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A Note on the Kolmogorov-Feller Weak Law of Large Numbers

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Abstract In this paper, the Kolmogorov-Feller type weak law of large numbers are obtained, which extend and improve the related known works in the literature.

Keywords Kolmogorov-Feller type weak law of large numbers; negatively associated random variables; independent identically distributed random variables

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1. Introduction

The celebrated Kolmogorov-Feller weak law of large numbers (WLLN) provides a necessary and sufficient condition in the i.i.d. case, the point being that the mean does not exist.

Theorem 1.1 ([1, VII.7]) *Let $X, X_1, X_2, \ldots$ be independent identically distributed (i.i.d.) random variables with partial sums $S_n = \sum_{i=1}^n X_i, n \geq 1$. Then*

$$\frac{S_n - nEXI(|X| \leq n)}{n} \xrightarrow{P} 0 \quad as \quad n \to \infty$$

if and only if $xP(|X| > x) \to 0$ as $x \to \infty$.

Gut [2] gave the following example:

Example 1.2 Suppose that $X, X_1, X_2, \ldots$ are independent random variables with common density

$$f(x) = \begin{cases} \frac{1}{2x^2}, & \text{for } |x| > 1, \\ 0, & \text{otherwise.} \end{cases}$$

The mean does not exist, in this case the Feller condition becomes

$$nP(|X| > n) = n \int_n^\infty \frac{1}{x^2} \mathrm{d}x = 1.$$

But

$$\frac{S_n}{n \log n} \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty.$$

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In other words, a weak law exists, but, with another normalization.

Motivated by this example, Gut [2] provided the following general Kolmogorov-Feller weak law of large numbers.

Theorem 1.3 *Let $X, X_1, X_2, \ldots$ be i.i.d. random variables with partial sums $S_n, n \geq 1$. Further, let $b(x)$ be an increasing and regular varying function at infinity with index $1/\rho$ for some $\rho \in (0, 1]$. Finally, set $b_n = b(n), n \geq 1$. Then*

$$\frac{S_n - nEXI(|X| \leq b_n)}{b_n} \xrightarrow{P} 0 \quad as \quad n \to \infty$$

if and only if $nP(|X| > b_n) \to 0$ as $n \to \infty$.

Motivated by Theorem 1.3, we provide the following more general Kolmogorov-Feller type weak law of large numbers–Theorems 2.4 and 2.5 which extend and improve Theorem 1.3.

A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets A and B of $\{1, 2, \ldots, n\}$,

$$\mathrm{Cov}(f_1(X_i, i \in A), f_2(X_j, j \in B)) \leq 0,$$

whenever f_1 and f_2 are coordinatewise increasing and such that the covariance exists. An infinite family of random variables $\{X_i, i \geq 1\}$ is NA if for every positive integer $n \geq 2$, $\{X_i, 1 \leq i \leq n\}$ is NA. This definition was introduced by Alam and Saxena [3] and carefully studied by Block et al. [4] and Joag-Dev and Proschan [5]. NA sequences have many good properties and extensive applications in multivariate statistical analysis and reliability theory. We refer to Joag-Dev and Proschan [5] for fundamental properties, Matula [6] for the Kolmogorov type strong law of large numbers and the three series theorem, Su et al. [7] for a moment inequality, a weak invariance principle and an example to show that there exists infinite families of non-degenerate non-independent strictly stationary NA random variables, Shao [8] for the Rosenthal type maximal inequality and the Kolmogorov exponential inequality, Qiu and Yang [9] for strong laws of large numbers, and so on.

Throughout this paper, we assume that $\{X, X_n, n \geq 1\}$ is a sequence of identically distributed random variables, $\{k_n, n \geq 1\}$ is a sequence of positive integers such that $\lim_{n\to\infty} k_n = \infty, S_{k_n} = \sum_{i=1}^{k_n} X_i$, C always stands for a positive constant which may differ from one place to another.

2. Main results and proofs

In order to prove the main result of this paper, we present the following Lemmas:

Lemma 2.1 *Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed NA random variables. Then for any $t > 0$*

$$nP(|X| > t) \leq -2 \ln \big(1 - P(\max_{1\leq j\leq n} |X_j| > t)\big). \tag{2.1}$$

Proof Since $1 - x \leq e^{-x}$, we have

$$P(\max_{1\leq j\leq n} X_j > t) = 1 - P(\max_{1\leq j\leq n} X_j \leq t) = 1 - P(X_1 \leq t, X_2 \leq t, \ldots, X_n \leq t)$$

$$\geq 1 - \prod_{j=1}^{n} P(X_j \leq t) = 1 - \{P(X \leq t)\}^n$$
$$= 1 - \{1 - P(X > t)\}^n \geq 1 - \exp\{-nP(X > t)\},$$

therefore $nP(X > t) \leq -\ln(1 - P(\max_{1\leq j\leq n} X_j > t))$. Replacing the X_j by $-X_j$ and repeating the above argument will establish

$$nP(-X > t) \leq -\ln\big(1 - P(\max_{1\leq j\leq n} -X_j > t)\big).$$

Hence, (2.1) holds. □

Lemma 2.2 *Let $X, X_1, X_2, \ldots, X_n$ be identically distributed NA random variables. Then for any $t > 0$*

$$P(\max_{1\leq j\leq n} |S_j| > t) \geq 1 - e^{-\frac{1}{2}nP(|X|>2t)}. \tag{2.2}$$

Proof By Lemma 2.1 and $P(\max_{1\leq j\leq n} |S_j| > t) \geq P(\max_{1\leq j\leq n} |X_j| > 2t)$, (2.2) holds. □

Lemma 2.3 *Let $X, X_1, X_2, \ldots, X_n$ be symmetric i.i.d. random variables. Then for any $t > 0$*

$$P(|S_n| > t) \geq \frac{1}{2}\big(1 - e^{-\frac{1}{2}nP(|X|>t)}\big). \tag{2.3}$$

Proof Note that independent random variables are NA random variables, by Lemma 2.1 and 5.7.b of [10], (2.3) holds. □

Now we present the main result of this paper.

Theorem 2.4 *Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed NA random variables, $\{b_n, n \geq 1\}$ be a sequence of increasing positive reals.*

(i) The following statements are equivalent:

$$k_n P(|X| > b_n) \to 0 \quad \text{as} \quad n \to \infty, \tag{2.4}$$

$$\frac{\max_{1\leq j\leq k_n} |X_j|}{b_n} \xrightarrow{P} 0 \text{ as } n \to \infty. \tag{2.5}$$

(ii) If

$$\frac{k_n}{b_n^2} = o(1), \quad \frac{k_n}{b_n^2} \sum_{i=1}^{n} \frac{b_i^2 - b_{i-1}^2}{k_{i-1}} = O(1), \tag{2.6}$$

where $b_0 = 0, k_0 = 1$, then (2.4), (2.5) and the following statement are equivalent:

$$\frac{\max_{1\leq j\leq k_n} |S_j - jEXI(|X| \leq b_n)|}{b_n} \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty. \tag{2.7}$$

Proof (i) (2.4)$\Longrightarrow$(2.5) is obvious. By Lemma 2.1, we have that (2.5)$\Longrightarrow$(2.4).

(ii) (2.4)$\Longrightarrow$(2.7). For $1 \leq j \leq k_n, n \geq 1$, set

$$Y_j^{(n)} = -b_n I(X_j < -b_n) + X_j I(|X_j| \leq b_n) + b_n I(X_j > b_n).$$

Note that for $\forall\varepsilon > 0$

$$(\max_{1\leq j\leq k_n} |S_j - jEXI(|X| \leq b_n)| > \varepsilon b_n)$$

$$= (\max_{1\le j\le k_n} |S_j - jEXI(|X|\le b_n)| > \varepsilon b_n \text{ and } |X_i|\le b_n \text{ for all } i\le k_n)\cup$$
$$(\max_{1\le j\le k_n} |S_j - jEXI(|X|\le b_n)| > \varepsilon b_n \text{ and } |X_i| > b_n \text{ for at least one } i\in\{1,2,\ldots,k_n\})$$
$$\subseteq \Big(\max_{1\le j\le k_n}\Big|\sum_{i=1}^{j}\big(Y_i^{(n)} - EY_i^{(n)} - b_nP(X_i<-b_n) + b_nP(X_i>b_n)\big)\Big| > \varepsilon b_n\Big)\cup$$
$$\Big(\bigcup_{i=1}^{k_n}(|X_i|>b_n)\Big).$$

By (2.4)

$$P\Big(\bigcup_{i=1}^{k_n}(|X_i|>b_n)\Big) \le \sum_{i=1}^{k_n} P(|X_i|>b_n) = k_nP(|X|>b_n)\to 0 \text{ as } n\to\infty.$$

Since $Y_1^{(n)} - EY_1^{(n)}, Y_2^{(n)} - EY_2^{(n)}, \ldots, Y_{k_n}^{(n)} - EY_{k_n}^{(n)}$ are NA random variables for every $n\ge 1$, by (2.4), Theorem 2 of Shao [8], (2.6) and Toeplitz Lemma [11], for n large enough, we have

$$P\Big(\max_{1\le j\le k_n}\Big|\sum_{i=1}^{j}\big(Y_i^{(n)} - EY_i^{(n)} - b_nP(X_i<-b_n) + b_nP(X_i>b_n)\big)\Big| > \varepsilon b_n\Big)$$
$$\le P\Big(\max_{1\le j\le k_n}\Big|\sum_{i=1}^{j}\big(Y_i^{(n)} - EY_i^{(n)}\big)\Big| + k_nb_nP(|X|>b_n) > \varepsilon b_n\Big)$$
$$\le P\Big(\max_{1\le j\le k_n}\Big|\sum_{i=1}^{j}\big(Y_i^{(n)} - EY_i^{(n)}\big)\Big| > \varepsilon b_n/2\Big)$$
$$\le 4\varepsilon^{-2}b_n^{-2}\sum_{i=1}^{k_n}E|Y_i^{(n)}|^2 \le C\frac{k_n}{b_n^2}\left\{E|X|^2I(|X|\le b_n) + b_n^2P(|X|>b_n)\right\}$$
$$= C\frac{k_n}{b_n^2}\sum_{i=1}^{n}E|X|^2I(b_{i-1}<|X|\le b_i) + Ck_nP(|X|>b_n)$$
$$\le C\frac{k_n}{b_n^2}\sum_{i=1}^{n}b_i^2\{P(|X|>b_{i-1}) - P(|X|>b_i)\} + Ck_nP(|X|>b_n)$$
$$\le C\frac{k_n}{b_n^2}\sum_{i=1}^{n}(b_i^2 - b_{i-1}^2)P(|X|>b_{i-1}) + Ck_nP(|X|>b_n)$$
$$= C\frac{k_n}{b_n^2}\sum_{i=1}^{n}\frac{b_i^2 - b_{i-1}^2}{k_{i-1}}k_{i-1}P(|X|>b_{i-1}) + Ck_nP(|X|>b_n)\to 0,\quad n\to\infty.$$

Therefore, (2.7) holds.

(2.7)$\Longrightarrow$(2.4). By Lemma 2.2 and $P(|X - m(X)|>\varepsilon)\le 4P(|X-a|>\varepsilon/2)$ for every constant a and $\varepsilon>0$, we have

$$P(\max_{1\le j\le k_n}|S_j - jEXI(|X|\le b_n)| > \varepsilon b_n) \ge 1 - e^{-\frac{1}{2}k_nP(|X-EXI(|X|\le b_n)|>2\varepsilon b_n)}$$
$$\ge 1 - e^{-\frac{1}{8}k_nP(|X|>4\varepsilon b_n + |m(X)|)},$$

where $m(X)$ denotes the median of X. Therefore, (2.4) holds by (2.7). $\Box$

Theorem 2.5 *Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables, $\{b_n, n \geq 1\}$ be a sequence of increasing positive reals. Then*

(i) (2.4) and (2.5) are equivalent.

(ii) If (2.6) holds, then (2.4), (2.5), (2.7) and the following statement are equivalent:

$$\frac{S_{k_n} - k_n EXI(|X| \leq b_n)}{b_n} \xrightarrow{P} 0 \text{ as } n \to \infty. \tag{2.8}$$

Proof From the proof of Theorem 2.4, it is enough to prove that (2.7)⟹(2.8) and (2.8)⟹(2.4). (2.7)⟹(2.8) is obvious. We prove that (2.8)⟹(2.4). By the weak symmetrization inequalities [10] and Lemma 2.3, we have

$$\begin{aligned} &2P(|S_{k_n} - k_n EXI(|X| \leq b_n)| > \varepsilon b_n) \geq P(|S^S_{k_n}| > 2\varepsilon b_n) \\ &\geq \frac{1}{2}\big(1 - e^{-\frac{1}{2}k_n P(|X^S| > 2\varepsilon b_n)}\big) \geq \frac{1}{2}\big(1 - e^{-\frac{1}{4}k_n P(|X| > 2\varepsilon b_n + |m(X)|)}\big), \end{aligned}$$

where X^S denotes the symmetrized version of X, $S^S_{k_n} = X^S_1 + X^S_2 + \cdots + X^S_{k_n}$, $m(X)$ denotes the median of X. Therefore, (2.4) holds by (2.8). □

Remark 2.6 Suppose that $b(x)$ is an increasing and regular varying function at infinity with index $1/\rho$ for some $\rho \in (0, 1]$, and set $b_n = b(n), k_n = n, n \geq 1$. Then (2.6) holds. Therefore, Theorem 1.3 is obtained from Theorem 2.5.

We present two examples to illustrate Theorem 2.5.

Example 2.7 In Example 1.2, we take $b_n = n$, $k_n = [\sqrt{n}]$, where $[x]$ denotes the greatest integer not exceeding x. Thus,

$$k_n P(|X| > b_n) = [\sqrt{n}\,] \int_n^\infty \frac{1}{x^2} \mathrm{d}x = \frac{[\sqrt{n}\,]}{n} \to 0 \text{ as } n \to \infty$$

and $\frac{k_n}{b_n^2} = o(1)$ and

$$\frac{k_n}{b_n^2} \sum_{i=1}^n \frac{b_i^2 - b_{i-1}^2}{k_{i-1}} \leq \frac{[\sqrt{n}\,]}{n^2} \Big\{1 + \sum_{i=2}^n \frac{2i}{[\sqrt{i-1}\,]}\Big\} \leq 4 \frac{[\sqrt{n}\,]}{n^2} \sum_{i=1}^n \sqrt{i} \leq 4.$$

Therefore, by Theorem 2.5, we have

$$\frac{\max_{1 \leq j \leq [\sqrt{n}]} |S_j - jEXI(|X| \leq n)|}{n} \xrightarrow{P} 0 \text{ as } n \to \infty.$$

Example 2.8 Suppose that $X, X_1, X_2, \ldots$ are independent random variables with common density

$$f(x) = \begin{cases} \frac{2(\ln 3)^2}{x(\ln x)^3}, & \text{for } x > 3, \\ 0, & \text{otherwise}. \end{cases}$$

The mean does not exist. Let $k_n = n$. If we take $b_n = b(n), n \geq 1$, where $b(x)$ is an arbitrary increasing and regular varying function with index $1/\rho$ for some $\rho \in (0, 1]$, then

$$nP(|X| > b_n) = n \int_{b_n}^\infty \frac{2(\ln 3)^2}{x(\ln x)^3} \mathrm{d}x = C \frac{n}{(\ln b_n)^2} \to \infty \text{ as } n \to \infty,$$

therefore, in this case, by Theorem 1.3

$$\frac{S_n - nEXI(|X| \le b_n)}{b_n} \overset{P}{\nrightarrow} 0 \quad \text{as} \quad n \to \infty.$$

But, if we take $b(x) = \exp(x), b_n = b(n)$, then

$$nP(|X| > b_n) = n\int_{b_n}^{\infty} \frac{2(\ln 3)^2}{x(\ln x)^3}\mathrm{d}x = C\frac{1}{n} \to 0 \text{ as } n \to \infty$$

and $\frac{k_n}{b_n^2} = \frac{n}{e^{2n}} = o(1)$. Since $f(x) = e^x/x^2, x \in [2, \infty)$ is an increasing function, we have

$$\begin{aligned}\frac{k_n}{b_n^2}\sum_{i=1}^{n}\frac{b_i^2 - b_{i-1}^2}{k_{i-1}} =& \frac{n}{b_n^2}\Big\{\frac{b_2^2}{2} + \frac{b_3^2}{2\cdot 3} + \cdots + \frac{b_{n-1}^2}{(n-1)(n-2)} + \frac{b_n^2}{n-1}\Big\}\\ \le& 4\frac{n}{b_n^2}\Big\{\frac{b_2^2}{2^2} + \frac{b_3^2}{3^2} + \cdots + \frac{b_{n-1}^2}{(n-1)^2} + \frac{b_n^2}{n-1}\Big\}\\ \le& 4\frac{n}{b_n^2}\Big\{\frac{b_n^2}{n} + \frac{b_n^2}{n-1}\Big\} < 17.\end{aligned}$$

Therefore, by Theorem 2.5, we have

$$\frac{\max_{1\le j\le n}|S_j - jEXI(|X| \le e^n)|}{e^n} \overset{P}{\to} 0 \text{ as } n \to \infty.$$

In other words, a weak law exists, but, with another normalization.

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