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Some Sets of GCF_ϵ Expansions Whose Parameter ϵ Fetch the Marginal Value

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Abstract Let $\epsilon:\mathbb{N}\to\mathbb{R}$ be a parameter function satisfying the condition $\epsilon(k)+k+1>0$ and let $T_\epsilon:(0,1]\to(0,1]$ be a transformation defined by

$$T_\epsilon(x)=\frac{-1+(k+1)x}{1+k-k\epsilon x}\ \text{for}\ x\in\Big(\frac{1}{k+1},\frac{1}{k}\Big].$$

Under the algorithm T_ϵ, every $x\in(0,1]$ is attached an expansion, called generalized continued fraction (GCF_ϵ) expansion with parameters by Schweiger. Define the sequence $\{k_n(x)\}_{n\geq1}$ of the partial quotients of x by $k_1(x)=\lfloor 1/x\rfloor$ and $k_n(x)=k_1(T_\epsilon^{n-1}(x))$ for every $n\geq2$. Under the restriction $-k-1<\epsilon(k)<-k$, define the set of non-recurring GCF_ϵ expansions as

$$\mathcal{F}_\epsilon=\{x\in(0,1]:k_{n+1}(x)>k_n(x)\ \text{for infinitely many}\ \ n\}.$$

It has been proved by Schweiger that $\mathcal{F}_\epsilon$ has Lebesgue measure 0. In the present paper, we strengthen this result by showing that

$$\begin{cases}\dim_H\mathcal{F}_\epsilon\geq\frac12, & \text{when}\ \epsilon(k)=-k-1+\rho\ \text{for a constant}\ 0<\rho<1;\\ \frac{1}{s+2}\leq\dim_H\mathcal{F}_\epsilon\leq\frac1s, & \text{when}\ \epsilon(k)=-k-1+\frac{1}{k^s}\ \text{for any}\ s\geq1\end{cases}$$

where $\dim_H$ denotes the Hausdorff dimension.

Keywords GCF_ϵ expansions; metric properties; Hausdorff dimension

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1. Introduction

In 2003, Schweiger [1] introduced a new class of continued fractions with parameters, called generalized continued fractions (GCF_ϵ), which are induced by the transformations $T_\epsilon:(0,1]\to(0,1]$

$$T_\epsilon(x):=\frac{-1+(k+1)x}{1+\epsilon-k\epsilon x}\ \text{when}\ x\in\Big(\frac{1}{k+1},\frac1k\Big]=:B(k) \tag{1}$$

where the parameter $\epsilon:\mathbb{N}\to\mathbb{R}$ satisfies

$$\epsilon(k)+k+1>0,\quad \text{for all}\ k\geq1. \tag{2}$$

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For any $x \in (0,1]$, its partial quotients $\{k_n\}_{n\geq 1}$ in the GCF$_\epsilon$ expansion are defined as

$$k_1 = k_1(x) := \Big\lfloor \frac{1}{x} \Big\rfloor, \quad \text{and} \quad k_n = k_n(x) := k_1\big(T_\epsilon^{n-1}(x)\big).$$

By the algorithm (1), it follows [1] that

$$x = \frac{A_n + B_n T_\epsilon^n(x)}{C_n + D_n T_\epsilon^n(x)} \quad \text{for all } n \geq 1,$$

where the numbers A_n, B_n, C_n, D_n are given by the recursive relations

$$\begin{pmatrix} C_n & D_n \\ A_n & B_n \end{pmatrix} = \begin{pmatrix} C_{n-1} & D_{n-1} \\ C_{n-1} & B_{n-1} \end{pmatrix} \begin{pmatrix} k_n + 1 & k_n \epsilon(k_n) \\ 1 & 1 + \epsilon(k_n) \end{pmatrix} \quad n \geq 1, \tag{3}$$

$$\text{with} \quad \begin{pmatrix} C_0 & D_0 \\ A_0 & B_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

A well known example of the generalized continued fraction is in the case that the parameter function $\epsilon \equiv 0$. In this case, the algorithm (1) becomes

$$T(x) = -1 + (k+1)x \text{ when } x \in \Big(\frac{1}{k+1}, \frac{1}{k}\Big].$$

Then every $x \in (0,1]$ can be expanded into a series with the form

$$x = \frac{1}{k_1(x)+1} + \cdots + \frac{1}{(k_1+1)(k_2(x)+1)\cdots(k_n(x)+1)} + \cdots.$$

Actually this is the Engel series expansion which was well studied in the literature, see Erdös, Rényi & Szüsz [2], Rényi [3], Galambos [4] and Liu, Wu [5], etc.

Schweiger [1] studied the arithmetical as well as the ergodic properties of GCF$_\epsilon$ map. At the same time, he showed that with different choices of the parameter functions ϵ, the stochastic properties of the partial quotients differ greatly. Concerning the properties of the partial quotients, by the condition shared by the parameter $\epsilon(k)$ (see (2)), it is clear that

$$k_{n+1}(x) \geq k_n(x) \quad \text{for all } n \geq 1,$$

i.e., the partial quotients sequence of x is non-decreasing. We investigated the metrical properties of $\{k_n\}_{n\geq 1}$ further in [8] and proved that when $-1 < \epsilon(k) \leq 1$, for almost all $x \in (0,1]$,

$$\lim_{n\to\infty} \frac{\log k_n(x)}{n} = 1,$$

and when $\epsilon(k) = -1$, this equality is no longer true. It was also shown [7] that the partial quotients in the GCF$_\epsilon$ expansions share a 0-1 law and the central limit theorem under the restriction of $-1 < \epsilon(k) \leq 1$. These results showed that when $-1 < \epsilon(k) \leq 1$, the metric properties of GCF$_\epsilon$ and Engel series expansion are very similar. However, in this paper we will see that the situation changes radically when $-k-1 < \epsilon(k) < -k-\rho$ for a constant $0 < \rho < 1$. This is because in this case, T_ϵ has two fixed points $-\frac{1}{\epsilon}$ and $\frac{1}{k}$ in every interval $B(k) := \big(\frac{1}{k+1}, \frac{1}{k}\big]$. So all $B(k)$ can be divided into two subintervals as:

$$B(k^-) =: \Big[\frac{1}{k+1}, -\frac{1}{\epsilon(k)}\Big] \quad \text{and} \quad \mathrm{B}(\mathrm{k}^+) =: \Big(-\frac{1}{\epsilon(\mathrm{k})}, \frac{1}{\mathrm{k}}\Big].$$

such that $TB(k^+) = B(k^+)$. Therefore if $(k_1^-, k_2^-, \dots, k_n^-, k^+)$ is an admissible block, then $k_n < k$. And it is easy to see that, the set defined by

$$\mathcal{F}_\epsilon = \bigcap_{n=1}^{\infty} \bigcup_{k_1 \le \cdots \le k_n} B(k_1^-, k_2^-, \dots, k_n^-) \tag{4}$$

is a complementary set of the ultimately recurring GCF_ϵ expansion. That is

$$\mathcal{F}_\epsilon := \{x \in (0,1] : k_{n+1}(x) > k_n(x) \text{ for infinitely many } n\}.$$

We define the cylinder set as follows. For any non-decreasing integer vector $(k_1, \dots, k_n)$, define the n-th order cylinders as follows

$$B(k_1, \dots, k_n) = \{x \in (0,1] : k_j(x) = k_j, \ \forall 1 \le j \le n\}.$$

an nth order cylinder, which is the set of points whose partial quotients begin with $(k_1, \dots, k_n)$. Then the following results have been obtained in section 3 of [1]:

$$|B(k_1, k_2, \dots, k_n)| = \frac{B_n C_n - A_n D_n}{C_n(k_n C_n + D_n)}; \tag{5}$$

$$\left|B(k_1^-, k_2^-, \dots, k_n^-)\right| = \frac{B_n C_n - A_n D_n}{C_n(-\epsilon(k_n) C_n + D_n)}; \tag{6}$$

$$\lambda(\mathcal{F}_\epsilon) = \lambda\Big(\bigcap_{n=1}^{\infty} \bigcup_{k_1 < \cdots < k_n} B(k_1^-, k_2^-, \dots, k_n^-)\Big) = 0, \tag{7}$$

where $-k-1 < \epsilon(k) < -k-1+\rho$ for a constant $0 < \rho < 1$.

In this paper, we strengthen the result (7) by showing that

Theorem 1.1 *Let $\mathcal{F}_\epsilon$ be the set defined above. Then*

$$\begin{cases} \dim_H \mathcal{F}_\epsilon \ge \frac{1}{2}, & \text{when } \epsilon(k) = -k-1+\rho \text{ for a constant } 0<\rho<1; \\ \frac{1}{s+2} \le \dim_H \mathcal{F}_\epsilon \le \frac{1}{s}, & \text{when } \epsilon(k) = -k-1+\frac{1}{k^s} \text{ for any } s \ge 1 \end{cases}$$

where $\dim_H$ denotes the Hausdorff dimension.

2. Preliminary

In this section, we present some simple facts about the generalized continued fractions for later use.

The first lemma concerns the relationships between A_n, B_n, C_n, D_n which are recursively defined by (3).

Lemma 2.1 ([1,8]) *For all $n \ge 1$,*

(i) $C_n = (k_n + 1)C_{n-1} + D_{n-1} > 0$;

(ii) $D_n = k_n \epsilon(k_n) C_{n-1} + (1 + \epsilon(k_n)) D_{n-1}$, *and* $D_n \ge 0$ *when* $\epsilon \ge 0$; $D_n < 0$ *when* $\epsilon < 0$;

(iii) $k_n C_n + D_n = (k_n C_{n-1} + D_{n-1})(k_n + 1 + \epsilon(k_n))$;

(iv) $B_n C_n - A_n D_n = (B_N C_N - A_N D_N) \prod_{i=N+1}^{n} (k_i + 1 + \epsilon(k_i)) > 0, \ \forall 0 \le N < n$.

The following lemmas are especially aimed for $\epsilon(k) = -k-1+\frac{1}{k^s}$.

Lemma 2.2 *If $\epsilon(k) = -k - 1 + \frac{1}{k^s}$, then when $k_n \geq 2$,*

$$k_nC_n + D_n = \frac{k_nC_{n-1} + D_{n-1}}{k_n^s} > 0; \quad -\epsilon(k_n)C_n + D_n \geq \frac{C_n}{2} \geq 1.$$

Proof By Lemma 2.1 (iii) and the condition $\epsilon(k) = -k - 1 + \frac{1}{k^s}$, noticing that $k_n \geq k_{n-1}$, we have

$$k_nC_n + D_n = \frac{k_nC_{n-1} + D_{n-1}}{k_n^s} \geq \frac{k_{n-1}C_{n-1} + D_{n-1}}{k_n^s} \geq \cdots \geq \frac{k_1C_1 + D_1}{k_n^s k_{n-1}^s \cdots k_2^s} > 0.$$

This also gives that

$$D_n \geq -k_nC_n. \tag{8}$$

Using Lemma 2.1 (i) and (8), we get

$$\begin{aligned} C_n &\geq (k_n+1)C_{n-1} - k_{n-1}C_{n-1} \geq (k_n + 1 - k_{n-1})C_{n-1} \\ &\geq C_{n-1} \geq \cdots \geq C_1 = k_1 + 1 \geq 2. \end{aligned}$$

Thus $\frac{C_n}{2} \geq 1$ is proved.

Using (8) again, we can find that when $k_n \geq 2$,

$$-\epsilon(k_n)C_n + D_n \geq (k_n + 1 - \frac{1}{k_n^s})C_n - k_nC_n = (1 - \frac{1}{k_n^s})C_n \geq \frac{1}{2}C_n. \ \square$$

The next lemma will be used for estimating the lower bound of $\dim_H \mathcal{F}_\epsilon$.

Lemma 2.3 ([1,8]) *Let $\epsilon(k) = -k - 1 + \frac{1}{k^s}$. Then when $k_n \geq 2$,*

$$C_n + D_n \leq 0; \quad k_nC_n + D_n \leq -\epsilon(k_n)C_n + D_n \leq C_n \leq k_nk_{n-1}\cdots k_1.$$

Proof By Lemma 2.1 (i) (ii), we have

$$\begin{aligned} C_n + D_n &= (k_n+1)C_{n-1} + D_{n-1} + k_n(-k_n - 1 + \frac{1}{k_n^s})C_{n-1} + (-k_n + \frac{1}{k_n^s})D_{n-1} \\ &= C_{n-1} + D_{n-1} + (-k_n + \frac{1}{k_n^s})(k_nC_{n-1} + D_{n-1}) \\ &\leq C_{n-1} + D_{n-1} - (k_nC_{n-1} + D_{n-1}) \leq 0. \end{aligned}$$

Then by Lemma 2.1 (i), we have

$$C_n = k_nC_{n-1} + (C_{n-1} + D_{n-1}) \leq k_nC_{n-1} \leq \cdots \leq k_nk_{n-1}\cdots k_1. \tag{9}$$

By the condition $\epsilon(k) = -k - 1 + \frac{1}{k^s}$, we have,

$$k_n < -\epsilon(k_n) = -k_n - 1 + \frac{1}{k_n^s}.$$

Thus

$$k_nC_n + D_n \leq -\epsilon(k_n)C_n + D_n = (k_nC_n + D_n) + (1 - \frac{1}{k_n^s})C_n. \tag{10}$$

Then using the first equality of Lemma 2.2, we get

$$-\epsilon(k_n)C_n + D_n = \frac{k_nC_{n-1} + D_{n-1}}{k_n^s} + (1 - \frac{1}{k_n^s})C_n = C_n - \frac{C_{n-1}}{k_n^s} \leq C_n. \tag{11}$$

So the second result follows from (10), (11) and (9). $\square$

Now we focus on the properties of the point set $\mathcal{F}_\epsilon$ with $\epsilon(k) = -k - 1 + \frac{1}{k_n^s}$ for any $s \geq 1$. From now on until the end of this paper, we fix a point $x \in \mathcal{F}_\epsilon$ and let $k_n = k_n(x)$ be the nth partial quotient of x. The numbers A_n, B_n, C_n, D_n are recursively defined by (3) for x.

3. The Hausdorff dimension of $E_\epsilon(\alpha)$

The proof of Theorem 1.1 is divided into two parts: one for upper bound, the other for lower bound.

3.1. Upper bound

Fix $\delta > 0$. Since $\sum_{k_n=1}^{\infty} \left(\frac{1}{k_n^s}\right)^{\frac{1+\delta}{s}} = \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}}$ converges, there exists M large enough so that for all $k_j \geq M$,

$$\sum_{k_n=k_j}^{\infty} \left(\frac{1}{k_n^s}\right)^{\frac{1+\delta}{s}} \leq 1. \tag{12}$$

From (4), we can see that $\bigcup_{k_1 \leq \cdots \leq k_n} B(k_1^-, k_2^-, \ldots, k_n^-)$ is a natural covering of $\mathcal{F}_\epsilon$ for any $n \geq 1$. Then the $\frac{1+\delta}{s}$-dimensional Hausdorff measure of $\mathcal{F}_\epsilon$ can be estimated as

$$\mathcal{H}^{\frac{1+\delta}{s}}(\mathcal{F}_\epsilon) \leq \liminf_{n\to\infty} \sum_{\substack{k_{i+1} \geq k_i \\ 1 \leq i \leq n-1}} \left|B(k_1^-, k_2^-, \ldots, k_n^-)\right|^{\frac{1+\delta}{s}}.$$

Under the condition $\epsilon(k) = -k - 1 + \frac{1}{k_s}$, by Lemma 2.1 (iv), we have

$$B_nC_n - A_nD_n = \frac{1}{(k_1k_2\cdots k_n)^s}. \tag{13}$$

On the other hand, by Lemma 2.2, we have $C_n(-\epsilon(k_n)C_n + D_n) \geq 2$. Then using (6), we get

$$\begin{aligned}\left|B(k_1^-, k_2^-, \ldots, k_n^-)\right| &= \frac{B_nC_n - A_nD_n}{C_n(-\epsilon(k_n)C_n + D_n)} \leq \frac{1}{2(k_1k_2\cdots k_n)^s} \\ &\leq \frac{1}{2(k_1k_2\cdots k_N)^s} \frac{1}{k_{N+1}^s} \frac{1}{k_{N+2}^s} \cdots \frac{1}{k_n^s}.\end{aligned}$$

Thus by (12), we have

$$\begin{aligned}\mathcal{H}^{\frac{1+\delta}{s}}(\mathcal{F}_\epsilon) &\leq \liminf_{n\to\infty} \sum_{\substack{k_{i+1} \geq k_i \\ 1 \leq i \leq n-1}} \left|B(k_1^-, k_2^-, \ldots, k_n^-)\right|^{\frac{1+\delta}{s}} \\ &\leq \liminf_{n\to\infty} \sum_{\substack{k_{i+1} \geq k_i \\ 1 \leq i \leq N-1}} \left(\frac{1}{2(k_1\cdots k_N)^s}\right)^{\frac{1+\delta}{s}} \sum_{k_{N+1}\geq k_N} \left(\frac{1}{k_{N+1}^s}\right)^{\frac{1+\delta}{s}} \cdots \sum_{k_n \geq k_{n-1}} \left(\frac{1}{k_{n-1}^s}\right)^{\frac{1+\delta}{s}} \\ &\leq \liminf_{n\to\infty} \sum_{\substack{k_{i+1} \geq k_i \\ 1 \leq i \leq N-1}} \left(\frac{1}{2(k_1\cdots k_N)^s}\right)^{\frac{1+\delta}{s}} < \infty\end{aligned}$$

which gives that $\dim_H E_\epsilon(\alpha) \leq \frac{1+\delta}{s}$. Since this is true for all $\delta > 0$, we get $\dim_H E_\epsilon(\alpha) \leq \frac{1}{s}$ for $\epsilon(k) = -k - 1 + \frac{1}{k^s}$ and any $s \geq 1$.

3.2. Lower bound

In order to estimate the lower bound, we recall the classical dimensional result concerning a specially defined Cantor set.

Lemma 3.1 ([6]) *Let $I = E_0 \supset E_1 \supset E_2 \supset \cdots$ be a decreasing sequence of sets, with each E_n, a union of a finite number of disjoint closed intervals. If each interval of E_{n-1} contains at least m_n intervals of E_n $(n = 1, 2, \ldots)$ which are separated by gaps of at least η_n, where $0 < \eta_{n+1} < \eta_n$ for each n. Then the lower bound of the Hausdorff dimension of E can be given by the following inequality:*

$$\dim_H \Big(\cap_{n\geq 1} E_n \Big) \geq \liminf_{n\to\infty} \frac{\log(m_1 m_2 \cdots m_{n-1})}{-\log(m_n \eta_n)}.$$

Now for each $n \geq 1$, let $E = \{x \in (0, 1] : 2^n < k_n(x) < 2^{n+1}, \forall n \geq 1\}$. Clearly, if $x \in E$, then $k_n(x) > k_{n-1}(x)$ for all $n \geq 1$. This implies that $E \subset \mathcal{F}_\epsilon$.

For each $n \geq 1$, let E_n be the collection of cylinders

$$E_n = \{B_n(k_1, \ldots, k_n) : 2^i < k_i(x) < 2^{i+1}, 1 \leq i \leq n\}. \tag{14}$$

Then $E = \bigcap_{n=1}^{\infty} E_n$, and E fulfills the construction of the Cantor set in Lemma 3.1. Now we specify the integers $\{m_n, n \geq 1\}$ and the real numbers $\{\eta_n, n \geq 1\}$.

Due to the definition of E_n, each interval of E_{n-1} contains $m_n = 2^n - 1 \geq 2^{n-1}$ intervals of E_n, and

$$m_1 m_2 \cdots m_{n-1} = 2^{1+2+\cdots+(n-2)} = 2^{\frac{(n-2)(n-1)}{2}}; \tag{15}$$

and any two of intervals in E_n are separated by at least one interval $B(k_1^-, \ldots, k_{n-1}^-, k_n^+)$.

By (5) and (6), we have

$$\begin{aligned} |B(k_1^-, \ldots, k_{n-1}^-, k_n^+)| =& |B(k_1, \ldots, k_{n-1}, k_n)| - B|(k_1^-, \ldots, k_{n-1}^-, k_n^-)| \\ =& \frac{B_nC_n - A_nD_n}{C_n(k_nC_n + D_n)} - \frac{B_nC_n - A_nD_n}{C_n(-\epsilon(k_n)C_n + D_n)} \\ =& \frac{(B_nC_n - A_nD_n)(-\epsilon(k_n) - k_n)}{(k_nC_n + D_n)(-\epsilon(k_n)C_n + D_n)}. \end{aligned}$$

By Lemma 2.3 and the equality (13), the above equality gives that

$$|B(k_1^-, \ldots, k_{n-1}^-, k_n^+)| \geq \frac{1}{(k_1k_2 \cdots k_n)^{s+2}}.$$

In view of (14), the partial quotients k_n satisfy that $2^n < k_n(x) < 2^{n+1}$ for all $n \geq 1$. Therefore,

$$|B(k_1^-, \ldots, k_{n-1}^-, k_n^+)| \geq \frac{1}{(2^{2+3+\cdots+(n+1)})^{s+2}} = \frac{1}{2^{\frac{n(n+1)(s+2)}{2}}} =: \eta_k. \tag{16}$$

As a result of (15) and (16), we get

$$\liminf_{n\to\infty} \frac{\log_2(m_1 \cdots m_{n-1})}{-\log_2(m_n\eta_n)} = \frac{1}{s+2}.$$

Combining this with Lemma 3.1, we get when $\epsilon(k) = -k - 1 + \frac{1}{k^s}$ for any $s \geq 1$,

$$\dim_H \mathcal{F}_\epsilon \geq \dim_H E \geq \frac{1}{s+2}.$$

Using the same method of proof, we can get $\dim_H \mathcal{F}_\epsilon \geq \frac{1}{2}$ when $\epsilon(k) = -k - 1 + \rho$ for a constant $0 < \rho < 1$.

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