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The Majorization Order on Monomials and Termination of the Successive Difference Substitutions

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Abstract We introduce a concept for the majorization order on monomials. With the help of this order, we derive a necessary condition on the positive termination of a general successive difference substitution algorithm (KSDS) for an input form f.

Keywords successive difference substitution algorithm; majorization order on monomials; termination; positive semi-definite form

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1. Introduction

The first successive difference substitution algorithm (SDS) based on the matrix

$$A_n = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ & 1 & \ddots & \vdots \\ & & \ddots & 1 \\ 0 & & & 1 \end{pmatrix}$$

originates from proving homogeneous symmetric inequalities. It was developed by Yang in [1–3], and improved subsequently in [4,5]. In particular, Yao established a new successive difference substitution algorithm based on the matrix

$$G_n = \begin{pmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n} \\ & \frac{1}{2} & \ddots & \vdots \\ & & \ddots & \frac{1}{n} \\ 0 & & & \frac{1}{n} \end{pmatrix}.$$

His method is named as NEWTSDS, which has many interesting properties [5]. These results illustrate that SDS may be an effective tool for solving many problems in real algebra.

However, it is still very hard to find necessary and/or sufficient conditions on the termination of SDS and NEWTSDS. In this paper, we will study the termination of a general successive

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difference substitution algorithm (KSDS) by the majorization order on monomials. Our main result is as follows:

Main result A necessary condition of positively terminating of KSDS for an input f is that, for an arbitrary ordering of variables, every monomial of f with negative coefficient is majorized by at least one monomial of f with positive coefficient.

The paper is organized as follows. In Section 2, we introduce KSDS and present some background materials. In Section 3, we discuss necessary conditions on the termination of KSDS using the majorization order on monomials. The future research directions are outlined in Section 4.

2. General successive difference substitution - KSDS

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. We set $|\alpha| = \alpha_1 + \cdots + \alpha_n$. A form (i.e., a homogeneous polynomial) f of degree d can be written as

$$f(x_1, \ldots, x_n) = \sum_{|\alpha|=d} C_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} = \sum_{|\alpha|=d} C_\alpha X^\alpha,\ C_\alpha \in \mathbb{R}.$$

The next definition is given in [5].

Definition 2.1 *A form f is said to be trivially positive if the coefficient C_α of every monomial X^α is nonnegative. It is said to be trivially negative if $f(1, 1, \ldots, 1) < 0$ (i.e., the sum of coefficients of f is less than zero).*

Definition 2.2 *A form $f(x_1, \ldots, x_n) \in \mathbb{R}[x_1, \ldots, x_n]$ is positive semi-definite on $\mathbb{R}_+^n$ if it satisfies*

$$\forall\ (x_1, \ldots, x_n) \in \mathbb{R}_+^n,\ f(x_1, \ldots, x_n) \geq 0,$$

where $\mathbb{R}_+^n = \{(x_1, \ldots, x_n) \mid x_1 \geq 0, \ldots, x_n \geq 0\}$. We denote by PSD the set of all the positive semi-definite forms on $\mathbb{R}_+^n$. Furthermore, a positive semi-definite form f is said to be positive definite on $\mathbb{R}_+^n$ if $f > 0$ for $(x_1, \ldots, x_n) \neq (0, \ldots, 0)$. The set of all the positive definite forms is denoted by PD.

There are two obvious results describing the relation between trivially positive (negative) and PSD:

(1) If a form f is trivially positive, then $f \in$ PSD.

(2) If a form f is trivially negative, then $f \notin$ PSD.

Given positive real numbers $q_1, \ldots, q_n$, we consider the matrix

$$K_n = \begin{pmatrix} q_1 & q_2 & \cdots & q_n \\ & q_2 & \ddots & \vdots \\ & & \ddots & q_n \\ 0 & & & q_n \end{pmatrix}. \tag{2.1}$$

Notice that $K_n = A_n$ if $q_1 = q_2 = \cdots = q_n = 1$, and that $K_n = G_n$ if $q_1 = 1$, $q_2 = \frac{1}{2}, \ldots, q_i = \frac{1}{i}, \ldots, q_n = \frac{1}{n}$. So K_n is a general form of the matrices including A_n and G_n.

Suppose that S_n is a symmetric group of degree n. For $\sigma \in S_n$, let P_σ be an $n \times n$ permutation matrix corresponding to σ. For example, suppose that $\sigma = (1)(23)$ is a permutation. Then it corresponds to the matrix

$$P_{(1)(23)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

in which the second and third rows are permuted from the identity matrix.

Using the notation in [5], we introduce a few terminologies.

Definition 2.3 *The $n \times n$ matrix B_σ with $\sigma \in S_n$ is defined by*

$$B_\sigma = P_\sigma K_n.$$

As an example, let us consider again $\sigma = (1)(23)$. Then

$$B_{(1)(23)} = P_{(1)(23)} K_3 = \begin{pmatrix} q_1 & q_2 & q_3 \\ 0 & 0 & q_3 \\ 0 & q_2 & q_3 \end{pmatrix}.$$

Definition 2.4 *Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ and $X = (x_1, \ldots, x_n)^T$. Define*

$$\mathrm{SDS}_K(f) = \bigcup_{\sigma \in S_n} f(B_\sigma X).$$

The set $\mathrm{SDS}_K(f)$ is called the set of difference substitution for f based on the matrix K_n.

It is easy to show the following equivalence relations [5]

$$f \in \mathrm{PSD} \Longleftrightarrow \mathrm{SDS}_K(f) \subset \mathrm{PSD} \;\text{ i.e., }\; f \notin \mathrm{PSD} \Longleftrightarrow \exists g \in \mathrm{SDS}_K(f),\;\; g \notin \mathrm{PSD}.$$

Repeatedly using the above two equivalence relations and Definition 2.1, we have an algorithm for testing positive semi-definite polynomials, which is called the successive difference substitution algorithm based on the matrix K_n (KSDS) in [5].

Algorithm KSDS

Input: A form $f \in \mathbb{Q}[x_1, x_2, \ldots, x_n]$.

Output: "$f \in$ PSD "or "$f \notin$ PSD ".

K1: Let $F = \{f\}$.

K2: Compute $T := \bigcup_{g \in F} \mathrm{SDS}_K(g)$, Temp:=$T \setminus \{$ tivially positive polynomials of $T\}$.

  K21: If Temp=$\emptyset$, then return "$f \in$ PSD".

  K22: Else if there are trivially negative forms in Temp then return "$f \notin$ PSD ".

  K23: Else let $F =$ Temp and go to step K2.

There is a fundamental question on the algorithm KSDS. Namely, under what conditions does the algorithm terminate? This question is very hard to answer. Quite recently, Yang and Yao [4,5] obtained some results about the termination of SDS and NEWTSDS. Their results lead to the following definition.

Definition 2.5 *The algorithm KSDS is positively terminating if the output is "$f \in$ PSD" for the input f. The algorithm KSDS is negatively terminating if the output is "$f \notin$ PSD" for the input f. Otherwise, KSDS is not terminating for f.*

According to Definition 2.5, it is easy to get the following assertions.

Lemma 2.6 *(1) The algorithm KSDS is positively terminating for an input f if and only if there exists a positive integer m such that all of the coefficients of the polynomial*

$$f(B_{\sigma_m}\cdots B_{\sigma_2}B_{\sigma_1}X),\quad \forall\sigma_i\in S_n,\ i=1,\ldots,m$$

are positive.

(2) The algorithm KSDS is negatively terminating if and only if there exist m permutations $\sigma_1,\ldots,\sigma_m\in S_n$ such that

$$f\left(B_{\sigma_m}\cdots B_{\sigma_2}B_{\sigma_1}(1,1,\ldots,1)^T\right)<0.$$

3. Majorization order on monomials and the main result

Given two monomials

$$X^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}\ \text{ and }\ X^\beta=x_1^{\beta_1}\cdots x_n^{\beta_n}$$

with $|\alpha|=|\beta|$, we cannot order them unless some further conditions are imposed. For example, let $\alpha=(3,1,1),\beta=(2,1,2)$ and $x_1\ge x_2\ge x_3\ge 0$, then we have

$$x_1^3x_2x_3-x_1^2x_2x_3^2=x_1^2x_2x_3(x_1-x_3)\ge 0.$$

This example inspires us to use a majorization order on monomials for our analysis of the termination of KSDS.

Before that, we first introduce the majorization between two vectors given in [6–8].

Definition 3.1 *Let $\alpha=(\alpha_1,\ldots,\alpha_n)$ and $\beta=(\beta_1,\ldots,\beta_n)$, where $\alpha,\beta\in\mathbb{N}^n$ with $|\alpha|=|\beta|$. If*

$$\sum_{i=1}^k\alpha_i\ge\sum_{i=1}^k\beta_i\ \ \text{for all}\ \ k\in\{1,\ldots,n-1\},$$

then we say that α majorizes β, which is denoted as $\alpha\succeq\beta$.

Note that "$\succeq$" is a partial order. With the help of Definition 3.1, we construct the definition of majorization order on monomials.

Definition 3.2 (Majorization order on monomials) *Let X^α and X^β be two monomials with $|\alpha|=|\beta|$. Suppose that σ is a permutation on the set $\{1,2,\ldots,n\}$. If*

$$(\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(n)})\succeq(\beta_{\sigma(1)},\ldots,\beta_{\sigma(n)})\ \ \text{or, briefly,}\ \ \alpha_\sigma\succeq\beta_\sigma,$$

then we say that X^α majorizes X^β with respect to the permutation σ, which is denoted as $(X^\alpha)_\sigma\succeq(X^\beta)_\sigma$ or $X_\sigma^{\alpha_\sigma}\succeq X_\sigma^{\beta_\sigma}$. Here $(X^\alpha)_\sigma=x_{\sigma(1)}^{\alpha_{\sigma(1)}}\cdots x_{\sigma(n)}^{\alpha_{\sigma(n)}}=X_\sigma^{\alpha_\sigma}$.

Our definition of the majorization order on monomials evolves from the definition of the majorization on symmetric polynomials given in [6–8].

We need a few comments on the notation. Note that X^α, $(X^\alpha)_\sigma$ and $X_\sigma^{\alpha_\sigma}$ stand for the same monomial. Furthermore, there is

$$(X^\alpha)_\sigma \succeq (X^\beta)_\sigma \Longleftrightarrow (X_\sigma^{\alpha_\sigma})_I \succeq (X_\sigma^{\beta_\sigma})_I,$$

where I is the identical permutation and can be omitted. For example

$$(x_1^3x_2^4x_3)_{(21)(3)} \succeq (x_1^4x_2^2x_3^2)_{(21)(3)} \Longleftrightarrow x_2^4x_1^3x_3 \succeq x_2^2x_1^4x_3^2 \Longleftrightarrow (4,3,1) \succeq (2,4,2).$$

It is easy to see that, with respect to the permutation $\sigma = (1)(2)(3)$, the monomials $x_1^3x_2^4x_3$ and $x_1^4x_2^2x_3^2$ do not majorize each other. So the majorization order on monomials is a partial order. Moreover, the following three basic properties hold.

Lemma 3.3 *For a given permutation $\sigma \in S_n$ and $\alpha, \beta, \gamma \in \mathbb{N}^n$ with $|\alpha| = |\beta| = |\gamma|$, we have*

(1) $(X^\alpha)_\sigma \succeq (X^\alpha)_\sigma$.

(2) $(X^\alpha)_\sigma \succeq (X^\beta)_\sigma \wedge (X^\beta)_\sigma \succeq (X^\alpha)_\sigma \Longrightarrow X^\alpha = X^\beta$.

(3) $(X^\alpha)_\sigma \succeq (X^\beta)_\sigma \wedge (X^\beta)_\sigma \succeq (X^\gamma)_\sigma \Longrightarrow (X^\alpha)_\sigma \succeq (X^\gamma)_\sigma$.

Proof Straightforward. □

Lemma 3.4 *Let $\sigma \in S_n$ be a given permutation. For the monomial X^α and X^β with $|\alpha| = |\beta|$, we have $X^\alpha \geq X^\beta$ under the condition $x_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)} \geq 0$ if and only if $(X^\alpha)_\sigma \succeq (X^\beta)_\sigma$.*

Proof $\Rightarrow$. Let $x_{\sigma(1)} = \cdots = x_{\sigma(j)} = 2$, and let $x_{\sigma(j+1)} = \cdots = x_{\sigma(n)} = 1$. Then

$$2^{\alpha_{\sigma(1)}+\alpha_{\sigma(2)}+\cdots+\alpha_{\sigma(j)}} \geq 2^{\beta_{\sigma(1)}+\beta_{\sigma(2)}+\cdots+\alpha_{\sigma(j)}}.$$

Thus

$$\alpha_{\sigma(1)} + \alpha_{\sigma(2)} + \cdots + \alpha_{\sigma(j)} \geq \beta_{\sigma(1)} + \beta_{\sigma(2)} + \cdots + \alpha_{\sigma(j)}.$$

Let $j = 1, 2, \ldots, n-1$ successively. Then we immediately have

$$(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)}) \succeq (\beta_{\sigma(1)}, \ldots, \beta_{\sigma(n)}).$$

$\Leftarrow$. It is trivial if $x_i = 0$ for some $i = 1, \ldots, n$. So we assume that $x_i \neq 0$ for all $i = 1, \ldots, n$. Then

$$\frac{X^\alpha}{X^\beta} = \prod_{i=1}^{n-1} \Big(\frac{x_{\sigma(i)}}{x_{\sigma(i+1)}}\Big)^{\sum_{j=1}^{i}(\alpha_{\sigma(j)}-\beta_{\sigma(j)})} \geq 1. \quad \square$$

Lemma 3.5 *Let $M = (p_{ij})$ be an $n \times n$ matrix, in which $p_{ij} > 0$ if $i \leq j$ else $p_{ij} = 0$. For a monomial $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$, consider linear substitution $(x_1, \ldots, x_n)^T = M(t_1, \ldots, t_n)^T$, namely,*

$$\begin{aligned}&(p_{11}t_1 + p_{12}t_2 + \cdots + p_{1n}t_n)^{\alpha_1}(p_{22}t_2 + \cdots + p_{2n}t_n)^{\alpha_2}\cdots(p_{nn}t_n)^{\alpha_n}\\ &\quad= \sum_{|(j_1,\ldots,j_n)|=|\alpha|} C_{(j_1,\ldots,j_n)}t_1^{j_1}t_2^{j_2}\cdots t_n^{j_n}.\end{aligned}$$

Then $C_{(j_1,\ldots,j_n)} \neq 0 \Longleftrightarrow (t_1^{\alpha_1}\cdots t_n^{\alpha_n})_I \succeq (t_1^{j_1}\cdots t_n^{j_n})_I$.

Proof $\Rightarrow$. Consider the expansion

$$\sum_{|(j_1,\ldots,j_n)|=|\alpha|} C_{(j_1,\ldots,j_n)} t_1^{j_1} t_2^{j_2} \cdots t_n^{j_n}.$$

If $C_{(j_1,\ldots,j_n)} \neq 0$, then we have the following results:

The term $t_1^{j_1}$ can be obtained by expanding $(p_{11}t_1 + p_{12}t_2 + \cdots + p_{1n}t_n)^{\alpha_1}$. It follows that $j_1 \le \alpha_1$. Analogously, $t_2^{j_2}$ can be obtained by expanding $(p_{11}t_1 + p_{12}t_2 + \cdots + p_{1n}t_n)^{\alpha_1}$ or $(p_{22}t_2 + \cdots + p_{2n}t_n)^{\alpha_2}$ and therefore $j_2 \le (\alpha_1 - j_1) + \alpha_2$, namely, $j_1 + j_2 \le \alpha_1 + \alpha_2$. By the same token, we have

$$(j_1, \ldots, j_n) \preceq (\alpha_1, \ldots, \alpha_n).$$

Namely, $(t_1^{\alpha_1} \cdots t_n^{\alpha_n})_I \succeq (t_1^{j_1} \cdots t_n^{j_n})_I$.

$\Leftarrow$. It is easy to see that the converse implications are also true. $\square$

Theorem 3.6 *Suppose that*

$$f(x_1, \ldots, x_n) = \sum_{|\alpha|=d} C_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} = \sum_{|\alpha|=d} C_\alpha X^\alpha, \quad \textit{where } \ C_\alpha \neq 0.$$

is a homogeneous polynomial of degree d in $\mathbb{R}[x_1, \ldots, x_n]$. For a term $C_\lambda X^\lambda$ of f, if the monomial X^λ is not majorized by any other monomial of f with respect to $\sigma \in S_n$ then the coefficient of the monomial $(X_\sigma)^\lambda$ of $f(B_\sigma K_n^{m-1} X)$ is $(q_{\sigma(1)}^{\lambda_1} \cdots q_{\sigma(n)}^{\lambda_n})^m C_\lambda$.

Proof According to (2.1), we know that K_n^m is an upper triangular matrix and the diagonal elements are $q_1^m, \ldots, q_n^m$. Let

$$K_n^m = \begin{pmatrix} q_1^m & p_{12} & \cdots & p_{1n} \\ & q_2^m & \cdots & p_{2n} \\ & & \ddots & \vdots \\ 0 & & & q_n^m \end{pmatrix}, \quad \text{where } \ p_{ij} > 0, \ 1 \le i < j \le n.$$

Let

$$X' = \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = K_n^m X = \begin{pmatrix} q_1^m x_1 + p_{12}x_2 + \cdots + p_{1n}x_n \\ q_2^m x_2 + \cdots + p_{2n}x_n \\ \cdots \\ q_n^m x_n \end{pmatrix}. \tag{3.1}$$

By Definition 2.3 and (3.1), we have the following result.

$$f(B_\sigma K_n^{m-1} X) = f(P_\sigma K_n^m X) = f(P_\sigma X') = f(x'_{\sigma(1)}, \ldots, x'_{\sigma(n)}) = \sum_{|\alpha|=d} C_\alpha (X'_\sigma)^\alpha. \tag{3.2}$$

Notice that the monomial X^λ is not majorized by any other monomial of f with respect to σ. By Lemmas 3.3 and 3.5, the monomial $(X_\sigma)^\lambda$ of $f(B_\sigma K_n^{m-1} X)$ is only generated by expanding $(X'_\sigma)^\lambda$. By (3.2), we get that the coefficient of $(X_\sigma)^\lambda$ is $(q_{\sigma(1)}^{\lambda_1} \cdots q_{\sigma(n)}^{\lambda_n})^m C_\lambda$. $\square$

By Lemma 2.6 and Theorem 3.6, we immediately have the following main result.

Theorem 3.7 *A necessary condition of positively terminating of KSDS for an input form f is* *that, for an arbitrary ordering of variables, every monomial of f with a negative coefficient is majorized by at least one monomial of f with a positive coefficient.*

Proof We argue by contradiction. Suppose that there is a term $C_\lambda X^\lambda$ ($C_\lambda < 0$) of f, in which X^λ is not majorized by any other monomial of f with respect to σ. Then, by Theorem 3.6, the coefficient of X_σ^λ is always negative after expanding the polynomial $f(B_\sigma K_n^{m-1}X)$. This is a contradiction with Lemma 2.6. □

For example, let us consider the cyclic polynomial

$$f = x_1^4x_2^2 - x_1^3x_2x_3^2 + x_2^4x_3^2 - x_1^2x_2^3x_3 + x_1^2x_3^4 - x_1x_2^2x_3^3.$$

Note that the monomial $x_1^3x_2x_3^2$ in f has a negative coefficient, which is not majorized by any other monomials $x_1^4x_2^2$, $x_2^4x_3^2$, $x_1^2x_3^4$ in f with positive coefficients in the ordering x_1, x_3, x_2. Choose the following matrix A_3, and let the permutation $\sigma = (1)(23)$.

$$A_3 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_{(1)(23)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Expanding the polynomial $f(P_{(1)(23)}A_3^mX)$, we see that the coefficient of $x_1^3x_2^2x_3$ is always -1 by Theorem 3.6. So SDS (based on A_3) is not positively terminating for the input f. By other methods, we can prove that $\forall X \in \mathbb{R}_+^3$, $f \geq 0$. So SDS is not negatively terminating either.

On the other hand, using Jordan normal form, we can compute $P_{(1)(23)}A_3^m$

$$P_{(1)(23)}A_3^m = \begin{pmatrix} 1 & m & m(m-1)/2 \\ 0 & 0 & 1 \\ 0 & 1 & m \end{pmatrix}.$$

The coefficient of $x_1^3x_2^2x_3$ is still -1 by expanding $f(P_{(1)(23)}A_3^mX)$. Thus, the results obtained by the above two methods are compatible.

4. Conclusion

There are many interesting questions arising from the family of successive difference substitutions. For example, what is a necessary and sufficient condition for the positive termination of the algorithm KSDS? What is a necessary and sufficient condition for the negative termination of KSDS? Some research directions are listed below:

(1) Yang and Yao proved that a necessary and sufficient condition on the negative termination of SDS and NEWTSDS is $f \notin$ PSD (see [4,5]). So we put forward a conjecture for KSDS.

Conjecture The algorithm KSDS is negatively terminating if and only if $f \notin$ PSD.

(2) For the positive termination of NEWTSDS, Yao has proved the following result in [5].

Theorem 4.1 *Let $f(X) \in \mathbb{R}[x_1, \ldots, x_n]$. If $(\forall X \in \mathbb{R}_+^n, X \neq 0)$ $f(X) > 0$, then there exists*

$m > 0$ *such that the coefficients of*

$$f(B_{\sigma_1}B_{\sigma_2}\cdots B_{\sigma_m}X),\quad \forall\sigma_i \in S_n\ (B_{\sigma_i} = P_{\sigma_i}G_n)$$

are all positive.

In other words, NEWTSDS is positively terminating for a form in PD. However, it appears more difficult to study the positive termination of KSDS.

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