{-2/q}) = \lambda^{2/q} \Gamma(1 - 2/q)$, $q > 2$. $\square$

5. Concluding remarks

In this paper, we introduce a new family of exponential-slashed-elliptical distributions by replacing uniform assumption with exponential variables underlying the definition of the slashed-elliptical distribution to support extended kurtosis coefficient for univariate and multivariate distributions. The result family is quite flexible, including all the elliptical distributions as its special cases, and has heavier tails than its parent elliptical distribution, so it is useful for modeling data sets that may have heavy-tails and/or outliers. A real data is presented to show better performance of the exponential-slashed-normal distribution, one special case of the exponential-slashed-elliptical distribution, when compared to its parent normal and slash distributions.

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