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A Class of Projectively Flat Spherically Symmetric Finsler Metrics

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Abstract In this paper, we study spherically symmetric Finsler metrics. Analyzing the solution of the projectively flat equation, we construct a new class of projectively flat Finsler metrics.

Keywords projectively flat; Finsler metric; spherically symmetric

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1. Introduction

It is an important problem in Finsler geometry to study and characterize projectively flat Finsler metrics on an open domain in $\mathbb{R}^m$. Hilbert's 4th Problem is to characterize the distance functions on an open subset in $\mathbb{R}^m$ such that straight lines are geodesics [5]. Regular distance functions with straight geodesics are projectively flat Finsler metrics. A Finsler metric $F = F(x,y)$ on an open subset $U \subset \mathbb{R}^m$ is projectively flat if and only if it satisfies the following equation:

$$F_{x^i y^j} y^i = F_{x^i}. \tag{1.1}$$

Shen discussed the classification problem on projective Finsler metrics of constant flag curvature [14], the first author provided the projective factor of a class of projectively flat general (α,β)-metrics [12] and studied a necessary and sufficient condition for a class of Finsler metric to be projectively flat [13]. Li proved that locally projectively flat Finsler metrics with constant flag curvature $\mathbf{K}$ are totally determined by their behaviors at the origin by solving some nonlinear PDEs. The classifications when $\mathbf{K}=0$, $\mathbf{K}=-1$, $\mathbf{K}=1$ are given in an algebraic way [15].

On the other hand, the study of spherically symmetric Finsler metrics has attracted a lot of attentions. Many known Finsler metrics are spherically symmetric [1,4,7,14,15]. A Finsler metric F is said to be spherically symmetric (orthogonally invariant in an alternative terminology in [6]) if F satisfies

$$F(Ax, Ay) = F(x,y) \tag{1.2}$$

for all $A \in O(m)$, equivalently, if the orthogonal group $O(m)$ acts as isometrics of F. Such metrics were first introduced by Rutz [16].

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It was pointed out in [6] that a Finsler metric F on $\mathbb{B}^m(\mu)$ is a spherically symmetric if and only if there is a function $\phi : [0,\mu) \times \mathbb{R} \to \mathbb{R}$ such that

$$F(x,y) =\mid y \mid \phi(\mid x \mid, \frac{\langle x,y\rangle}{\mid y \mid}) \tag{1.3}$$

where $(x,y) \in T\mathbb{R}^m(\mu)\backslash\{0\}$. Additionally, the spherically symmetric Finsler metric of the form (1.3) can be rewritten as the following form [8]

$$F =\mid y \mid \phi(\frac{\mid x \mid^2}{2}, \frac{\langle x,y\rangle}{\mid y \mid}).$$

Spherically symmetric Finsler metrics are the simplest and most important general (α,β)-metrics [4]. Mo, Zhou and Zhu classified the projective spherically symmetric Finsler metrics with constant flag curvature in [2, 9, 10]. A lot of spherically symmetric Finsler metrics with nice curvature properties had been investigated by Mo, Huang and et al [3, 6–11].

An important example of non-Riemmannian projectively flat Finsler metrics is the Funk metric

$$\Theta = \frac{\sqrt{(1-\mid x \mid^2)\mid y \mid^2 + \langle x,y\rangle^2}}{1-\mid x \mid^2} + \frac{\langle x,y\rangle}{1-\mid x \mid^2} \tag{1.4}$$

on the unit ball $\mathbb{B}^m(1)$ where $y \in T_x\mathbb{B}^m \subset \mathbb{R}^m$. Here $\mid \cdot \mid$ and $\langle \cdot,\cdot\rangle$ denote the standard Euclidean norm and inner product respectively. By simple calculation, the Funk metric Θ could also be expressed in the form $\Theta = \Theta_1 + \Theta_2$, where

$$\Theta_1 =\mid y \mid h(t)s, \quad \Theta_2 =\mid y \mid \sqrt{g(t)+g'(t)s^2}$$

where

$$g(t) = h(t) = \frac{1}{1-2t}, \quad t = \frac{\mid x \mid^2}{2}, \quad s = \frac{\langle x,y\rangle}{\mid y \mid}.$$

We can verify that Θ_1 and Θ_2 satisfy (1.1) by direct calculations. It is easy to see that if Θ_1 and Θ_2 satisfy (1.1), then $a\Theta_1 + b\Theta_2$ is also a solution of (1.1) where a, b are non-negative constants.

Inspired by the above statements, we try to find the solutions of the projectively flat Eq. (1.1) in the following forms:

$$F =\mid y \mid \phi(\frac{\mid x \mid^2}{2}, \frac{\langle x,y\rangle}{\mid y \mid}),$$

$$\phi(\frac{\mid x \mid^2}{2}, \frac{\langle x,y\rangle}{\mid y \mid}) = \sum_{i=0}^{n} \phi_i(\frac{\mid x \mid^2}{2})\frac{\langle x,y\rangle^i}{\mid y \mid^i} + \sum_{r\neq 0,1}[\sum_{j=0}^{l} \phi_j(\frac{\mid x \mid^2}{2})\frac{\langle x,y\rangle^j}{\mid y \mid^j}]^{\frac{1}{r}}.$$

Through calculations, we have the following conclusion:

Theorem 1.1 *Let $\phi(t,s)$ be a function defined by*

$$\begin{aligned}\phi(t,s) =&\phi_0(t) + \phi_1(t)s + \frac{1}{2}\phi_0'(t)s^2 + \sum_{j=2}^{n}(-1)^{j-1}\frac{(2j-3)!!}{(2j)!}\phi_0^{(j)}(t)s^{2j} + \\ &b\sum_{r\neq 0,1}(h_0(t) + \frac{1}{2}h_0'(t)s^2)^{\frac{1}{r}}, \quad n \in \{1,2,\ldots\}, \ r \in Z\end{aligned}$$

where b, C_1, C_2 *are constants and* h_0 *is a differentiable function which satisfies*

$$h_0(t) = \left(\frac{r}{-3C_1tr + 4C_1t - 3C_2r + 4C_2}\right)^{\frac{r}{3r-4}}$$

and ϕ_1 *is any continuous function,* ϕ_0 *is a polynomial function of degree* N *where* $N \leqslant n$ *which satisfies*

$$\begin{aligned}\phi_0(t)+&\phi_0'(t)(t-\frac{3}{2}s^2)+\sum_{j=2}^{n}(-1)^{j-1}\frac{(2j-3)!!}{(2j-2)!}\phi_0^{(j)}(t)s^{2j-2}(t-s^2\frac{1+2j}{2j})+\\ &b\sum_{r\neq 0,1}(h_0(t)+\frac{1}{2}h_0'(t)s^2)^{\frac{1}{r}-2}(h_0^2(t)+\frac{h_0(t)h_0'(t)}{r}t)+\\ &b\sum_{r\neq 0,1}(h_0(t)+\frac{1}{2}h_0'(t)s^2)^{\frac{1}{r}-2}(\frac{1}{2}-\frac{1}{r})s^2\cdot\\ &[2h_0(t)h_0'(t)+(h_0'(t))^2s^2(\frac{1}{2}+\frac{1}{r})-\frac{(h_0'(t))^2}{r}t]>0 \end{aligned}\tag{1.5}$$

when $m = 2$. *Moreover, the additional inequality holds*

$$\begin{aligned}\phi_0(t)-&\frac{1}{2}\phi_0'(t)+\sum_{j=2}^{n}(-1)^{j-1}\frac{(2j-3)!!}{(2j-1)!}\phi_0^{(j)}(t)s^{2j}(\frac{1}{2j}-1)+\\ &b\sum_{r\neq 0,1}(h_0(t)+\frac{1}{2}h_0'(t)s^2)^{\frac{1}{r}-1}[h_0(t)+(\frac{1}{2}-\frac{1}{r})h_0'(t)s^2]>0\end{aligned}\tag{1.6}$$

when $m \geq 3$. $\phi_0^{(j)}$ *denotes the* j*-order derivative for* $\phi_0(t)$ *and then the following spherically symmetric Finsler metric on* $\mathbb{B}^m(\mu)$, $F = \mid y \mid \phi(\frac{|x|^2}{2}, \frac{\langle x,y\rangle}{|y|})$ *is projectively flat.*

Remark 1.2 Let us take a look at a special case, namely when $b = 1, n = m = 2$. After setting $\phi_0(t) = 0, \phi_1(t) = h_0(t) = \frac{1}{1-2t}$, we obtain the Funk metric.

2. Solutions of the PDE

In this section, we are going to construct a lot of projectively flat Finsler metrics which contain the Funk metric. From [8], we have the following lemma.

Lemma 2.1 *Let* $F = \mid y \mid \phi(\frac{|x|^2}{2}, \frac{\langle x,y\rangle}{|y|})$ *be a spherically symmetric Finsler metric on* $\mathbb{B}^m(\mu)$. F *is projectively flat if and only if* ϕ *satisfies*

$$s\phi_{ts} + \phi_{ss} - \phi_t = 0 \tag{2.1}$$

where $t = \frac{|x|^2}{2}$ *and* $s = \frac{\langle x,y\rangle}{|y|}$.

Consider the spherically symmetric Finsler metric $F = \mid y \mid \phi(\frac{|x|^2}{2}, \frac{\langle x,y\rangle}{|y|})$ on $\mathbb{B}^m(\mu)$ where $\phi = \phi(t,s)$ is given by $\phi(t,s) = \sum_{j=0}^{l}\phi_j(t)s^j$. By a direct calculation we get

$$\phi_t(t,s) = \sum_{j=0}^{l}\phi_j'(t)s^j, \tag{2.2}$$

$$\phi_{ts}(t,s)=\sum_{j=0}^{l} j\phi'_j(t)s^{j-1}, \tag{2.3}$$

$$\phi_{ss}(t,s)=\sum_{j=0}^{l} j(j-1)\phi_j(t)s^{j-2}. \tag{2.4}$$

Substituting (2.2), (2.3) and (2.4) into (2.1) yields the following equation

$$\sum_{j=0}^{l}(j-1)\phi'_j(t)s^j+\sum_{j=0}^{l-2}(j+2)(j+1)\phi_{j+2}s^j=0, \tag{2.5}$$

which is equivalent to

$$\sum_{j=0}^{l-2}[(j-1)\phi'_j(t)+(j+2)(j+1)\phi_{j+2}(t)]s^j+\sum_{j=l-1}^{l}(j-1)\phi'_j(t)s^j=0. \tag{2.6}$$

By (2.6), $F=\mid y\mid \phi(\frac{|x|^2}{2},\frac{\langle x,y\rangle}{|y|})$ is projectively flat if and only if

$$\begin{cases} (j-1)\phi'_j(t)+(j+2)(j+1)\phi_{j+2}(t)=0, & j=0,1,2\dots,l-2,\\ (j-1)\phi'_j(t)=0, & j=l-1,l. \end{cases} \tag{2.7}$$

When $j=0$, from the first equation of (2.7) we get

$$\phi'_0(t)=2\phi_2(t). \tag{2.8}$$

Similarly, taking $j=1$ and $j=2$ gives

$$\phi_3(t)=0,\quad \phi'_2(t)+12\phi_4(t)=0. \tag{2.9}$$

If $k=j+2$, the first equation of (2.7) is equivalent to

$$k(k-1)\phi_k(t)+(k-3)\phi'_{k-2}(t)=0. \tag{2.10}$$

It is easy to see the recurrence fomula on $\phi_k(t)$ and $\phi'_k(t)$ satisfies

$$\phi_k(t)=-\frac{k-3}{k(k-1)}\phi'_{k-2}(t). \tag{2.11}$$

If $k=$ odd, $k\geq 3$, then by (2.11) we have

$$\phi_k(t)=(-1)^{\frac{k-3}{2}}\frac{(k-3)(k-5)\cdots 2}{k(k-1)\cdots 4}\phi_3^{(\frac{k-3}{2})}(t)=0. \tag{2.12}$$

If $k=$ even, $k\geq 4$, then we have

$$\phi_k(t)=(-1)^{\frac{k-4}{2}}\frac{24(k-3)!!}{k!}\phi_4^{(\frac{k-4}{2})}(t)=(-1)^{\frac{k-2}{2}}\frac{(k-3)!!}{k!}\phi_0^{(\frac{k}{2})}(t). \tag{2.13}$$

Then we discuss (2.13) in two different cases.

Case 1 If $k=$ odd ≥ 5, setting $l=2n+1$, by the second equation of (2.7) we have

$$\phi_{2n+1}(t)=0,\quad \phi_{2n}(t)=\text{constant}. \tag{2.14}$$

From (2.1), (2.12), (2.13), (2.14) we have

$$\phi(t,s)=\phi_0(t)+\phi_1(t)s+\phi_2(t)s^2+\cdots+\phi_{2n-1}(t)s^{2n-1}+\phi_{2n}(t)s^{2n}+\phi_{2n+1}(t)s^{2n+1}$$

$$= \phi_0(t) + \phi_1(t)s + \frac{1}{2}\phi_0'(t)s^2 + \sum_{j=2}^{n}(-1)^{j-1}\frac{(2j-3)!!}{(2j)!}\phi_0^{(j)}(t)s^{2j}. \tag{2.15}$$

Case 2 If $k =$ even ≥ 4, setting $l = 2n+2$, by the second equation of (2.7) we have

$$\phi_{2n+2}(t) = \text{constant}, \quad \phi_{2n+1}(t) = 0. \tag{2.16}$$

From (2.1), (2.12), (2.13), (2.16) we have

$$\phi(t,s) = \phi_0(t) + \phi_1(t)s + \frac{1}{2}\phi_0'(t)s^2 + \sum_{j=1}^{n}(-1)^{j}\frac{(2j-1)!!}{(2j+2)!}\phi_0^{(j+1)}(t)s^{2j+2}. \tag{2.17}$$

The case $l \in \{1,2,3\}$ is similar. Through the above analysis, we obtain the following proposition.

Proposition 2.2 *Let $F =\mid y \mid \phi(\frac{|x|^2}{2}, \frac{\langle x,y\rangle}{|y|})$ be a spherically symmetric Finsler metric on $\mathbb{B}^m(\mu)$. $F =\mid y \mid \phi(\frac{|x|^2}{2}, \frac{\langle x,y\rangle}{|y|})$ in the form $F =\mid y \mid \sum_{j=0}^{l}\phi_j(t)s^j$ is a solution of the projectively flat Eq.(2.1) if and only if $\phi(t,s)$ satisfies*

$$\begin{aligned}\phi(t,s) &= \phi_0(t) + \phi_1(t)s + \phi_2(t)s^2 + \cdots + \phi_l(t)s^l \\ &= \phi_0(t) + \phi_1(t)s + \frac{1}{2}\phi_0'(t)s^2 + \sum_{j=2}^{n}(-1)^{j-1}\frac{(2j-3)!!}{(2j)!}\phi_0^{(j)}(t)s^{2j}\end{aligned}$$

and $\phi_0^{(n)} =$ constant.

If the solution of (2.1) has the following form

$$\phi(t,s) = (\sum_{j=0}^{l} h_j(t)s^j)^{\frac{1}{r}}, \quad h_l \neq 0,\ r \in Z - \{0,1\}, \tag{2.18}$$

we have

$$r\phi^{r-1}\phi_t = \sum_{j=0}^{l} h_j'(t)s^j, \tag{2.19}$$

$$r\phi^{r-1}\phi_s = \sum_{j=0}^{l} jh_j(t)s^{j-1}, \tag{2.20}$$

$$r(r-1)\phi^{r-2}\phi_s\phi_t + r\phi^{r-1}\phi_{ts} = \sum_{j=0}^{l} jh_j'(t)s^{j-1}. \tag{2.21}$$

Putting together (2.19), (2.20), (2.21), we have

$$\begin{aligned} r\phi^{2r-1}\phi_{ts} &= \phi^r \sum_{j=0}^{l} jh_j'(t)s^{j-1} - (1-\frac{1}{r})(\sum_{j=0}^{l} h_j'(t)s^j)(\sum_{i=0}^{l} ih_i(t)s^{i-1}) \\ &= (\sum_{i=0}^{l} h_i(t)s^i)(\sum_{j=0}^{l} jh_j'(t)s^{j-1}) - (1-\frac{1}{r})(\sum_{j=0}^{l} h_j'(t)s^j)(\sum_{i=0}^{l} ih_i(t)s^{i-1}) \\ &= \sum_{k=1}^{2l}\sum_{i+j=k}[j-(1-\frac{1}{r})i]h_i(t)h_j'(t)s^{k-1}, \end{aligned} \tag{2.22}$$

here we used the following lemma.

Lemma 2.3 ([8]) *We have the following equations*

$$\sum_{i=1}^{m} a_i \sum_{j=1}^{m} b_j = \sum_{k=1}^{2m} \sum_{i+j=k} a_i b_j, \quad \sum_{i=1}^{m} a_i \sum_{j=1}^{m} b_j - \sum_{i=1}^{m} c_i \sum_{j=1}^{m} d_j = \sum_{k=1}^{2m} \sum_{i+j=k} a_i b_j.$$

Differentiating (2.20), we get

$$r(r-1)\phi^{r-2}(\phi_s)^2 + r\phi^{r-1}\phi_{ss} = \sum_{j=0}^{l} j(j-1)h_j(t)s^{j-2}.$$

Similarly, by using Lemma 2.3 we have

$$\begin{aligned} r\phi^{2r-1}\phi_{ss} &= \phi^r \sum_{j=0}^{l} j(j-1)h_j(t)s^{j-2} - r(r-1)\phi^{2r-2}(\phi_s)^2 \\ &= (\sum_{i=0}^{l} h_i(t)s^i)(\sum_{j=0}^{l} j(j-1)h_j(t)s^{j-2}) - (1-\frac{1}{r})(\sum_{i=0}^{l} ih_i(t)s^{i-1})(\sum_{j=0}^{l} jh_j(t)s^{j-1}) \\ &= \sum_{k=2}^{2l} \sum_{i+j=k} j[j-1-(1-\frac{1}{r})i]h_i(t)h_j(t)s^{k-2} \\ &= \sum_{k=0}^{2l-2} \sum_{i+j=k} (j+1)[j-(1-\frac{1}{r})(i+1)]h_{i+1}(t)h_{j+1}(t)s^k. \end{aligned} \tag{2.23}$$

From (2.19) and Lemma 2.3 we have

$$\begin{aligned} r\phi^{2r-1}\phi_t &= \phi^r \sum_{j=0}^{l} h'_j(t)s^j = (\sum_{i=0}^{l} h_i(t)s^i)(\sum_{j=0}^{l} h'_j(t)s^j) \\ &= \sum_{k=0}^{2l} \sum_{i+j=k} h_i(t)h'_j(t)s^k. \end{aligned} \tag{2.24}$$

Combining (2.22), (2.23), (2.24), we have the following

$$\begin{aligned} r\phi^{2r-1}(s\phi_{ts} + \phi_{ss} - \phi_t) = & \sum_{k=0}^{2l-2} \sum_{i+j=k} [j-(1-\frac{1}{r})-1]h_i(t)h'_j(t)s^k + \\ & \sum_{k=0}^{2l-2} \sum_{i+j=k} (j+1)[j-(1-\frac{1}{r})(i+1)]h_{i+1}(t)h_{j+1}(t)s^k + \\ & \sum_{k=2l-1}^{2l} \sum_{i+j=k} [j-(1-\frac{1}{r})i-1]h_i(t)h'_j(t)s^k. \end{aligned} \tag{2.25}$$

As $F = \mid y \mid \phi(\frac{|x|^2}{2}, \frac{\langle x,y\rangle}{|y|})$ on $\mathbb{B}^m(\mu)$ is projectively flat, by using (2.1), we obtain the following equations

$$\begin{cases} [j-(1-\frac{1}{r})i-1]h_i(t)h'_j(t) + (j+1)[j-(1-\frac{1}{r})(i+1)]h_{i+1}(t)h_{j+1}(t) = 0, \\ \qquad\qquad k = 0, 1, \ldots, 2l-2, \\ (j-i+\frac{i}{r}-1)h_i(t)h'_j(t) = 0, \quad k = 2l-1, 2l. \end{cases}$$

Here we focus on a special case $l = 2$ and $h_1(t) = 0$, then

$$\begin{cases} h_0(t)h_0'(t) - 2h_0(t)h_2(t) = 0, \\ h_0(t)h_2'(t) + (-2 + \frac{4}{r})h_2^2(t) + (-3 + \frac{2}{r})h_2(t)h_0'(t) = 0. \end{cases} \tag{2.26}$$

From the first equation of (2.26), we know $h_0(t) = 0$ or $h_0'(t) = 2h_2(t)$. If $h_0(t) = 0$, then $\phi(t,s) = 0$, thus we consider $h_0'(t) = 2h_2(t)$. In this situation we have

$$\phi(t,s) = (h_0(t) + \frac{1}{2}h_0'(t)s^2)^{\frac{1}{r}} \tag{2.27}$$

and from the second equation of (2.26), we know that $h_0(t)$ satisfies

$$\frac{1}{2}h_0(t)h_0''(t) + (-2 + \frac{2}{r})(h_0'(t))^2 = 0. \tag{2.28}$$

The general solution $h_0(t)$ of (2.28) is given by

$$h_0(t) = (\frac{r}{-3C_1tr + 4C_1t - 3C_2r + 4C_2})^{\frac{r}{3r-4}} \tag{2.29}$$

where $C_1,\ C_2$ are constants. Combining Proposition 2.2, (2.27) and (2.29), we have the following proposition by applying the fundamental property of projectively flat equation.

Proposition 2.4 *Let $F = \mid y \mid \phi(\frac{|x|^2}{2}, \frac{\langle x,y\rangle}{|y|})$ be a spherically symmetric Finsler metric on $\mathbb{B}^m(\mu)$. $\phi(t,s) = \sum_{i=0}^{n}\phi_i(t)s^i + \sum_{r\neq 0,1}(\sum_{j=0}^{l}h_j(t)s^j)^{\frac{1}{r}}$ is a solution of the projectively flat Eq. (2.1) if and only if*

$$\phi(t,s) = \phi_0(t) + \phi_1(t)s + \frac{1}{2}\phi_0'(t)s^2 + \sum_{j=2}^{n}(-1)^{j-1}\frac{(2j-3)!!}{(2j)!}\phi_0^{(j)}(t)s^{2j} + b\sum_{r\neq 0,1}(h_0(t) + \frac{1}{2}h_0'(t)s^2)^{\frac{1}{r}}$$

and

$$h_0(t) = (\frac{r}{-3C_1tr + 4C_1t - 3C_2r + 4C_2})^{\frac{r}{3r-4}}$$

where b, C_1, C_2 are constants.

3. Proof of the Theorem

In Proposition 2.4, $\phi(t,s)$ cannot ensure that $F = \mid y \mid \phi(\frac{|x|^2}{2}, \frac{\langle x,y\rangle}{|y|})$ is a Finsler metric. In order to obtain projectively flat Finsler metric, $\phi(t,s)$ in Proposition 2.2 needs to satisfy the necessary and sufficient condition for $F = \alpha\phi(\|\beta_x\|_\alpha, \frac{\beta}{\alpha})$ to be a Finsler metric for any α and β with $\|\beta_x\|_\alpha < b_0$ given by Yu and Zhu [4]. In particular, considering $F = \mid y \mid \phi(\frac{|x|^2}{2}, \frac{\langle x,y\rangle}{|y|})$, then F is a Finsler metric if and only if the positive function ϕ satisfies

$$\phi - s\phi_s > 0, \quad \phi - s\phi_s + (t - s^2)\phi_{ss} > 0$$

when $m \geq 3$ or $\phi - s\phi_2 + (t - s^2)\phi_{ss} > 0$, when $m = 2$. By a direct calculation we have

$$\begin{aligned} \phi - s\phi_s =& \phi_0(t) - \frac{1}{2}\phi_0'(t) + \sum_{j=2}^{n}(-1)^{j-1}\frac{(2j-3)!!}{(2j-1)!}\phi_0^{(j)}(t)s^{2j}(\frac{1}{2j} - 1) + \\ & b\sum_{r\neq 0,1}(h_0(t) + \frac{1}{2}h_0'(t)s^2)^{\frac{1}{r}-1}[h_0(t) + (\frac{1}{2} - \frac{1}{r})h_0'(t)s^2], \end{aligned} \tag{2.30}$$

$$\begin{aligned}
&\phi - s\phi_s + (t-s^2)\phi_{ss} \\
&= \phi_0(t) + \phi_0'(t)(t - \frac{3}{2}s^2) + \sum_{j=2}^{n}(-1)^{j-1}\frac{(2j-3)!!}{(2j-2)!}\phi_0^{(j)}(t)s^{2j-2}(t - s^2\frac{1+2j}{2j}) + \\
&\quad b\sum_{r\neq 0,1}(h_0(t) + \frac{1}{2}h_0'(t)s^2)^{\frac{1}{r}-2}(h_0^2(t) + \frac{h_0(t)h_0'(t)}{r}t) + \\
&\quad b\sum_{r\neq 0,1}(h_0(t) + \frac{1}{2}h_0'(t)s^2)^{\frac{1}{r}-2}(\frac{1}{2} - \frac{1}{r})s^2[2h_0(t)h_0'(t) + (h_0'(t))^2\times \\
&\quad s^2(\frac{1}{2} + \frac{1}{r}) - \frac{(h_0'(t))^2}{r}t].
\end{aligned} \tag{2.31}$$

The proof of the Theorem is completed by combining proposition 2.4, (2.30), (2.31) and the fundamental property of the projectively flat equation (1.1). □

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