Journal of Mathematical Research with Applications
May, 2015, Vol. 35, No. 3, pp. 237–255
DOI:10.3770/j.issn:2095-2651.2015.03.001
Http://jmre.dlut.edu.cn

Toeplitz Operators with Unbounded Symbols on Segal-Bargmann Space

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Abstract In this paper, we construct a function φ in $L^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$ which is unbounded on any neighborhood of each point in $\mathbb{C}^n$ such that T_φ is a trace class operator on the Segal-Bargmann space $H^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$. In addition, we also characterize the Schatten p-class Toeplitz operators with positive measure symbols on $H^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$.

Keywords Segal-Bargmann space; Toeplitz operator; unbounded function; Schatten class

MR(2010) Subject Classification 32A37; 47B38; 47B35; 47B07

1. Introduction

Let $\mathbb{C}^n$ be the n-dimensional complex Euclidean space and $\mathbb{B}_n$ be the open unit ball of $\mathbb{C}^n$. For any two points $z = (z_1, z_2, \dots, z_n)$ and $w = (w_1, w_2, \dots, w_n)$ in $\mathbb{C}^n$, we write

$$\langle z, w\rangle = z_1\overline{w_1} + z_2\overline{w_2} + \cdots + z_n\overline{w_n}$$

and

$$|z|^2 = \langle z, z\rangle = |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2.$$

For $k = (k_1, k_2, \dots, k_n)$ an n-tuple non-negative integers, we write

$$k! = k_1!k_2!\cdots k_n!,\ \ \|k\| = k_1 + k_2 + \cdots + k_n,\ \ z^k = z_1^{k_1} z_2^{k_2}\cdots z_n^{k_n}.$$

For each coordinate z_j, we write $z_j = x_j + iy_j$ where x_j, y_j are real numbers. Then, $z = (z_1, z_2, \dots, z_n)$ can also be denoted as $z = (x_1, y_1, x_2, y_2, \dots, x_n, y_n)$.

Throughout the paper, we fix a positive parameter α and consider the Gaussian measure

$$\mathrm{d}V_\alpha(z) = (\frac{\alpha}{\pi})^n e^{-\alpha|z|^2}\mathrm{d}V(z)$$

where $\mathrm{d}V$ is the usual Euclidean Volume measure on $\mathbb{C}^n = \mathbb{R}^{2n}$.

For any $p > 0$, write

$$L^p(\mathbb{C}^n, \mathrm{d}V_\alpha) = \Big\{f \text{ is an entire function on } \mathbb{C}^n \Big| \int_{\mathbb{C}^n} |f(z)|^p \mathrm{d}V_\alpha(z) < +\infty\Big\},$$

Received October 10, 2014; Accepted January 16, 2015
Supported by the National Natural Science Foundation of China (Grant No. 11271092) and the Natural Science Foundation of Guangdong Province (Grant No. S2011010005367).

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and

$$\|f\|_p=\Big[(\frac{\alpha}{\pi})^n\int_{\mathbb{C}^n}|f(z)|^pe^{-\alpha|z|^2}\mathrm{d}V(z)\Big]^{\frac{1}{p}}.$$

The space defined as follows

$$H^p(\mathbb{C}^n,\mathrm{d}V_\alpha)=\{f \text{ is an entire function on } \mathbb{C}^n|f\in L^p(\mathbb{C}^n,\mathrm{d}V_\alpha)\},$$

is called Segal-Bargmann space. In particular, $H^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$ is a Hilbert space with the following inner product inherited from $L^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$:

$$\langle f,g\rangle=\int_{\mathbb{C}^n}f(z)\overline{g(z)}\mathrm{d}V_\alpha(z),$$

and we denote by $\|\cdot\|_2$ the norm in $H^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$.

For any $f\in L^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$, we define the integral operator $P:L^2(\mathbb{C}^n,\mathrm{d}V_\alpha)\to H^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$ as

$$P(f)(z)=\int_{\mathbb{C}^n}f(w)\overline{K_z(w)}\mathrm{d}V_\alpha(w),$$

where $K_z(w)=e^{\alpha\bar{z}w}$ is the reproducing kernel of $H^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$. Then, P is the orthogonal projection from $L^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$ to $H^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$. For more details, we refer to [1–4].

Given $\varphi\in L^\infty(\mathbb{C}^n)$, we define a linear operator $T_\varphi:H^2(\mathbb{C}^n,\mathrm{d}V_\alpha)\to H^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$ by

$$T_\varphi(f)(z)=P(\varphi f)(z)=\int_{\mathbb{C}^n}\varphi(w)f(w)\overline{K_z(w)}\mathrm{d}V_\alpha(w),\quad f\in H^2(\mathbb{C}^n,\mathrm{d}V_\alpha).$$

We call T_φ as the Toeplitz operator on $H^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$ with symbol φ. It is obvious that T_φ is bounded with $\|T_\varphi\|\le\|\varphi\|_\infty$. Furthermore, for any complex numbers a and b and any bounded functions φ and ψ, we can easily find that $T_{\overline{\varphi}}=T_\varphi^*$, $T_{a\varphi+b\psi}=aT_\varphi+bT_\psi$ and $T_\varphi\ge0$ whenever $\varphi\ge0$.

For any $z\in\mathbb{C}^n$ and $r>0$, we use $B(z,r)=\{w\in\mathbb{C}^n:|w-z|<r\}$ to denote the Euclidean ball centered at z with radius r. Then,

$$V(B(z,r))=\int_{B(z,r)}\mathrm{d}V(w)=\frac{(\pi r^2)^n}{n!}.$$

We refer to [5] for the specifical proof.

The Toeplitz operators on Segal-Bargmann spaces have been researched by both mathematician and physician for many years, since the Segal-Bargmann space is closely related to quantum mechanics. Specifically, the Fock boson annihilation and creation operators in quantum mechanics can be represented as the operators $\frac{\mathrm{d}}{\mathrm{d}z}$ and M_z in Segal-Bargmann Space, and the normalized reproducing kernel of Segal-Bargmann Space also corresponds to the coherent states in quantum mechanics, moreover, the C^*-algebra generated by Weyl operators of boson quantum mechanics consists of the uniform limits of almost-periodic Toeplitz operators on Segal-Bargmann space [1,6,7]. To investigate more applications in physics, it is significative to make certain some unknown properties of Toeplitz operators on Segal-Bargmann spaces.

Naturally, just as considering the Toeplitz operators on classical Hardy space and Bergman space, we want to make clear the boundedness and compactness of Toeplitz operators on Segal-Bargmann space at first. Clearly, T_φ is bounded if φ is essentially bounded, in which case

$\|T_\varphi\| \le \|\varphi\|_\infty$. But it is easy to check that the converse is false by the counter-example given by [1] (for $n = 1$, we just need to set $\varphi(z) = \frac{1}{\sqrt{|z|}}$). In fact, Isralowitz and Zhu [3] equivalently characterize the boundedness and compactness of Toeplitz operators with some special symbols on Fock space in dimension one. Alexander and Dror [8] also studied the boundedness and compactness of the Toeplitz operators defined on generalized Bargmann-Fock spaces by Carleson measures and vanishing Carleson measures. But we are eager to find the direct relationship between the boundedness or compactness of Toeplitz operators and their symbols, and we wonder whether there exists any bounded or compact Toeplitz operator with unbounded symbol on Segal-Bargmann space like on Bergman space.

Motivated by [9–11], in the second section, we construct a class of function in $L^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$ which are unbounded on any neighborhood of each point in $\mathbb{C}^n$, such that the Toeplitz operators with these symbols are not only bounded but also compact on $H^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$. By the process of constructing, we also find there exists a function φ in $L^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$ which is unbounded on any neighborhood of each point in $\mathbb{C}^n$ such that T_φ is a trace class operator on $H^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$. Furthermore, we obtain the equivalent characterizations of Schatten p-class Toeplitz operators with positive symbols on $H^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$ in the latter two sections. We also find the characterizations of Schatten class Toeplitz operators in terms of the Berezin transform on Segal-Bargmann space is different from Bergman space setting. Just as the theory in dimension one, the cut-off phenomenon that is often seen in Bergman space theory disappears in Segal-Bargmann space [12,13], the results given in [3] about the Schatten p-class Toeplitz operators on the Fock space are generalized.

2. Trace class Toeplitz operators with unbounded symbols

At the beginning of this section, we give the following sufficient condition of the compact Toeplitz operator on $H^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$.

Proposition 2.1 *Suppose φ in $L^\infty(\mathbb{C}^n)$ vanishes at infinity. Then T_φ is compact on $H^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$.*

Proof Since φ is essentially bounded and vanishes at infinity, it is obvious that for arbitrary $\epsilon > 0$, there exist positive constants M and λ such that $|\varphi(z)| \le M$ a.e. and $|\varphi(z)| < \epsilon$ for all $|z| > \lambda$. Set

$$\chi_1(w) = \begin{cases} 1, & \text{if } |w| > \lambda; \\ 0, & \text{if } |w| \le \lambda \end{cases}$$

and

$$\chi_2(w) = \begin{cases} 0, & \text{if } |w| > \lambda; \\ 1, & \text{if } |w| \le \lambda. \end{cases}$$

Assume $\{f_j\} \subset H^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$ with $\|f_j\| \le 1$ is a sequence which weakly converges to zero as $j \to \infty$. Then

$$T_\varphi f_j(z) = P(\varphi f_j)(z) = \int_{\mathbb{C}^n} \varphi(w) f_j(w) \overline{K_z(w)} \mathrm{d}V_\alpha(w),$$

from which it follows

$$\begin{aligned}\|T_\varphi f_j\|^2 =&\int_{\mathbb{C}^n}\Big|\int_{\mathbb{C}^n}\varphi(w)f_j(w)\overline{K_z(w)}\mathrm{d}V_\alpha(w)\Big|^2\mathrm{d}V_\alpha(z)\\ =&\int_{\mathbb{C}^n}\Big|\int_{\{w:|w|>\lambda\}}\varphi(w)f_j(w)\overline{K_z(w)}\mathrm{d}V_\alpha(w)+\\ &\int_{\{w:|w|\le\lambda\}}\varphi(w)f_j(w)\overline{K_z(w)}\mathrm{d}V_\alpha(w)\Big|^2\mathrm{d}V_\alpha(z)\\ \le&2\int_{\mathbb{C}^n}\Big\{|\int_{\{w:|w|>\lambda\}}\varphi(w)f_j(w)\overline{K_z(w)}\mathrm{d}V_\alpha(w)|^2+\\ &|\int_{\{w:|w|\le\lambda\}}\varphi(w)f_j(w)\overline{K_z(w)}\mathrm{d}V_\alpha(w)|^2\Big\}\mathrm{d}V_\alpha(z)\\ =&2\int_{\mathbb{C}^n}\Big\{|\int_{\mathbb{C}^n}\varphi(w)\chi_1(w)f_j(w)\overline{K_z(w)}\mathrm{d}V_\alpha(w)|^2+\\ &|\int_{\mathbb{C}^n}\varphi(w)\chi_2(w)f_j(w)\overline{K_z(w)}\mathrm{d}V_\alpha(w)|^2\Big\}\mathrm{d}V_\alpha(z)\\ =&2\int_{\mathbb{C}^n}|P(\varphi\chi_1f_j)(z)|^2\mathrm{d}V_\alpha(z)+2\int_{\mathbb{C}^n}|P(\varphi\chi_2f_j)(z)|^2\mathrm{d}V_\alpha(z)\\ \le&2(\|\varphi\chi_1f_j\|^2+\|\varphi\chi_2f_j\|^2).\end{aligned}$$

Note

$$\begin{aligned}\|\varphi\chi_1f_j\|^2&=\int_{\mathbb{C}^n}|\varphi(w)\chi_1(w)f_j(w)|^2\mathrm{d}V_\alpha(w)=\int_{\{w:|w|>\lambda\}}|\varphi(w)f_j(w)|^2\mathrm{d}V_\alpha(w)\\ &\le\epsilon^2\int_{\mathbb{C}^n}|f_j(w)|^2\mathrm{d}V_\alpha(w)=\epsilon^2\|f_j\|^2\le\epsilon^2\end{aligned}$$

and

$$\|\varphi\chi_2f_j\|^2=\int_{\mathbb{C}^n}|\varphi(w)\chi_2(w)f_j(w)|^2\mathrm{d}V_\alpha(w)=\int_{\{w:|w|\le\lambda\}}|\varphi(w)f_j(w)|^2\mathrm{d}V_\alpha(w)\le\epsilon^2M^2.$$

Therefore, $\|T_\varphi f_j\|\to0$ as $j\to\infty$, and this implies that T_φ is compact on $H^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$.

Now, we turn to introduce a new circular-like cone to construct a function φ in $L^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$ which is unbounded on any neighborhood of each point in $\mathbb{C}^n$. For $\delta>0$, $\xi\in\mathbb{C}^n$, denote

$$\Omega(\xi,\delta)=\{z\in B(0,|\xi|):[1-(1-|\frac{z}{\xi}|)^\delta]^{\frac12}\cdot\frac{|z-\xi|}{|\xi|}<Re\langle\frac{\xi}{|\xi|},\frac{\xi-z}{|\xi|}\rangle,\mathrm{Re}\langle z,\xi\rangle>0\}.$$

Then, $\Omega(\xi,\delta)$ is an open set of $B(0,|\xi|)$ which is a circular-like cone with vertex ξ.

For any $r>0$, write $B(0,r)$ in $\mathbb{C}^n$ as $B(r)$ and $\partial B(r)$ its boundary in this section, and denote by $\mathrm{d}\sigma_r$ the area measure on the sphere $\partial B(r)$. Obviously, $\sigma_r(\partial B(r))=O(r^{2n-1})$. Assume b_1, b_2 are arbitrary positive numbers. It is clear that we can choose some $\delta=\delta(b_1,b_2)>0$ such that

$$\sigma_r[\Omega(\xi,\delta(b_1,b_2))\cap\partial B(r)]<d\cdot(|\xi|^2-r^2)^{b_1}e^{-b_2r^2}$$

for any $0<r<|\xi|$ and $\xi\in\mathbb{C}^n$, where d is a constant which is independent of ξ and r. For simplicity, we write $\Omega(\xi,\delta(b_1,b_2))$ as $\Omega(\xi,b_1,b_2)$.

Theorem 2.2 *Assume* $b_1>0$, $b_2>0$. *For arbitrary* $\xi\in\mathbb{C}^n$, *let* $U_\xi(z)=(|\xi|^2-|z|^2)^{-\frac{b_1}{2}}$, $z\in\mathbb{C}^n$

and $\chi_{\Omega(\xi,b_1,b_2)}(z)$ *is the characteristic function of the set* $\Omega(\xi,b_1,b_2)$, $\varphi(z)=\chi_{\Omega(\xi,b_1,b_2)}(z)\cdot U_\xi(z)$. *Then* T_φ *is a compact operator on* $H^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$.

Proof Suppose $\{f_j\}\subset H^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$ with $\|f_j\|\le 1$ is a sequence which weakly converges to zero as $j\to\infty$. Then, $f_j(w)\to 0$ uniformly on $\overline{\Omega(\xi,b_1,b_2)}$. That is, for any $\epsilon>0$, there is a $K_0>0$ such that $|f_j(w)|<\epsilon$ for arbitrary $w\in\overline{\Omega(\xi,b_1,b_2)}$ when $j>K_0$. Thus,

$$\begin{aligned}
\|T_\varphi f_j\|^2 &= \int_{\mathbb{C}^n}|P(\varphi f_j)(z)|^2\mathrm{d}V_\alpha(z)=\int_{\mathbb{C}^n}\Big|\int_{\mathbb{C}^n}\varphi(w)f_j(w)\overline{K_z(w)}\mathrm{d}V_\alpha(w)\Big|^2\mathrm{d}V_\alpha(z)\\
&=\int_{\mathbb{C}^n}\Big|\int_{\Omega(\xi,b_1,b_2)}(|\xi|^2-|w|^2)^{-\frac{b_1}{2}}e^{\alpha z\overline{w}}f_j(w)\mathrm{d}V_\alpha(w)\Big|^2\mathrm{d}V_\alpha(z)\\
&\le \epsilon^2\int_{\mathbb{C}^n}\int_{\Omega(\xi,b_1,b_2)}(|\xi|^2-|w|^2)^{-b_1}|e^{\alpha z\overline{w}}|^2\mathrm{d}V_\alpha(w)\mathrm{d}V_\alpha(z)\\
&=\epsilon^2\int_{\Omega(\xi,b_1,b_2)}(|\xi|^2-|w|^2)^{-b_1}\Big[\int_{\mathbb{C}^n}|K_w(z)|^2\mathrm{d}V_\alpha(z)\Big]\mathrm{d}V_\alpha(w)\\
&=\epsilon^2\int_{\Omega(\xi,b_1,b_2)}(|\xi|^2-|w|^2)^{-b_1}e^{\alpha|w|^2}\mathrm{d}V_\alpha(w)\\
&=(\frac{\alpha}{\pi})^n\epsilon^2\int_{\Omega(\xi,b_1,b_2)}(|\xi|^2-|w|^2)^{-b_1}\mathrm{d}V(w)\\
&=C_0(\frac{\alpha}{\pi})^n\epsilon^2\int_0^{|\xi|}\int_{\Omega(\xi,b_1,b_2)\cap\partial B(r)}(|\xi|^2-r^2)^{-b_1}\mathrm{d}\sigma_r\mathrm{d}r\\
&=C_0(\frac{\alpha}{\pi})^n\epsilon^2\int_0^{|\xi|}\sigma_r[\Omega(\xi,b_1,b_2)\cap\partial B(r)](|\xi|^2-r^2)^{-b_1}\mathrm{d}r\\
&\le C_0d(\frac{\alpha}{\pi})^n\epsilon^2\int_0^{|\xi|}e^{-b_2r^2}\mathrm{d}r=\frac{\sqrt{b_2\pi}}{2b_2}(\frac{\alpha}{\pi})^nC_0\mathrm{d}\epsilon^2,
\end{aligned}$$

where C_0 is a positive constant. Hence, $\|T_\varphi f_j\|\to 0$ as $j\to\infty$. The proof of this theorem has been completed. □

As we know, the set consisting of the points in $\mathbb{C}^n$ with rational coordinate components is a countable dense subset of $\mathbb{C}^n$, we denote this set as $\{\xi_j\}_{j=1}^\infty$. Consequently, from Theorem 2.2, we can construct a class of functions $\{\varphi_j\}$ in $L^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$ by the characteristic functions on the corresponding circular-like cones $\{\Omega(\xi_j,b_1,b_2)\}$, such that T_{φ_j} is compact on $H^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$ for each $j\in\mathbf{Z}_+$. Further, we also construct a function φ in $L^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$ which is unbounded on any neighborhood of each point in $\mathbb{C}^n$, such that T_φ is a trace class operator on $H^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$.

Theorem 2.3 *There is a function* φ *in* $L^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$ *which is unbounded on any neighborhood of each point in* $\mathbb{C}^n$, *such that* T_φ *is a compact operator on* $H^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$.

Proof For arbitrary $\xi\in\mathbb{C}^n$ and $r>0$, it is enough to construct a function φ in $L^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$ which satisfies $\mathrm{esssup}_{z\in B(\xi,r)}|\varphi(z)|=\infty$ induces a compact Toeplitz operator on $H^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$. Assume $b_1>0$, $b_2>0$ and $U_{\xi_j}(z)$ is the function in Theorem 2.2. For each ξ_j, set $\varphi_j(z)=\chi_{\Omega(\xi_j,b_1,b_2)}(z)\cdot U_{\xi_j}(z)$, then T_{φ_j} is a compact operator on $H^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$ by Theorem 2.2. For any

$f \in H^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$, we have

$$\begin{aligned}
\|T_{\varphi_j} f\|^2 &= \int_{\mathbb{C}^n} \Big| \int_{\mathbb{C}^n} \varphi_j(w) f(w) \overline{K_z(w)} \mathrm{d}V_\alpha(w) \Big|^2 \mathrm{d}V_\alpha(z) \\
&= \int_{\mathbb{C}^n} \Big| \int_{\Omega(\xi_j, b_1, b_2)} (|\xi_j|^2 - |w|^2)^{-\frac{b_1}{2}} e^{\alpha z \overline{w}} f(w) \mathrm{d}V_\alpha(w) \Big|^2 \mathrm{d}V_\alpha(z) \\
&\le \|f\|^2 \int_{\mathbb{C}^n} \int_{\Omega(\xi_j, b_1, b_2)} (|\xi_j|^2 - |w|^2)^{-b_1} |e^{\alpha z \overline{w}}|^2 \mathrm{d}V_\alpha(w) \mathrm{d}V_\alpha(z) \\
&\le \frac{\sqrt{b_2 \pi}}{2b_2} (\frac{\alpha}{\pi})^n C_0 d \|f\|^2 = C_1^2 \|f\|^2,
\end{aligned}$$

where the last inequality comes from the computation in Theorem 2.2 and $C_1 = [\frac{\sqrt{b_2\pi}}{2b_2}(\frac{\alpha}{\pi})^n C_0 d]^{\frac{1}{2}}$ is a positive constant relative to the dimension n. Consequently, $\|T_{\varphi_j}\| \le C_1$. For any positive integers M and N, without loss of generality, assume $M < N$ and take $T_M = \sum_{j=1}^{M} \frac{1}{2^j} T_{\varphi_j}$. Then,

$$\Big\| \sum_{j=M}^{N} \frac{1}{2^j} T_{\varphi_j} f \Big\| \le \sum_{j=M}^{N} \frac{1}{2^j} \|T_{\varphi_j} f\| \le C_1 \|f\| \sum_{j=M}^{N} \frac{1}{2^j}$$

for any $f \in H^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$, which implies that

$$\|T_N - T_M\| = \Big\| \sum_{j=M}^{N} \frac{1}{2^j} T_{\varphi_j} \Big\| \le C_1 \sum_{j=M}^{N} \frac{1}{2^j}.$$

Thus, $\sum_{j=1}^{\infty} \frac{1}{2^j} T_{\varphi_j}$ converges to T in norm. Obviously, T is a compact operator. Moreover, it is not difficult to check that $\varphi_j \in L^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$ and $\|\varphi_j\| \le C_1$. In fact,

$$\begin{aligned}
\|\varphi_j\|_2^2 &= \int_{\mathbb{C}^n} |\varphi_j(z)|^2 \mathrm{d}V_\alpha(z) = (\frac{\alpha}{\pi})^n \int_{\Omega(\xi_j, b_1, b_2)} (|\xi_j|^2 - |z|^2)^{-b_1} e^{-\alpha |z|^2} \mathrm{d}V(z) \\
&= C_0 (\frac{\alpha}{\pi})^n \int_0^{|\xi_j|} \int_{\Omega(\xi_j, b_1, b_2) \cap \partial B(r)} (|\xi_j|^2 - r^2)^{-b_1} e^{-\alpha r^2} \mathrm{d}\sigma_r \mathrm{d}r \\
&\le C_0 d (\frac{\alpha}{\pi})^n \int_0^{|\xi_j|} e^{(-\alpha - b_2) r^2} \mathrm{d}r \le C_0 d (\frac{\alpha}{\pi})^n \int_0^{+\infty} e^{-b_2 r^2} \mathrm{d}r \\
&\le \frac{\sqrt{b_2 \pi}}{2b_2} (\frac{\alpha}{\pi})^n C_0 d = C_1^2,
\end{aligned}$$

thus $\sum_{j=1}^{\infty} \frac{1}{2^j} \varphi_j$ converges to a L^2-function φ, and

$$\begin{aligned}
\|T_\varphi f\| &= \|T_{\sum_{j=1}^{\infty} \frac{1}{2^j} \varphi_j} f\| = \Big\| \sum_{j=1}^{\infty} \frac{1}{2^j} T_{\varphi_j} f \Big\| \le \sum_{j=1}^{\infty} \frac{1}{2^j} \|T_{\varphi_j} f\| \\
&\le C_1 \|f\| \sum_{j=1}^{\infty} \frac{1}{2^j} = C_1 \|f\|
\end{aligned}$$

for any $f \in H^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$. This indicates that $\|T_\varphi\| \le C_1$. Moreover, assume p is an arbitrary polynomial, then

$$\|(T_\varphi - T_M) p\| = \|T_{\sum_{j=M+1}^{\infty} \frac{1}{2^j} \varphi_j} p\| = \Big\| \sum_{j=M+1}^{\infty} \frac{1}{2^j} T_{\varphi_j} p \Big\|$$

$$\le \sum_{j=M+1}^{\infty} \frac{1}{2^j}\|T_{\varphi_j}p\| \le C_1\|p\| \sum_{j=M+1}^{\infty} \frac{1}{2^j} \to 0, \quad M \to \infty.$$

Therefore, $T = T_\varphi$. In other words, T is a Toeplitz operator with symbol $\varphi = \sum_{j=1}^{\infty} \frac{1}{2^j}\varphi_j$. Since $\{\xi_j\}$ is dense in $\mathbb{C}^n$, it is clear that for arbitrary $\xi \in \mathbb{C}^n$ and $r > 0$, $\mathrm{esssup}_{z\in B(\xi,r)}|\varphi(z)| = \infty$. This completes the proof. □

What's more, we can construct a trace class operator with unbounded symbol on $H^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$.

Theorem 2.4 *Assume* $0 < \alpha < n$. *Then there is a function* φ *in* $L^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$ *which is unbounded on any neighborhood of each point in* $\mathbb{C}^n$, *such that* T_φ *is a trace class operator on* $H^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$.

Proof Assume $b_1 > 0$, $b_2 > n$, and let $e_k = \sqrt{\frac{\alpha^{\|k\|}}{k!}}z^k$. Then $\{e_k\}_{k_j\ge 0}$ is an orthonomal basis of $H^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$ (see [1,2]). For any $j \in \mathbf{Z}_+$, suppose $\lambda_j \in [\sqrt{|\xi_j|^2-1}, |\xi_j|)$, set $\varphi_j(z) = \chi_{\Omega_{\lambda_j}(\xi_j,b_1,b_2)}(z)\cdot U_{\xi_j}(z)$, where $\Omega_{\lambda_j}(\xi_j, b_1, b_2) = \{z \in \Omega(\xi_j, b_1, b_2) : |z| > |\lambda_j|\}$ and $U_{\xi_j}(z)$ is the function in Theorem 2.3. Then

$$\begin{aligned}
|\langle T_{\varphi_j}e_k, e_k\rangle| &= \langle T_{\varphi_j}e_k, e_k\rangle = \int_{\mathbb{C}^n} P(\varphi_j e_k)(z)\overline{e_k(z)}\mathrm{d}V_\alpha(z) \\
&= \int_{\mathbb{C}^n}\int_{\mathbb{C}^n} \varphi_j(w)e_k(w)\overline{K_z(w)}\mathrm{d}V_\alpha(w)\overline{e_k(z)}\mathrm{d}V_\alpha(z) \\
&= \frac{\alpha^{\|k\|}}{k!}\int_{\mathbb{C}^n} \varphi_j(w)w^k\overline{[\int_{\mathbb{C}^n} z^k\overline{K_w(z)}\mathrm{d}V_\alpha(z)\ ]}\mathrm{d}V_\alpha(w) \\
&= \frac{\alpha^{\|k\|}}{k!}\int_{\mathbb{C}^n} \varphi_j(w)w^k\overline{w^k}\mathrm{d}V_\alpha(w) \\
&= (\frac{\alpha}{\pi})^n\frac{\alpha^{\|k\|}}{k!}\int_{\Omega_{\lambda_j}(\xi_j,b_1,b_2)} (|\xi_j|^2-|w|^2)^{-\frac{b_1}{2}}e^{-\alpha|w|^2}w^k\overline{w^k}\mathrm{d}V(w) \\
&\le C_0(\frac{\alpha}{\pi})^n\frac{\alpha^{\|k\|}}{k!}\int_{\lambda_j}^{|\xi_j|}\int_{\Omega_{\lambda_j}(\xi_j,b_1,b_2)\cap\partial B(r)} (|\xi_j|^2-|w|^2)^{-\frac{b_1}{2}}e^{-\alpha r^2}r^{2\|k\|}\mathrm{d}\sigma_r\mathrm{d}r \\
&\le C_0 d(\frac{\alpha}{\pi})^n\frac{\alpha^{\|k\|}}{k!}\int_{\lambda_j}^{+\infty} (|\xi_j|^2-r^2)^{\frac{b_1}{2}}e^{(-b_2-\alpha)r^2}r^{2\|k\|}\mathrm{d}r \\
&\le C_0 d(\frac{\alpha}{\pi})^n\frac{\alpha^{\|k\|}}{k!}\int_{\lambda_j}^{+\infty} e^{(-b_2-\alpha)r^2}r^{2\|k\|}\mathrm{d}r \\
&\le C_0 d(\frac{\alpha}{\pi})^n\frac{\alpha^{\|k\|}}{k!}\int_{0}^{+\infty} e^{-nr^2}r^{2\|k\|}\mathrm{d}r,
\end{aligned}$$

where C_0 is a positive constant independent of α. By changing variable $t = r^2$ and using the integration by parts, we have

$$\begin{aligned}
&\int_0^{+\infty} e^{-nr^2}r^{2\|k\|}\mathrm{d}r \le \frac{(\|k\|-\frac{1}{2})\cdot(\|k\|-\frac{3}{2})\cdot\cdots\cdot\frac{3}{2}\cdot\frac{1}{2}}{n^{\|k\|-1}}\int_0^{+\infty} e^{-nt}t^{\frac{1}{2}}\mathrm{d}t \\
&= \frac{(\|k\|-\frac{1}{2})\cdot(\|k\|-\frac{3}{2})\cdot\cdots\cdot\frac{3}{2}\cdot\frac{1}{2}}{n^{\|k\|-1}}\Big(\int_0^1 e^{-nt}t^{\frac{1}{2}}\mathrm{d}t + \int_1^{+\infty} e^{-nt}t^{\frac{1}{2}}\mathrm{d}t\Big) \\
&\le \frac{(\|k\|-\frac{1}{2})\cdot(\|k\|-\frac{3}{2})\cdot\cdots\cdot\frac{3}{2}\cdot\frac{1}{2}}{n^{\|k\|-1}}\Big(\int_0^1 e^{-nt}\mathrm{d}t + \int_1^{+\infty} e^{-nt}t\mathrm{d}t\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(\|k\|-\frac{1}{2})\cdot(\|k\|-\frac{3}{2})\cdot\cdots\cdot\frac{3}{2}\cdot\frac{1}{2}}{n^{\|k\|-1}}\Big(\frac{1-e^{-n}}{n}+\frac{1}{n^2}\Big)\\
&\le 2\frac{(\|k\|-\frac{1}{2})\cdot(\|k\|-\frac{3}{2})\cdot\cdots\cdot\frac{3}{2}\cdot\frac{1}{2}}{n^{\|k\|-1}}.
\end{aligned}
$$

It follows that

$$|\langle T_{\varphi_j}e_k,e_k\rangle|\le M_0\frac{\alpha^{\|k\|}}{k!}\frac{(\|k\|-\frac{1}{2})\cdot(\|k\|-\frac{3}{2})\cdot\cdots\cdot\frac{3}{2}\cdot\frac{1}{2}}{n^{\|k\|-1}},$$

where $M_0=2C_0d(\frac{\alpha}{\pi})^n$ is a constant only dependent on n. Set $T=\sum_{j=1}^{\infty}\frac{1}{2^j}T_{\varphi_j}$ and $\varphi=\sum_{j=1}^{\infty}\frac{1}{2^j}\varphi_j$, then $T=T_\varphi$ is compact by Theorem 2.3. Note T is positive, we get

$$
\begin{aligned}
\sum_{k\in N^n}|\langle Te_k,e_k\rangle| &= \sum_{k\in N^n}\langle Te_k,e_k\rangle=\sum_{k\in N^n}\sum_{j=1}^{\infty}\frac{1}{2^j}\langle T_{\varphi_j}e_k,e_k\rangle\\
&\le M_0\sum_{k\in N^n}\sum_{j=1}^{\infty}\frac{1}{2^j}\frac{\alpha^{\|k\|}}{k!}\frac{(\|k\|-\frac{1}{2})\cdot(\|k\|-\frac{3}{2})\cdot\cdots\cdot\frac{3}{2}\cdot\frac{1}{2}}{n^{\|k\|-1}}\\
&= M_0\sum_{k\in N^n}\frac{\alpha^{\|k\|}}{k!}\frac{(\|k\|-\frac{1}{2})\cdot(\|k\|-\frac{3}{2})\cdot\cdots\cdot\frac{3}{2}\cdot\frac{1}{2}}{n^{\|k\|-1}}\\
&= nM_0\sum_{m=0}^{\infty}\Big(\frac{\alpha}{n}\Big)^m\sum_{\|k\|=m}\frac{(m-\frac{1}{2})\cdot(m-\frac{3}{2})\cdot\cdots\cdot\frac{3}{2}\cdot\frac{1}{2}}{k!}\\
&\le nM_0\sum_{m=0}^{\infty}\Big(\frac{\alpha}{n}\Big)^m\sum_{\|k\|=m}\frac{(m+1)!}{k!}.
\end{aligned}
$$

By using the inductive method similar to [11], we easily see that

$$\sum_{\|k\|=m}\frac{(m+1)!}{k!}=O(m+1).$$

Consequently, there is a positive constant M_1 which is dependent on n such that

$$\sum_{k\in N^n}|\langle Te_k,e_k\rangle|\le nM_0\sum_{m=0}^{\infty}\Big(\frac{\alpha}{n}\Big)^m\sum_{\|k\|=m}\frac{(m+1)!}{k!}\le M_1\sum_{m=0}^{\infty}\Big(\frac{\alpha}{n}\Big)^m(m+1)<\infty$$

for each $0<\alpha<n$. This shows that T is a trace class operator on $H^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$. The theorem has been proved. □

3. Toeplitz operators in S_p with $p\ge 1$

In fact, we can also define Toeplitz operators on $H^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$ with more general symbols. More specifically, if μ is a complex Borel measure on $\mathbb{C}^n$, we define the Toeplitz operator T_μ as

$$T_\mu(f)(z)=\int_{\mathbb{C}^n}f(w)\overline{K_z(w)}e^{-\alpha|w|^2}\mathrm{d}\mu(w),\quad z\in\mathbb{C}^n.$$

Here, we notice that there is an extra weight factor $e^{-\alpha|w|^2}$ in our definition of T_μ compared to the traditional definition of Toeplitz operators on weighted Bergman spaces which was begun in [14]. Since the kernel function $K_z(w)$ is unbounded for any fixed $w\ne 0$, it is not clear when the integrals above will converge from the loose definition of T_μ, even if the measure μ is finite.

Suppose that μ is a Borel measure that satisfies the condition

$$\int_{\mathbb{C}^n} |K_z(w)| e^{-\alpha|w|^2} \mathrm{d}|\mu|(w) < \infty \tag{1}$$

for all $z \in \mathbb{C}^n$. Then because of the exponential form of the kernel function, it is clear that condition (1) is equivalent to

$$\int_{\mathbb{C}^n} |K_z(w)|^2 e^{-\alpha|w|^2} \mathrm{d}|\mu|(w) < \infty \tag{2}$$

for all $z \in \mathbb{C}^n$.

If μ satisfies condition (1) or (2), then we can easily get that the Toeplitz operator T_μ is well-defined on a dense subset of $H^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$. Therefore, all measures used in the following sections will be assumed to satisfy condition (1), so that Toeplitz operators are well-defined. We also define a function $\widetilde{\mu}$ on $\mathbb{C}^n$ as follows:

$$\widetilde{\mu}(z) = \int_{\mathbb{C}^n} |k_z(w)|^2 e^{-\alpha|w|^2} \mathrm{d}\mu(w), \quad z \in \mathbb{C}^n, \tag{3}$$

where $k_z(w) = K_z(w)/\sqrt{K_z(z)} = e^{\alpha \bar{z} w - \frac{\alpha}{2}|z|^2}$, called normalized reproducing kernel in $H^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$. Thus, we can write

$$\widetilde{\mu}(z) = \int_{\mathbb{C}^n} \frac{|K_z(w)|^2}{K_z(z) K_w(w)} \mathrm{d}\mu(w) = \int_{\mathbb{C}^n} e^{-\alpha|z-w|^2} \mathrm{d}\mu(w),$$

called the Berezin transform of μ. Again, we have included the extra weight factor $e^{-\alpha|w|^2}$ in (3) compared to the traditional definition of the Berezin transform in the Bergman space setting. If the Toeplitz operator T_μ happens to be a bounded operator on $H^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$, then for any $z \in \mathbb{C}^n$, we have $\widetilde{\mu}(z) = \langle T_\mu k_z, k_z \rangle$.

If $\mathrm{d}\mu(z) = (\frac{\alpha}{\pi})^n \varphi(z) \mathrm{d}V(z)$, we get $T_\mu = T_\varphi$ and we will write $\widetilde{\varphi}$ for $\widetilde{\mu}$. In this case, we call $\widetilde{\varphi}$ the Berezin transform of φ and

$$\widetilde{\varphi}(z) = (\frac{\alpha}{\pi})^n \int_{\mathbb{C}^n} \varphi(w) e^{-\alpha|z-w|^2} \mathrm{d}V(w),$$

which implies that all our results can be formulated in terms of the density function φ if the measure μ is absolutely continuous as above. If μ is a locally finite Borel measure, the function $z \to \mu(B(z,r))$ is the constant $\frac{(\pi r^2)^n}{n!}$ times the average of μ over $B(z,r)$. Thus we will call $\mu(B(z,r))$ an averaging function of μ. The following pointwise estimate for functions in $H^p(\mathbb{C}^n, \mathrm{d}V_\alpha)$ will be used to prove the main result in this section.

Lemma 3.1 *For any $r > 0$ and $p > 0$, there exists a positive constant C such that*

$$|f(z)|^p e^{-\alpha|z|^2} \leq C \int_{B(z,r)} |f(w)|^p \mathrm{d}V_\alpha(w)$$

for any entire function f on $\mathbb{C}^n$ and $z \in \mathbb{C}^n$.

Proof By a change of variables to the integral on the right of the lemma, we obtain

$$I(z) = \int_{B(z,r)} |f(w)|^p \mathrm{d}V_\alpha(w) = (\frac{\alpha}{\pi})^n \int_{B(z,r)} |f(w)|^p e^{-\alpha|w|^2} \mathrm{d}V(w)$$

$$\begin{aligned}&=(\frac{\alpha}{\pi})^n\int_{|u|<r}|f(z+u)|^pe^{-\alpha|z+u|^2}\mathrm{d}V(u)\\&=(\frac{\alpha}{\pi})^ne^{-\alpha|z|^2}\int_{B(0,r)}|f(z+u)|^p|e^{-\alpha\overline{z}u}|^2e^{-\alpha|u|^2}\mathrm{d}V(u)\\&=e^{-\alpha|z|^2}\int_{B(0,r)}|f(z+u)e^{-\frac{2\alpha\overline{z}u}{p}}|^p\mathrm{d}V_\alpha(u).\end{aligned}$$

Let $h(u)=f(z+u)e^{-\frac{2\alpha\overline{z}u}{p}}$. We can easily get

$$|h(0)|^p\le\frac{1}{V_\alpha(B(0,r))}\int_{B(0,r)}|h(u)|^p\mathrm{d}V_\alpha(u)$$

by the subharmonicity of $|h(u)|^p$. Then,

$$I(z)\ge V_\alpha(B(0,r))|h(0)|^pe^{-\alpha|z|^2}=V_\alpha(B(0,r))|f(z)|^pe^{-\alpha|z|^2}.$$

Thus, the result holds with $C=V_\alpha(B(0,r))^{-1}$.

The following elementary estimate will also be needed on several occasions later.

Lemma 3.2 *For any $r>0$, there exists a positive constant $C=C(r)$ such that $\mu(B(z,r))\le C\widetilde{\mu}(z)$ for all $z\in\mathbb{C}^n$.*

Proof For given $z\in\mathbb{C}^n$, we have

$$\begin{aligned}\mu(B(z,r))&=\int_{B(z,r)}\mathrm{d}\mu(w)=e^{\alpha r^2}\int_{B(z,r)}e^{-\alpha r^2}\mathrm{d}\mu(w)\le e^{\alpha r^2}\int_{B(z,r)}e^{-\alpha|z-w|^2}\mathrm{d}\mu(w)\\&\le e^{\alpha r^2}\int_{\mathbb{C}_n}e^{-\alpha|z-w|^2}\mathrm{d}\mu(w)=e^{\alpha r^2}\widetilde{\mu}(z).\end{aligned}$$

This gives the desired result.

Suppose $z^{(1)},z^{(2)},\dots,z^{(n)}$ are different points in $\mathbb{C}^n$ that are linearly independent, the set of points $m_1z^{(1)}+m_2z^{(2)}+\cdots+m_nz^{(n)}$ is called the lattice generated by $\{z^{(1)},z^{(2)},\dots,z^{(n)}\}$, where m_i $(1\le i\le n)$ are arbitrary integers. For example, for any integer i $(1\le i\le 2n)$ and $r>0$, we set

$$\xi_i^r=\overbrace{(0,0,\dots,0,r,0,\dots,0)}^{2n},$$

whose ith coordinate component is r and other coordinate components are zeros, as we know $\{\xi_i^1\}_{i=1}^{2n}$ is a standard orthonormal basis of $\mathbb{R}^{2n}$, then the set $\{m_1\xi_1^r+m_2\xi_2^r+\cdots+m_{2n}\xi_{2n}^r|m_i\in\mathbb{Z},1\le i\le 2n\}$ is the lattice generated by $\{\xi_i^r\}_{i=1}^{2n}$. For convenience, we will write every such lattice as a sequence.

In this section, we are going to determine when a Toeplitz operator T_μ on $H^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$ belongs to Schatten class S_p concerns the case $p\ge1$, while the next section concerns the case $0<p\le1$. Background information about the Schatten class S_p can be found in [12] for example.

For any bounded linear operator T on $H^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$, we can define the Berezin transform $\widetilde{T}$ by

$$\widetilde{T}(z)=\langle Tk_z,k_z\rangle,\quad z\in\mathbb{C}^n,$$

where k_z are the normalized reproducing kernels in $H^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$. Let $\{e_k = \sqrt{\frac{\alpha^{\|k\|}}{k!}} z^k\}_{k_j \geq 0}$ be an orthonomal basis of $H^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$. If T is positive on $H^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$, then

$$\begin{aligned}\mathrm{tr}(T) &= \sum_{k \in N^n} \langle Te_k, e_k \rangle = \sum_{k \in N^n} \int_{\mathbb{C}^n} Te_k(z)\overline{e_k(z)}\mathrm{d}V_\alpha(z) \\ &= \sum_{k \in N^n} \int_{\mathbb{C}^n} \langle Te_k, K_z \rangle \overline{e_k(z)} \mathrm{d}V_\alpha(z) = \int_{\mathbb{C}^n} \Big\langle T \sum_{k=1}^{\infty} e_k \overline{e_k(z)}, K_z \Big\rangle \mathrm{d}V_\alpha(z) \\ &= \int_{\mathbb{C}^n} \langle TK_z, K_z \rangle \mathrm{d}V_\alpha(z) = \int_{\mathbb{C}^n} \langle Tk_z, k_z \rangle K_z(z) \mathrm{d}V_\alpha(z) \\ &= (\frac{\alpha}{\pi})^n \int_{\mathbb{C}^n} \widetilde{T}(z) \mathrm{d}V(z).\end{aligned}$$

In particular, T is the trace class S_1 if and only if the integral above converges. Consequently, we obtain the following trace formula for Toeplitz operators on Segal-Bargmann spaces.

Lemma 3.3 *Assume* $\mu \geq 0$. *Then* T_μ *is the trace class* S_1 *if and only if* μ *is finite on* $\mathbb{C}^n$. *Moreover,* $\mathrm{tr}(T_\mu) = \mu(\mathbb{C}^n)$.

Proof Since all integrands below are nonnegative, we use Fubini's theorem to obtain

$$\begin{aligned}\mathrm{tr}(T_\mu) &= (\frac{\alpha}{\pi})^n \int_{\mathbb{C}^n} \widetilde{\mu}(z) \mathrm{d}V(z) = (\frac{\alpha}{\pi})^n \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |k_z(w)|^2 e^{-\alpha|w|^2} \mathrm{d}\mu(w) \mathrm{d}V(z) \\ &= \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |e^{-\alpha \bar{z} w}|^2 e^{-\alpha(|w|^2+|z|^2)} \mathrm{d}\mu(w) e^{\alpha|z|^2} \mathrm{d}V_\alpha(z) \\ &= \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} |e^{-\alpha \bar{z} w}|^2 \mathrm{d}V_\alpha(z) e^{-\alpha|w|^2} \mathrm{d}\mu(w) \\ &= \int_{\mathbb{C}^n} K_w(w) e^{-\alpha|w|^2} \mathrm{d}\mu(w) = \mu(\mathbb{C}^n).\end{aligned}$$

This also shows that $\mathrm{tr}(T_\mu) < \infty$ if and only if $\mu(\mathbb{C}^n) < \infty$.

Lemma 3.4 *If* $p \geq 1$ *and* $\varphi \in L^p(\mathbb{C}^n, \mathrm{d}V)$, *then* $T_\varphi \in S_p$.

Proof By interpolation, we only need to prove the result in the case $p = 1$ (the case $p = +\infty$ is trivial). Suppose $\varphi \in L^1(\mathbb{C}^n, \mathrm{d}V)$ and let $\{e_k = \sqrt{\frac{\alpha^{\|k\|}}{k!}} z^k\}_{k_j \geq 0}$ be an orthonormal basis of $H^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$. Note

$$\langle T_\varphi e_k, e_k \rangle = \int_{\mathbb{C}^n} |e_k(z)|^2 \varphi(z) \mathrm{d}V_\alpha(z)$$

for any $k \in N^n$, it follows that

$$\begin{aligned}\|T_\varphi\|_{S_1} &= \sum_{k \in N^n} |\langle T_\varphi e_k, e_k \rangle| = \sum_{k \in N^n} \Big| \int_{\mathbb{C}^n} |e_k(z)|^2 \varphi(z) \mathrm{d}V_\alpha(z) \Big| \\ &\leq \sum_{k \in N^n} \int_{\mathbb{C}^n} |e_k(z)|^2 |\varphi(z)| \mathrm{d}V_\alpha(z) = \int_{\mathbb{C}^n} K_z(z) |\varphi(z)| \mathrm{d}V_\alpha(z) \\ &= (\frac{\alpha}{\pi})^n \int_{\mathbb{C}^n} |\varphi(z)| \mathrm{d}V(z) = (\frac{\alpha}{\pi})^n \|\varphi\|_{L^1}.\end{aligned}$$

The proof has been completed. Moreover, we should claim here that the condition $p \geq 1$ is sharp, since we can give an example to show that this lemma is false when $0<p<1$. Consider the set $K\subseteq\mathbb{R}^{2n}=\mathbb{C}^n$ given by

$$K=\bigcup_{k=1}^{\infty}\overbrace{[2k,2k+k^{-\frac{1}{2np}}]\times[2k,2k+k^{-\frac{1}{2np}}]\times\cdots\times[2k,2k+k^{-\frac{1}{2np}}]}^{2n}$$

and take $\varphi=\chi_K$ the characteristic function of K. It is easy to check that φ is in $L^p(\mathbb{C}^n,\mathrm{d}V)$ for $0<p<1$. However, $\sum_{k=1}^{\infty}\widehat{\varphi}_r(a_k)^p=+\infty$. Thus, T_φ is not in S_p by the equivalent descriptions in Theorem 4.5 which will be proved in the next section.

To find the necessary and sufficient condition of Schatten p-class Toeplitz operators, we still need the following lemma.

Lemma 3.5 *Suppose $r>0$, $\mu\geq 0$, and*

$$\widehat{\mu}_r(z)=\mu(B(z,r))/[\frac{(\pi r^2)^n}{n!}],\quad z\in\mathbb{C}^n.$$

If $\widehat{\mu}_r\in L^p(\mathbb{C}^n,\mathrm{d}V)$, then $T_{\widehat{\mu}_r}$ and T_μ are bounded on $H^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$. Moreover, there exists a positive constant C which is independent of μ such that $T_\mu\leq CT_{\widehat{\mu}_r}$.

Proof Since $\widehat{\mu}_r$ is in $L^p(\mathbb{C}^n,\mathrm{d}V)$, a simple application of Theorems 5.1 and 6.2 in [8] tells us that T_μ and $T_{\widehat{\mu}_r}$ are bounded. Moreover, given $f\in H^2(\mathbb{C}^n,\mathrm{d}V_\alpha)$, by using Fubini's theorem, we obtain

$$\begin{aligned}\frac{(\pi r^2)^n}{n!}\langle T_{\widehat{\mu}_r}f,f\rangle&=\frac{(\pi r^2)^n}{n!}\int_{\mathbb{C}^n}|f(z)|^2|\widehat{\mu}_r(z)|\mathrm{d}V_\alpha(z)=\int_{\mathbb{C}^n}|f(z)|^2\mu(B(z,r))\mathrm{d}V_\alpha(z)\\&=\int_{\mathbb{C}^n}|f(z)|^2\int_{\mathbb{C}^n}\chi_{B(z,r)}(w)\mathrm{d}\mu(w)\mathrm{d}V_\alpha(z)\\&=\int_{\mathbb{C}^n}\mathrm{d}\mu(w)\int_{\mathbb{C}^n}|f(z)|^2\chi_{B(w,r)}(z)\mathrm{d}V_\alpha(z)\\&=(\frac{\alpha}{\pi})^n\int_{\mathbb{C}^n}\Big[\int_{B(w,r)}|f(z)|^2e^{-\alpha|z|^2}\mathrm{d}V(z)\Big]\mathrm{d}\mu(w).\end{aligned}$$

Combining the above identity with Lemma 3.1, there exists a positive constant C_1 such that

$$\frac{(\pi r^2)^n}{n!}\langle T_{\widehat{\mu}_r}f,f\rangle\geq C_1(\frac{\alpha}{\pi})^n\int_{\mathbb{C}^n}|f(w)|^2e^{-\alpha|w|^2}\mathrm{d}\mu(w)=C_1(\frac{\alpha}{\pi})^n\langle T_\mu f,f\rangle,$$

which implies that $\langle T_\mu f,f\rangle\leq C\langle T_{\widehat{\mu}_r}f,f\rangle$ by setting $C=\frac{(\pi r)^{2n}}{C_1\alpha^n n!}$. This proves the desired result.

Now, we give the main result of this section.

Theorem 3.6 *Suppose $\mu\geq 0$, $p\geq 1$, $r>0$, and $\{a_j\}$ is the lattice in $\mathbb{C}^n$ generated by $\{\xi_i^r=(0,0,\dots,0,r,0,\dots,0)\}_{i=1}^{2n}$. Then the following conditions are equivalent:*

(a) The Toeplitz operator T_μ belongs to the Schatten class S_p.

(b) The function $\widetilde{\mu}(z)$ belongs to $L^p(\mathbb{C}^n,\mathrm{d}V)$.

(c) The function $\mu(B(z,r))$ belongs to $L^p(\mathbb{C}^n,\mathrm{d}V)$.

(d) The sequence $\{\mu(B(a_j,r))\}$ belongs to l^p.

Proof (a)⇒(b). By the Lemma 1.4.5 of [12] and trace formula in Lemma 3.3, we know $T_\mu\in S_p$

if and only if $T_\mu^p \in S_1$ if and only if $\mathrm{tr}(T_\mu^p) < \infty$, so condition (a) holds implies that $\widetilde{\mu}(z) \in L^p(\mathbb{C}^n, \mathrm{d}V)$ from the fact that

$$\mathrm{tr}(T_\mu^p) = (\frac{\alpha}{\pi})^n \int_{\mathbb{C}^n} \langle T_\mu^p k_z, k_z\rangle \mathrm{d}V(z) \geq (\frac{\alpha}{\pi})^n \int_{\mathbb{C}^n} \langle T_\mu k_z, k_z\rangle^p \mathrm{d}V(z) = (\frac{\alpha}{\pi})^n \int_{\mathbb{C}^n} \widetilde{\mu}(z)^p \mathrm{d}V(z)$$

where the first inequality holds by Proposition 6.3.3(1) of [12].

(b)⇒(c). It is obvious from Lemma 3.2.

(c)⇒(a). If the averaging function $\widehat{\mu}_r(z)$ is in $L^p(\mathbb{C}^n, \mathrm{d}V)$, then it follows from Lemma 3.4 that $T_{\widehat{\mu}_r(z)}$ is in S_p. Combining this with Lemma 3.5, we conclude that T_μ is in S_p. This means (c)⇒(a) holds. Hence conditions (a), (b) and (c) are equivalent.

To complete the proof, we will have to prove conditions (d) is equivalent to any other conditions, we choose to prove (b)⇔(d) here.

(b)⇒(d). Obviously, condition (b) holds that the function $\mu(B(z, 2r)) \in L^p(\mathbb{C}^n, \mathrm{d}V)$. Choose a positive integer m such that each point in the $\mathbb{C}^n$ belongs to at most m of the balls $B(a_j, r)$. Then

$$\begin{aligned} m \int_{\mathbb{C}^n} \mu(B(z, 2r))^p \mathrm{d}V(z) &\geq \sum_{j=1}^{\infty} \int_{B(a_j, r)} \mu(B(z, 2r))^p \mathrm{d}V(z) \\ \geq \sum_{j=1}^{\infty} \int_{B(a_j, r)} \mu(B(a_j, r))^p \mathrm{d}V(z) &= \frac{(\pi r^2)^n}{n!} \sum_{j=1}^{\infty} \mu(B(a_j, r))^p \end{aligned}$$

for each $z \in B(a_j, r)$, where the second inequality is deduced from the triangle inequality which makes $B(a_j, r) \subseteq B(z, 2r)$ for each $z \in B(a_j, r)$. This shows that conditions (b) implies (d).

(d)⇒(b). Suppose $\{z_j\}$ is the lattice generated by $\{\xi_i^{\frac{r}{2}} = (0, 0, \ldots, 0, \frac{r}{2}, 0, \ldots, 0)\}_{i=1}^{2n}$. In fact, for each point z_j that is not in the lattice $\{a_j\}$, the ball $B(z_j, r)$ is covered by finite adjacent balls $B(a_j, r)$. Hence, the condition $\sum_{j=1}^{\infty} \mu(B(a_j, r))^p < \infty$ implies that $\sum_{j=1}^{\infty} \mu(B(z_j, r))^p < \infty$. Therefore,

$$\begin{aligned} \int_{\mathbb{C}^n} \mu(B(z, \frac{r}{2}))^p \mathrm{d}V(z) &\leq \sum_{j=1}^{\infty} \int_{B(z_j, \frac{r}{2})} \mu(B(z, \frac{r}{2}))^p \mathrm{d}V(z) \leq \sum_{j=1}^{\infty} \int_{B(z_j, \frac{r}{2})} \mu(B(z_j, r))^p \mathrm{d}V(z) \\ &= \frac{(\pi r^2)^n}{4^n n!} \sum_{j=1}^{\infty} \mu(B(z_j, r))^p < \infty. \end{aligned}$$

This shows condition (d) implies (c), as the equivalence of (c) to (b) implies that if condition (c) holds for one positive radius, then it will hold for any other positive radius. This completes the proof of the theorem. □

4. Toeplitz operators in S_p with $0 < p \leq 1$

In this section, we will pay attention to the case $0 < p \leq 1$.

Lemma 4.1 *Suppose $\mu \geq 0$, $0 < p \leq 1$, $r > 0$, and $\{a_j\}$ is the lattice in $\mathbb{C}^n$ generated by $\{\xi_i^r\}_{i=1}^{2n}$. Then the following conditions are equivalent:*

(a) The function $\widetilde{\mu}(z)$ is in $L^p(\mathbb{C}^n, \mathrm{d}V)$.

(b) *The function* $\mu(B(z,r))$ *is in* $L^p(\mathbb{C}^n, \mathrm{d}V)$.

(c) *The sequence* $\{\mu(B(a_j,r))\}$ *is in* l^p.

Proof (c)⇒(a). Note

$$\widetilde{\mu}(z) = \int_{\mathbb{C}^n} e^{-\alpha|z-w|^2} \mathrm{d}\mu(w) \le \sum_{j=1}^{\infty} \int_{B(a_j,r)} e^{-\alpha|z-w|^2} \mathrm{d}\mu(w)$$

and

$$|z-w|^2 \ge (|z-a_j| - |a_j - w|)^2 \ge |z-a_j|^2 - 2r|z-a_j|$$

for any $w \in B(a_j,r)$, we have

$$\widetilde{\mu}(z) \le \sum_{j=1}^{\infty} e^{-\alpha|z-a_j|^2 + 2\alpha r|z-a_j|} \mu(B(a_j,r)).$$

For $0 < p \le 1$, Hölder inequality gives

$$\widetilde{\mu}(z)^p \le \sum_{j=1}^{\infty} e^{-p\alpha|z-a_j|^2 + 2p\alpha r|z-a_j|} \mu(B(a_j,r))^p.$$

Thus, we can easily get

$$\int_{\mathbb{C}^n} \widetilde{\mu}(z)^p \mathrm{d}V(z) \le \sum_{j=1}^{\infty} \mu(B(a_j,r))^p \int_{\mathbb{C}^n} e^{-p\alpha|z-a_j|^2 + 2p\alpha r|z-a_j|} \mathrm{d}V(z)$$

by using Fubini's theorem. By an obvious change of variables, the integral above equals

$$\int_{\mathbb{C}^n} e^{-p\alpha|u|^2 + 2p\alpha r|u|} \mathrm{d}V(z),$$

which is easily seen to be convergent. (c)⇒(a) holds.

(a)⇒(c). Since there exists a positive integer m such that z belongs to at most m of the balls $B(a_j,r)$ for any $z \in \mathbb{C}^n$, we have

$$m \int_{\mathbb{C}^n} \widetilde{\mu}(z)^p \mathrm{d}V(z) \ge \sum_{j=1}^{\infty} \int_{B(a_j,r)} \widetilde{\mu}(z)^p \mathrm{d}V(z).$$

Notice

$$\widetilde{\mu}(z) = \int_{\mathbb{C}^n} e^{-\alpha|z-w|^2} \mathrm{d}\mu(w) \ge \int_{B(a_j,r)} e^{-\alpha|z-w|^2} \mathrm{d}\mu(w) \ge e^{-4\alpha r^2} \mu(B(a_j,r)),$$

then

$$m \int_{\mathbb{C}^n} \widetilde{\mu}(z)^p \mathrm{d}V(z) \ge \sum_{j=1}^{\infty} \int_{B(a_j,r)} e^{-4p\alpha r^2} \mu(B(a_j,r))^p \mathrm{d}V(z) \ge \frac{(\pi r^2)^n}{n!} e^{-4p\alpha r^2} \sum_{j=1}^{\infty} \mu(B(a_j,r))^p.$$

Thus, $\widetilde{\mu}(z) \in L^p(\mathbb{C}^n, \mathrm{d}V)$ implies $\{\mu(B(a_j,r))\} \in l^p$. (c)⇒(a) holds.

(a)⇒(b). It is obvious from Lemma 3.2.

(b)⇒(c). If condition (b) holds, we consider the lattice generated by $\{\xi_i^{\frac{r}{2}}\}_{i=1}^{2n}$ and arrange it into a sequence $\{z_j\}$. Since there exists a positive integer m such that every point in $\mathbb{C}^n$ belongs to at most m of the balls $B(z_j, \frac{r}{2})$, we have

$$m\int_{\mathbb{C}^n}\mu(B(z,r))^p \mathrm{d}V(z) \geq \sum_{j=1}^{\infty}\int_{B(z_j,\frac{r}{2})}\mu(B(z,r))^p \mathrm{d}V(z).$$

The triangle inequality tells us that $\mu(B(z,r)) \geq \mu(B(z_j,\frac{r}{2}))$ for each $z \in B(z_j,\frac{r}{2})$. Therefore,

$$m\int_{\mathbb{C}^n}\mu(B(z,r))^p \mathrm{d}V(z) \geq \frac{(\pi r^2)^n}{4^n n!}\sum_{j=1}^{\infty}\mu(B(z_j,\frac{r}{2}))^p.$$

By the equivalence of condition (a) and (c), the function $\widetilde{\mu}$ belongs to $L^p(\mathbb{C}^n, \mathrm{d}V)$, and applying the equivalence of (a) and (c) once more, we conclude that $\{\mu(B(a_j,r))\}$ is in l^p. This completes the proof of the lemma. □

Lemma 4.2 *Suppose* $\mu \geq 0$, $0 < p \leq 1$, *and the function* $\widetilde{\mu}(z)$ *is in* $L^p(\mathbb{C}^n, \mathrm{d}V)$. *Then the Toeplitz operator* T_μ *is in the Schatten class* S_p.

Proof As we know $T_\mu \in S_p$ if and only if $T_\mu^p \in S_1$ if and only if $\mathrm{tr}(T_\mu^p) < \infty$. In order to prove $T_\mu \in S_p$, we just ought to show that $\mathrm{tr}(T_\mu^p) < \infty$. In fact,

$$\mathrm{tr}(T_\mu^p) = (\frac{\alpha}{\pi})^n\int_{\mathbb{C}^n}\langle T_\mu^p k_z, k_z\rangle \mathrm{d}V(z) \leq (\frac{\alpha}{\pi})^n\int_{\mathbb{C}^n}\langle T_\mu k_z, k_z\rangle^p \mathrm{d}V(z) = (\frac{\alpha}{\pi})^n\int_{\mathbb{C}^n}\widetilde{\mu}(z)^p \mathrm{d}V(z)$$

where the first inequality comes from Proposition 6.3.3(2) of [12]. Thus, the integral is convergent from the assumption that $\widetilde{\mu}(z) \in L^p(\mathbb{C}^n, \mathrm{d}V)$. This completes the proof of this Lemma. □

Lemma 4.3 *Suppose* $\varphi \geq 0$, $0 < p \leq 1$, *and* $T_\varphi \in S_p$. *Then* $\varphi \in L^p(\mathbb{C}^n, \mathrm{d}V)$.

Proof It is similar to the case when $n = 1$, we omit here [3]. Furthermore, the condition $0 < p \leq 1$ here is sharp because we can also give an counter-example to show that this conclusion is false as $p > 1$. Just take $\varphi(z) = \chi_{[0,1]}(|z|)|z|^{-\frac{2}{p}}$, it is not difficult to check that φ is not in $L^p(\mathbb{C}^n, \mathrm{d}V)$ when $p > 1$. However, as φ is radial, the operator T_φ is diagonal with respect to the standard orthonormal basis $\{\sqrt{\frac{\alpha^{\|k\|}}{k!}}z^k\}_{k_j\geq 0}$ of $H^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$, and one can easily check that $T_\varphi \in S_p$ for each $p > 1$.

To obtain the necessary and sufficient condition of Schatten p-class Toeplitz operators as $0 < p \leq 1$, we also need the following Lemma whose proof can be found in [12].

Lemma 4.4 *If* $0 < p \leq 1$, *then for any orthonormal basis* $\{e_k\}$ *of a separable Hilbert space* H *and any compact operator* T *on* H, *we have that* $\|T\|_{S_p}^p \leq \sum_{l\in N^n}\sum_{k\in N^n}|\langle Te_l, e_k\rangle|^p$.

Now, we are ready to characterize Toeplitz operator T_μ in S_p in the case of $0 < p \leq 1$. The careful reader will find that several key ideas in the proof of the following theorem are similar to the counterpart of the Bergman space theory.

Theorem 4.5 *Suppose* $\mu \geq 0$, $0 < p \leq 1$, $r > 0$, *and* $\{a_j\}$ *is the lattice in* $\mathbb{C}^n$ *generated by* $\{\xi_i^r\}_{i=1}^{2n}$. *Then the following conditions are equivalent:*

(a) The Toeplitz operator T_μ *is in the Schatten class* S_p.

(b) The function $\widetilde{\mu}(z)$ *is in* $L^p(\mathbb{C}^n, \mathrm{d}V)$.

(c) The function $\widehat{\mu}_r$ is in $L^p(\mathbb{C}^n, \mathrm{d}V)$.

(d) The sequence $\{\widehat{\mu}_r(a_j)\}$ is in l^p.

Proof We have proved the equivalence of (b), (c) and (d) in Lemma 4.1. Moreover, condition (b) implies condition (a) was proved in Lemma 4.2. Therefore, to complete our proof of this theorem, we just ought to show condition (a) implies any condition of (b), (c) and (d). We choose to prove condition (a) implies (d) here. In what follows, $C_1, C_2, \ldots$ will denote positive constants that only depend on p, α and r. For convenience, we will use the norm

$$|z|_\infty = \max\{|x_1|, |y_1|, |x_2|, |y_2|, \ldots, |x_n|, |y_n|\}$$

where $z = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n) \in \mathbb{R}^{2n}$. Denote by $B(z,r)$ the closed ball centered at z with radius r in this norm. If we can prove that condition (a) implies

$$\sum_{j=1}^{\infty} \mu(B(2a_j, r))^p < \infty,$$

then condition (d) will easily follow. To this end, fix some $R > 0$ and partition $\{2a_j\}$ into M subsequence such that the Euclidean distance between any two points in each subsequence is at least R. Let $\{\zeta_j\}$ be such a subsequence and let $\nu = \sum_{j=1}^{\infty} \mu\chi_j$, where χ_j is the characteristic function of $B(\zeta_j, r)$. Since $T_\mu \in S_p$ and $\mu \geq \nu$, we have $T_\nu \leq T_\mu$, and so $T_\nu \in S_p$ with

$$\|T_\nu\|_{S_p} \leq \|T_\mu\|_{S_p}. \tag{4}$$

Suppose $\{e_k\}$ is an orthonormal basis for $H^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$. Then we can construct an one-to-one mapping from $\{k = (k_1, k_2, \ldots, k_n) \in N^n\}$ to $N = \{0, 1, 2, \ldots\}$ because both of them are countable sets. Thus, we can define a bounded linear operator A on $H^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$ such that $Ae_k = k_{\zeta_{j_k}}$, where $k = (k_1, k_2, \ldots, k_n) \in N^n$ and j_k is a non-negative number depending on k. Let $T = A^* T_\nu A$ so that $\|T\|_{S_p} \leq \|T_\mu\|_{S_p}$. We split the operator T as $T = D + E$ where D is the diagonal operator defined on $H^2(\mathbb{C}^n, \mathrm{d}V_\alpha)$ by $Df = \sum_{k \in N^n} \langle Te_k, e_k\rangle \langle f, e_k\rangle e_k$ and $E = T - D$. By the triangle inequality, we have

$$\|T\|_{S_p}^p \geq \|D\|_{S_p}^p - \|E\|_{S_p}^p. \tag{5}$$

From the definition of D, we have

$$\begin{aligned}
\|D\|_{S_p}^p &= \sum_{k \in N^n} \langle Te_k, e_k\rangle^p = \sum_{k \in N^n} \langle T_\nu Ae_k, Ae_k\rangle^p = \sum_{j_k=1}^{\infty} \langle T_\nu k_{\zeta_{j_k}}, k_{\zeta_{j_k}}\rangle^p \\
&= \sum_{j_k=1}^{\infty} \Big(\int_{\mathbb{C}^n} e^{-\alpha|z-\zeta_{j_k}|^2} \mathrm{d}\nu(z)\Big)^p \geq \sum_{j_k=1}^{\infty} \Big(\int_{B(\zeta_{j_k}, r)} e^{-\alpha|z-\zeta_{j_k}|^2} \mathrm{d}\nu(z)\Big)^p \\
&\geq e^{-\alpha p r^2} \sum_{j_k=1}^{\infty} \nu(B(\zeta_{j_k}, r))^p = C_1 \sum_{j=1}^{\infty} \nu(B(\zeta_j, r))^p.
\end{aligned}$$

On the other hand, we have

$$\|E\|_{S_p}^p \leq \sum_{l \in N^n} \sum_{k \in N^n} |\langle Ee_l, e_k\rangle|^p = \sum_{l \in N^n} \sum_{k \in N^n} |\langle Te_l, e_k\rangle - \langle De_l, e_k\rangle|^p$$

$$
\begin{aligned}
&= \sum_{j_l \neq j_k} \langle T_\nu k_{\zeta_{j_l}}, k_{\zeta_{j_k}} \rangle^p = \sum_{j_l \neq j_k} \Big| \int_{\mathbb{C}^n} k_{\zeta_{j_l}}(z) \overline{k_{\zeta_{j_k}}(z)} e^{-\alpha |z|^2} \mathrm{d}\nu(z) \Big|^p \\
&= \sum_{j_l \neq j_k} \langle T_\nu k_{\zeta_{j_l}}, k_{\zeta_{j_k}} \rangle^p = \sum_{j_l \neq j_k} \Big| \int_{\mathbb{C}^n} e^{-\frac{\alpha |z-\zeta_{j_l}|^2}{2}} e^{-\frac{\alpha |z-\zeta_{j_k}|^2}{2}} e^{\alpha i \cdot \mathrm{Im}(z\overline{\zeta_{j_l}} + \overline{z}\zeta_{j_k})} \mathrm{d}\nu(z) \Big|^p \\
&\leq \sum_{j_l \neq j_k} \Big(\int_{\mathbb{C}^n} e^{-\frac{\alpha |z-\zeta_{j_l}|^2}{2}} e^{-\frac{\alpha |z-\zeta_{j_k}|^2}{2}} \mathrm{d}\nu(z) \Big)^p.
\end{aligned}
$$

If $j_l \neq j_k$, then $|\zeta_{j_l} - \zeta_{j_k}| \geq R$. Thus for $|z - \zeta_{j_l}| \leq \frac{R}{2}$ the triangle inequality gives us $|z - \zeta_{j_k}| \geq \frac{R}{2}$. Hence,

$$
e^{-\frac{\alpha |z-\zeta_{j_l}|^2}{2}} e^{-\frac{\alpha |z-\zeta_{j_k}|^2}{2}} \leq e^{-\frac{\alpha R^2}{16}} e^{-\frac{\alpha |z-\zeta_{j_l}|^2}{4}} e^{-\frac{\alpha |z-\zeta_{j_k}|^2}{4}}
$$

holds for each $z \in \mathbb{C}^n$.

Therefore, we have

$$
\|E\|_{S_p}^p \leq e^{-\frac{p\alpha R^2}{16}} \sum_{j_l \neq j_k} \Big(\int_{\mathbb{C}^n} e^{-\frac{\alpha |z-\zeta_{j_l}|^2}{4}} e^{-\frac{\alpha |z-\zeta_{j_k}|^2}{4}} \mathrm{d}\nu(z) \Big)^p.
$$

For each m in $\{0, 1, 2, \ldots\}$ and $j_l \in \mathbb{N}$, let

$$
E_{m,j_l} = \{z : r(2m-1) \leq |z - \zeta_{j_l}|_\infty < 2rm\}.
$$

Since $0 < p \leq 1$, we know that

$$
\begin{aligned}
&\sum_{j_l \neq j_k} \Big(\int_{\mathbb{C}^n} e^{-\frac{\alpha |z-\zeta_{j_l}|^2}{4}} e^{-\frac{\alpha |z-\zeta_{j_k}|^2}{4}} \mathrm{d}\nu(z) \Big)^p \leq \sum_{j_l \neq j_k} \sum_{m=0}^{\infty} \Big(\int_{E_{m,j_l}} e^{-\frac{\alpha |z-\zeta_{j_l}|^2}{4}} e^{-\frac{\alpha |z-\zeta_{j_k}|^2}{4}} \mathrm{d}\nu(z) \Big)^p \\
&\leq C_2 \sum_{m=0}^{\infty} e^{-\frac{p\alpha r^2 (2m-1)^2}{4}} \sum_{j_l \neq j_k} \Big(\int_{E_{m,j_l}} e^{-\frac{\alpha |z-\zeta_{j_k}|^2}{4}} \mathrm{d}\nu(z) \Big)^p
\end{aligned}
$$

for some constant C_2.

For any fixed m and j_l, we write $\mathbb{N} = \Omega^1_{m,j_l} \bigcup \Omega^2_{m,j_l}$, where

$$
\Omega^1_{m,j_l} = \{j_k \in \mathbb{N} : |\zeta_{j_l} - \zeta_{j_k}|_\infty \leq 2rm\}, \quad \Omega^2_{m,j_l} = \{j_k \in \mathbb{N} : |\zeta_{j_l} - \zeta_{j_k}|_\infty > 2rm\}.
$$

Thus, we have that

$$
\begin{aligned}
&\sum_{m=0}^{\infty} e^{-\frac{p\alpha r^2 (2m-1)^2}{4}} \sum_{j_l \neq j_k} \Big(\int_{E_{m,j_l}} e^{-\frac{\alpha |z-\zeta_{j_k}|^2}{4}} \mathrm{d}\nu(z) \Big)^p \\
&\leq \sum_{m=0}^{\infty} e^{-\frac{p\alpha r^2 (2m-1)^2}{4}} \sum_{j_l=1}^{\infty} \sum_{j_k \in \Omega^1_{m,j_l}} \Big(\int_{E_{m,j_l}} e^{-\frac{\alpha |z-\zeta_{j_k}|^2}{4}} \mathrm{d}\nu(z) \Big)^p + \\
&\sum_{m=0}^{\infty} e^{-\frac{p\alpha r^2 (2m-1)^2}{4}} \sum_{j_l=1}^{\infty} \sum_{j_k \in \Omega^2_{m,j_l}} \Big(\int_{E_{m,j_l}} e^{-\frac{\alpha |z-\zeta_{j_k}|^2}{4}} \mathrm{d}\nu(z) \Big)^p \\
&= S_1 + S_2,
\end{aligned}
$$

where

$$
S_1 = \sum_{m=0}^{\infty} e^{-\frac{p\alpha r^2 (2m-1)^2}{4}} \sum_{j_l=1}^{\infty} \sum_{j_k \in \Omega^1_{m,j_l}} \Big(\int_{E_{m,j_l}} e^{-\frac{\alpha |z-\zeta_{j_k}|^2}{4}} \mathrm{d}\nu(z) \Big)^p
$$

and

$$S_2=\sum_{m=0}^{\infty}e^{-\frac{p\alpha r^2(2m-1)^2}{4}}\sum_{j_l=1}^{\infty}\sum_{j_k\in\Omega^2_{m,j_l}}\Big(\int_{E_{m,j_l}}e^{-\frac{\alpha|z-\zeta_{j_k}|^2}{4}}\mathrm{d}\nu(z)\Big)^p.$$

From the definition of Ω^1_{m,j_l}, we know that $\mathrm{card}(\Omega^1_{m,j_l})\le C_3(m+1)^{2n}$ for some constant $C_3>0$, which implies that

$$\begin{aligned}S_1&\le C_4\sum_{m=0}^{\infty}(m+1)^{2n}e^{-\frac{p\alpha r^2(2m-1)^2}{4}}\sum_{j_l=1}^{\infty}\nu(E_{m,j_l})^p\\&\le C_4\sum_{m=0}^{\infty}(m+1)^{2n}e^{-\frac{p\alpha r^2(2m-1)^2}{4}}\sum_{j_l=1}^{\infty}\sum_{\{j_k:|2a_{j_k}-\zeta_{j_l}|=2rm\}}\nu(B(2a_{j_k},r))^p\\&\le C_5\sum_{m=0}^{\infty}(m+1)^{4n}e^{-\frac{p\alpha r^2(2m-1)^2}{4}}\sum_{j_k=1}^{\infty}\nu(B(2a_{j_k},r))^p\\&\le C_6\sum_{j_k=1}^{\infty}\nu(B(2a_{j_k},r))^p=C_6\sum_{j=1}^{\infty}\nu(B(\zeta_j,r))^p.\end{aligned}$$

We still have to estimate the sum S_2. Note that if $k\in\Omega^2_{m,j_l}$, then $z\in E_{m,j_l}$ implies that

$$|z-\zeta_{j_k}|_\infty\ge|\zeta_{j_l}-\zeta_{j_k}|_\infty-|z-\zeta_{j_l}|_\infty\ge|\zeta_{j_l}-\zeta_{j_k}|_\infty-2rm>0.$$

Therefore,

$$\begin{aligned}S_2&\le\sum_{m=0}^{\infty}e^{-\frac{p\alpha r^2(2m-1)^2}{4}}\sum_{j_l=1}^{\infty}\nu(E_{m,j_l})^p\sum_{j_k\in\Omega^2_{m,j_l}}e^{-\frac{p\alpha(|\zeta_{j_l}-\zeta_{j_k}|_\infty-2rm)^2}{4}}\\&\le C_7\sum_{m=0}^{\infty}e^{-\frac{p\alpha r^2(2m-1)^2}{4}}\sum_{j_l=1}^{\infty}\nu(E_{m,j_l})^p\sum_{j_k=1}^{\infty}e^{-\frac{p\alpha(2rj_k)^2}{4}}(m+j_k+1)^{2n}\\&\le C_8\sum_{m=0}^{\infty}(m+1)^{2n}e^{-\frac{p\alpha r^2(2m-1)^2}{4}}\sum_{j_l=1}^{\infty}\sum_{\{j_k:|2a_{j_k}-\zeta_{j_l}|=2rm\}}\nu(B(2a_{j_k},r))^p\\&\le C_9\sum_{m=0}^{\infty}(m+1)^{4n}e^{-\frac{p\alpha r^2(2m-1)^2}{4}}\sum_{j_k=1}^{\infty}\nu(B(2a_{j_k},r))^p\\&\le C_{10}\sum_{j=1}^{\infty}\nu(B(2a_j,r))^p=C_{10}\sum_{j=1}^{\infty}\nu(B(\zeta_j,r))^p.\end{aligned}$$

Now, we conclude that there is a positive constant C_{11} such that

$$\|E\|^p_{S_p}\le C_{11}e^{-\frac{p\alpha R^2}{16}}\sum_{j=1}^{\infty}\nu(B(\zeta_j,r))^p$$

from the estimates about S_1 and S_2. Moreover, combining (4) and (5) gives

$$\|T_\mu\|^p_{S_p}\ge\|T\|^p_{S_p}\ge\|D\|^p_{S_p}-\|E\|^p_{S_p}\ge(C_1-C_{11}e^{-\frac{p\alpha R^2}{16}})\sum_{j=1}^{\infty}\nu(B(\zeta_j,r))^p.$$

Since C_1 and C_{11} are not dependent on R, setting $R > 0$ large enough gives us

$$\sum_{j=1}^{\infty} \nu(B(\zeta_j, r))^p \leq C_{12} \|T_\mu\|_{S_p}^p.$$

Since this holds for each of the M subsequences of $\{2a_j\}$, we get

$$\sum_{j=1}^{\infty} \mu(B(2a_j, r))^p \leq C_{12} M \|T_\mu\|_{S_p}^p$$

for all positive Borel measures μ, which implies that the sequence $\{\widehat{\mu}_r(a_j)\}$ is in l^p. This means (a)⇒(d) holds, and thus completes the proof of the Theorem 4.5. □

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