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Local Spectral Properties under Perturbations

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Abstract It is shown that local spectral properties such as the single-valued extension property, Dunford's property (C), Bishop's property (β), the decomposition property (δ), or decomposability are stable under commuting perturbations whose spectra are finite.

Keywords Local spectral property; perturbation; algebraic operator

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1. Introduction

Are sums and products of commuting decomposable operators on Banach spaces decomposable? This is one of the most important open problems in the local spectral theory of operators on Banach spaces. Similarly, it is not known if local spectral properties such as Dunford's property (C), Bishop's property (β), or the decomposition property (δ) are preserved under sums and products of commuting operators. But it was shown in [3] that the single-valued extension property is not preserved under the sums and products of commuting operators; see also [2]. On the positive side, Sun [5] proved that the sum and the product of two commuting operators with Dunford's property (C) have the single-valued extension property.

Very recently, Aiena and Müller [1] showed that the (localized) single-valued extension property is stable under commuting Riesz perturbations. In this note, we show that local spectral properties such as the single-valued extension property, Dunford's property (C), Bishop's property (β), the decomposition property (δ), or decomposability are stable under commuting perturbations whose spectra are finite.

Throughout this note, $\mathcal{B}(X)$ will denote the set of all bounded linear operators on an infinite-dimensional complex Banach space X. For an operator $T \in \mathcal{B}(X)$, let $\sigma(T)$ denote its spectrum. An operator $T \in \mathcal{B}(X)$ is said to have the single valued extension property (SVEP for brevity), if for every open neighbourhood U the only analytic function $f : U \to X$ which satisfies $(T-\mu)f(\mu) = 0$ for all $\mu \in U$ is the function $f \equiv 0$.

For $T \in \mathcal{B}(X)$, the local resolvent set $\rho_T(x)$ of T at the point $x \in X$ is defined as the set of all $\lambda \in \mathbb{C}$ for which there exist an open neighborhood U_λ of λ and an analytic function

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$f : U_\lambda \to X$ such that

$$(T-\mu)f(\mu) = x \quad \text{for all} \quad \mu \in U_\lambda.$$

The local spectrum $\sigma_T(x)$ of T at x is then defined as $\sigma_T(x) = \mathbb{C}\backslash\rho_T(x)$. The local analytic solutions occurring in the definition of the local resolvent set will be unique for all $x \in X$ if and only if T has SVEP. For every subset F of $\mathbb{C}$, we define the local spectral subspace of T associated with F by

$$X_T(F) := \{x \in X : \sigma_T(x) \subseteq F\}.$$

Evidently, $X_T(F)$ is a hyperinvariant subspace of T, but not always closed. An operator $T \in \mathcal{B}(X)$ is said to have Dunford's property (C), if the local spectral subspace $X_T(F)$ is closed for every closed set $F \subseteq \mathbb{C}$.

Let $\mathcal{O}(U, X)$ denote the Fréchet algebra of all X-valued analytic functions on the open subset $U \subseteq \mathbb{C}$ endowed with uniform convergence on compact subsets of U. An operator $T \in \mathcal{B}(X)$ is said to satisfy Bishop's property (β) if for every open subset U of $\mathbb{C}$ and for any sequence $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{O}(U, X)$, $\lim_{n\to\infty}(T-\mu)f_n(\mu) = 0$ in $\mathcal{O}(U, X)$ implies $\lim_{n\to\infty} f_n(\mu) = 0$ in $\mathcal{O}(U, X)$. Dually, $T \in \mathcal{B}(X)$ is said to have the decomposition property (δ) if T^* satisfies (β). It is well known that T is decomposable, in the sense of Foiaş, if and only if T satisfies both (β) and (δ). We refer the reader to the seminal book [4] for further definitions. Moreover, it is shown that ([4])

$$\text{Bishop's property } (\beta) \Longrightarrow \text{Dunford's property } (C) \Longrightarrow \text{SVEP}.$$

2. Main result

We start with the following lemmas.

Lemma 2.1 *Let $T \in \mathcal{B}(X)$ and $\lambda \in \mathbb{C}$.*

(1) If T has SVEP, then so does $T + \lambda$;

(2) If T has Dunford's property (C), then so does $T + \lambda$;

(3) If T satisfies Bishop's property (β), then so does $T + \lambda$.

Proof The proofs of (1) and (3) can be easily derived from the notions involved in the statements, so the details are left to the reader.

(2) Since $x \in X_{T+\lambda}(F) \Longleftrightarrow \sigma_{T+\lambda}(x) \subseteq F \Longleftrightarrow \sigma_T(x) \subseteq F - \lambda \Longleftrightarrow x \in X_T(F - \lambda)$, we know that $X_{T+\lambda}(F) = X_T(F - \lambda)$. Hence Dunford's property (C) is transmitted form T to $T + \lambda$. $\square$

Lemma 2.2 *Let $T = T_1 \oplus T_2$ with respect to the direct decomposition $X = Y \oplus Z$.*

(1) T has SVEP if and only if both T_1 and T_2 have SVEP;

(2) T has Dunford's property (C) if and only if both T_1 and T_2 have Dunford's property (C);

(3) T satisfies Bishop's property (β) if and only if both T_1 and T_2 satisfy Bishop's property (β).

Proof The proofs of (1) and (3) are also left to the reader. Actually, a more general version of these results could be found in [4].

(2) Suppose that T has Dunford's property (C). Then by [4, Proposition 1.2.21], both T_1 and T_2 have Dunford's property (C).

Conversely, suppose that both T_1 and T_2 have Dunford's property (C). Since $x = y \oplus z \in X_{T_1 \oplus T_2}(F) \Longleftrightarrow \sigma_{T_1 \oplus T_2}(y \oplus z) = \sigma_{T_1}(y) \cup \sigma_{T_2}(z) \subseteq F \Longleftrightarrow \sigma_{T_1}(y) \subseteq F$ and $\sigma_{T_2}(z) \subseteq F \Longleftrightarrow y \in Y_{T_1}(F)$ and $z \in Z_{T_2}(F)$, we know that $X_T(F) = X_{T_1 \oplus T_2}(F) = Y_{T_1}(F) \oplus Z_{T_2}(F)$ is closed for every closed set $F \subseteq \mathbb{C}$. Therefore, T has Dunford's property (C). $\square$

Recall that an operator Q is called quasinilpotent if $\sigma(Q) = \{0\}$. We say that an operator is polynomially quasi-nilpotent if there exists a nonzero complex polynomial p such that $p(Q)$ is quasi-nilpotent. By the spectral mapping theorem for the ordinary spectrum, we infer that Q is polynomially quasi-nilpotent if and only if $\sigma(Q)$ is finite.

We are now ready for the main result of this note.

Theorem 2.3 *Let $T \in \mathcal{B}(X)$ and $Q \in \mathcal{B}(X)$ be a polynomially quasi-nilpotent operator commuting with T.*

(1) If T has SVEP, then so does $T + Q$;

(2) If T has Dunford's property (C), then so does $T + Q$;

(3) If T satisfies Bishop's property (β), then so does $T + Q$;

(4) If T satisfies decomposition property (δ), then so does $T + Q$;

(5) If T is decomposable, then so is $T + Q$.

Proof (2) Since Q is polynomially quasi-nilpotent, we suppose that $\sigma(Q) = \{\lambda_1, \dots, \lambda_n\}$. For $i = 1, \dots, n$, let P_i be the spectral projection associated with Q and the spectral set $\{\lambda_i\}$. From the classical spectral theory, it follows that $X_i := P_i(X)$ $(i = 1, \dots, n)$ is a closed invariant subspace of both Q and T, and that

$$X = X_1 \oplus \cdots \oplus X_n.$$

Let $Q_i = Q|_{X_i}$ and $T_i = T|_{X_i}$. Then we have $T_i Q_i = Q_i T_i$ and $\sigma(Q_i) = \{\lambda_i\}$. Since T has Dunford's property (C), it follows from [4, Proposition 1.2.21] that T_i $(i = 1, \dots, n)$ Dunford's property (C), and hence $T_i + \lambda_i$ $(i = 1, \dots, n)$ has Dunford's property (C) by Lemma 2.1. Since $\sigma(Q_i - \lambda_i) = \{0\}$ $(i = 1, \dots, n)$ and $T_i + Q_i = T_i + \lambda_i + Q_i - \lambda_i$ $(i = 1, \dots, n)$, we conclude from [4, Proposition 3.4.11] that $T_i + Q_i$ $(i = 1, \dots, n)$ has Dunford's property (C). Therefore $T + Q = (T_1 + Q_1) \oplus \cdots \oplus (T_n + Q_n)$ has Dunford's property (C) by Lemma 2.2.

(1) and (3) The proofs are similar to that of (2), we omit them here.

(4) Noting that T satisfies decomposition property (δ) if and only if T^* satisfies Bishop's property (β) and that Q is polynomially quasi-nilpotent if and only if Q^* is polynomially quasi-nilpotent, the assertion follows from (3).

(5) Since decomposability may be expressed as the conjunction of the properties (β) and (δ), we conclude from (3) and (4) the desired assertion. $\square$

Recall that an operator K is called *algebraic* if there exists a nonzero complex polynomial

p such that $p(K) = 0$. Evidently, algebraic operators are polynomially quasi-nilpotent.

Corollary 2.4 *Let $T \in \mathcal{B}(X)$ and $K \in \mathcal{B}(X)$ be an algebraic operator commuting with T.*

(1) If T has SVEP, then so does $T + K$;

(2) If T has Dunford's property (C), then so does $T + K$;

(3) If T satisfies Bishop's property (β), then so does $T + K$;

(4) If T satisfies decomposition property (δ), then so does $T + K$;

(5) If T is decomposable, then so is $T + K$.

Recall that an operator F is called power finite rank if there exists $n \in \mathbb{N}$ such that F^n is of finite rank. It is well known that power finite rank operators are algebraic [7].

Corollary 2.5 *Let $T \in \mathcal{B}(X)$ and $F \in \mathcal{B}(X)$ be a power finite rank operator commuting with T.*

(1) If T has SVEP, then so does $T + F$;

(2) If T has Dunford's property (C), then so does $T + F$;

(3) If T satisfies Bishop's property (β), then so does $T + F$;

(4) If T satisfies decomposition property (δ), then so does $T + F$;

(5) If T is decomposable, then so is $T + F$.

Corollary 2.6 *Let $T \in \mathcal{B}(X)$ and $F \in \mathcal{B}(X)$ be a finite rank operator commuting with T.*

(1) If T has SVEP, then so does $T + F$;

(2) If T has Dunford's property (C), then so does $T + F$;

(3) If T satisfies Bishop's property (β), then so does $T + F$;

(4) If T satisfies decomposition property (δ), then so does $T + F$;

(5) If T is decomposable, then so is $T + F$.

It is worth mentioning that the above corollary ceases to be true for non-commuting operators. Indeed, the sum of a decomposable operator and a rank-one operator may fail to have SVEP [6, Example 5.6.29].

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