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Existence of Simple $\mathrm{OA}_\lambda(3,5,v)'$s

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Abstract An orthogonal array of strength t, degree k, order v and index λ, denoted by $\mathrm{OA}_\lambda(t,k,v)$, is a $\lambda v^t \times k$ array on a v symbol set such that each $\lambda v^t \times t$ subarray contains each t-tuple exactly λ times. An $\mathrm{OA}_\lambda(t,k,v)$ is called simple and denoted by $\mathrm{SOA}_\lambda(t,k,v)$ if it contains no repeated rows. In this paper, it is proved that the necessary conditions for the existence of an $\mathrm{SOA}_\lambda(3,5,v)$ with $\lambda \geq 2$ are also sufficient with possible exceptions where $v=6$ and $\lambda \in \{3,7,11,13,15,17,19,21,23,25,29,33\}$.

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1. Introduction

Let t, k and v be all positive integers with $2 \leq t \leq k$. An orthogonal array, denoted by $\mathrm{OA}(N;t,k,v)$, is an $N \times k$ array with entries from a finite set V of v symbols such that each $N \times t$ subarray contains each t-tuple based on V equally often as a row. Clearly, the common frequency with which each of the t-tuples appears as a row in a subarray must be equal to N/v^t, which we will denote by λ and refer to as the index of the array. For this paper, the notation $\mathrm{OA}_\lambda(t,k,v)$ is often used. Here, the number k of columns is called the number of factors or degree. The number v of symbols is referred to as the order or the number of levels. t is termed as the strength of the orthogonal array. In arrays such as these, when $\lambda = 1$, the notation $\mathrm{OA}(t,k,v)$ is commonly used.

Orthogonal arrays have been extensively studied in the literature. They are of fundamental importance as ingredients in the construction of other useful combinatorial objects [1,2]. They are essential in statistics and they have important applications in coding theory, cryptography and computer science, as well as in drug screening. On this aspect, the interested reader may refer to [3–5]. It is well known that an $\mathrm{OA}(2,k,v)$ is equivalent to $k-2$ mutually orthogonal Latin squares (MOLS) of order v. Most results of orthogonal arrays can be attributed in a large

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degree to intelligent use of combinatorics, Galois fields and finite geometries etc.. Taking the advantage of finite fields, Bush established the following elegant results.

Lemma 1.1 ([6]) *If q is a prime power and $t < q$, then an $OA(t, q+1, q)$ exists. Moreover, if $q \geq 4$ is a power of 2, an $OA(3, q+2, q)$ exists.*

Lemma 1.2 ([7]) *If $OA(t, k, v_i)'s$ for $1 \leq i \leq m$ all exist, then an $OA(t, k, \prod_{i=1}^{m} v_i)$ exists.*

Recently, the authors Ji and Yin proved the following result.

Lemma 1.3 ([8]) *Let $v \geq 4$ be an integer. If $v \not\equiv 2 \pmod 4$, then an $OA(3, 5, v)$ exists.*

As a consequence, it is easy to see that an $\mathrm{OA}_\lambda(3, 5, v)$ with $v \geq 4$ and $v \not\equiv 2 \pmod 4$ exists for $\lambda \geq 2$: just take λ copies of an $\mathrm{OA}(3, 5, v)$ and superimpose them. The remaining cases, namely, $v = 3$ and $v \equiv 2 \pmod 4$ were settled by Li [9] which conclude the existence of $\mathrm{OA}_\lambda(3, 5, v)'$s with $\lambda \geq 2$.

By the definition of an OA, we see that the orthogonal arrays obtained in this way usually contain repeated rows. This raises the question whether there exist OAs of larger index without multiple rows from the viewpoint of design theory. Given a positive integer r, an OA of strength t is said to be an r-simple $\mathrm{OA}_\lambda(t, k, v)$ iff any two different rows agree in less than r entries. Specially, k-simple and $(t+1)$-simple orthogonal arrays are called simple and super-simple, denoted by $\mathrm{SOA}_\lambda(t, k, v)$ and $\mathrm{SSOA}_\lambda(t, k, v)$, respectively. In other words, if the array of degree k and strength t contains no repeated rows, it is referred to as an $\mathrm{SOA}_\lambda(t, k, v)$; if any $t+1$ columns of the array contains every $(t+1)$-tuple of symbols as a row at most once, we refer to as an $\mathrm{SSOA}_\lambda(t, k, v)$ (see [10,11]). The notion of simple and super-simple orthogonal arrays was presented in Hartman [12] under the name "r-simple transversal designs". From the definition of simple orthogonal arrays, we see that the number λv^t of rows in an $\mathrm{SOA}_\lambda(t, k, v)$ cannot exceed the total number v^k of k-tuple of symbols. This observation implies the necessary conditions for the existence of an $\mathrm{SOA}_\lambda(t, k, v)$ as follows.

Theorem 1.4 *An $SOA_\lambda(t, k, v)$ can exist only if $\lambda \leq v^{k-t}$.*

A complete solution to the existence of an $\mathrm{SOA}_\lambda(2, 4, v)$ and an $\mathrm{SOA}_\lambda(3, 4, v)$ was established by Hartman [12]. Here, we are mainly concerned with the existence of an $\mathrm{SOA}_\lambda(3, 5, v)$ with $\lambda \geq 2$. Let us fix an arbitrary column i of an $\mathrm{SOA}_\lambda(3, 5, v)$ and consider the subarray consisting of rows that have symbol x in this column. This subarray with the i-th column removed is an $\mathrm{SOA}_\lambda(2, 4, v)$. Hence, the existence of an $\mathrm{SOA}_\lambda(3, 5, v)$ implies the existence of an $\mathrm{SOA}_\lambda(2, 4, v)$. Motivated by this observation, the goal of this paper is to prove that the necessary conditions for the existence of an $\mathrm{SOA}_\lambda(3, 5, v)$ with $\lambda \geq 2$ are also sufficient with possible exceptions where $v = 6$ and $\lambda \in \{3, 7, 11, 13, 15, 17, 19, 21, 23, 25, 29, 33\}$.

2. Constructions of simple orthogonal arrays

In this section, we will present some approaches to construct simple orthogonal arrays. Given a positive integer n, we use the notation I_n to stand for the set of the first n positive integers in what follows.

Our first construction method involves the notion of difference matrix (DM). Let G be an abelian group of order v with the operation "+". A $k\times\lambda v$ matrix D with entries from G is called a difference matrix (DM) based on G, denoted by (v,k,λ)-DM, if it has the property that for all i and j with $1\le i,j\le k$, $i\ne j$, the vector difference between the ith and jth rows covers each element of G precisely λ times. Difference matrices were first defined by Bose and Bush [13], and are a simple but powerful tool for the construction of orthogonal arrays of strength two. The construction of OAs using DMs is of significance because difference matrices are much smaller in size than the orthogonal arrays that they induce. Here, we only concern with DMs of four rows. To obtain simple OAs of strength 3, degree 5, we need to introduce the notion of a simple DM below.

Suppose that $D=(d_{ij})$ $(i\in I_4, j\in I_{\lambda v})$ is a $(v,4,\lambda)$-DM over G. D is said to be simple, if D further satisfies that: for any j and j' with $1\le j<j'\le \lambda v$, $(d_{2j}-d_{1j}, d_{3j}-d_{2j}, d_{4j}-d_{3j})\ne (d_{2j'}-d_{1j'}, d_{3j'}-d_{2j'}, d_{4j'}-d_{3j'})$.

Example 2.1 The following array is a simple $(3,4,2)$-DM over $\mathbb{Z}_3$.

$$A=\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&1&1&2&2\\ 0&1&2&0&1&2\\ 0&2&2&1&0&1 \end{pmatrix}$$

Construction 2.2 *If a simple $(v,4,\lambda)$-DM exists, then an $SOA_\lambda(3,5,v)$ also exists.*

Proof Let $D=(d_{ij})$ $(i\in I_4, j\in I_{\lambda v})$ be the given simple $(v,4,\lambda)$-DM over G. For each column $(d_{1j},d_{2j},d_{3j},d_{4j})^T$, construct the following v^2 rows:

$$C(j,u,e)=(d_{1j}+u, d_{2j}+u, d_{3j}+u+e, d_{4j}+u+e, e),\quad e,u\in G.$$

Then we juxtapose the obtained rows to form a $\lambda v^3\times 5$ array A over G. Employing the same argument as the proof of Theorem 2.1 in [8], we can verify that the resulted array A is an $\mathrm{OA}_\lambda(3,5,v)$. By utilizing the simple property of the given DM, it is straightforward to show that A is simple. □

An orthogonal array $\mathrm{OA}_\lambda(t,k,v)$ is said to be completely reducible iff it is the union of λ orthogonal arrays $\mathrm{OA}(t,k,v)$ of index one. We can obtain an $\mathrm{SOA}_\lambda(t,k,v)$ from an $\mathrm{OA}(t,k,v)$ in the following way, where $1\le\lambda\le v^{k-t}$.

Construction 2.3 ([12]) *If an $OA(t,k,v)$ over G exists, then there is also a completely reducible simple $OA_{v^{k-t}}(t,k,v)$ over G.*

Construction 2.4 *If an $SSOA_\lambda(t,k,v)$ with $k\ge t+2$ over G exists, then an $SOA_{\lambda\mu}(t,k,v)$ over G also exists, where $1\le\mu\le v^{k-t-1}$.*

Proof Let A be an $\mathrm{SSOA}_\lambda(t,k,v)$ with $k\ge t+2$ over G. Suppose that B is an $\mathrm{OA}(k-t-1,k-t-1,v)$, which is formed by listing all $(k-t-1)$-tuples based on G. Let a and b be a

k-tuple and $(k-t-1)$-tuple over G, respectively. Their sum will be the tuple $c = a + b$ over G with the entries:

$$c_i = \begin{cases} a_i, & \text{if } i \in I_k \setminus I_{k-t-1}, \\ a_i + b_i, & \text{if } i \in I_{k-t-1}. \end{cases}$$

For a certain row B_i of B, the sum C_i of $A + B_i$ shall be the array of degree k and size λv^t consisting of all the tuples $a + B_i$ as rows, where a is any row in A. A permutation of symbols in an orthogonal array produces an orthogonal array with the same parameter. Thus, C_i is an $\mathrm{OA}_\lambda(t, k, v)$. Now, under the assumption $k \geq t + 2$, we may take μ disjoint rows B_i $(i = 1, 2, \cdots, \mu)$ from B and form μ $\mathrm{OA}_\lambda(t, k, v)'$s, C_i $(i = 1, 2, \ldots, \mu)$, where $1 \leq \mu \leq v^{k-t-1}$. Write C for the juxtaposition of these derived OAs from upper to bottom. Then C is an $\mathrm{OA}_{\lambda\mu}(t, k, v)$ over G. We only need to prove that C is simple. Let $c = a + b$ and $c' = a' + b'$ be two distinct rows in C. If $a = a'$, then $b \neq b'$ and thus, it is simple. Conversely, if a and a' are distinct, they agree in less than $t + 1$ entries. Hence, c and c' have less than $k - t - 1 + t + 1 = k$ entries in common. The proof is completed. $\Box$

The following example illustrates the idea in Construction 2.4.

Example 2.5 The transpose of the following array is an $\mathrm{SSOA}_2(3, 5, 2)$ over $\mathbb{Z}_2$, which was first presented in [10].

$$A_0 = \begin{array}{|cccccccccccccccc|}
\hline
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\
\hline
0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\
\hline
0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\
\hline
0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\
\hline
0&1&1&0&1&0&0&1&1&0&0&1&0&1&1&0 \\
\hline
\end{array}$$

Let B be an $\mathrm{OA}(1, 1, 2)$, i.e., B is a 2×1 array with the entries in the first and second row being 0 and 1, respectively. Then we construct the following two arrays A_1 and A_2:

$$A_1 = \begin{array}{|cccccccccccccccc|}
\hline
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\
\hline
0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\
\hline
0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\
\hline
0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\
\hline
0&1&1&0&1&0&0&1&1&0&0&1&0&1&1&0 \\
\hline
\end{array} \quad A_2 = \begin{array}{|cccccccccccccccc|}
\hline
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0 \\
\hline
0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\
\hline
0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\
\hline
0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\
\hline
0&1&1&0&1&0&0&1&1&0&0&1&0&1&1&0 \\
\hline
\end{array}$$

It can be checked that the superimposition $M = \binom{A_1^T}{A_2^T}$ is an $\mathrm{SOA}_4(3, 5, 2)$ over $\mathbb{Z}_2$. $\Box$

To continue with, suppose that $a = (a_1, a_2, \ldots, a_k)$ and $b = (b_1, b_2, \ldots, b_k)$, we define $a \otimes b = ((a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k))$. Let A and B be two arrays of degree k, size m and n, and orders v and h, respectively. Then $A \otimes B$ shall be the array of degree k, size mn and order vh consisting of all the k-tuple $a \otimes b$ as rows,whereby a is any row in A, and b in B.

Let A_1 and A_2 be an $\mathrm{SOA}_{\lambda_1}(t, k, v)$ and an $\mathrm{SOA}_{\lambda_2}(t, k, v)$ over the same symbol set, respectively. A_1 and A_2 are termed compatible if their superimposition is an $\mathrm{SOA}_{\lambda_1+\lambda_2}(t, k, v)$. We say that w simple OAs over the same symbol set are compatible if they are pairwise compatible. Suppose that A is an $\mathrm{OA}_\lambda(t, k, v)$. If the rows of A can be partitioned into μ subarrays such that each subarray contains no identical rows, then we call A a *μ-row-divisible $OA_\lambda(t,k,v)$*. Clearly, a simple OA is 1-row-divisible. The notion of "compatible" and "row-divisible" for super-simple OAs was first proposed in [10]. Here, we only consider the simple orthogonal arrays.

From the proof of Construction 2.4, we see that the resulted μ $\mathrm{OA}_\lambda(t,k,v)'$s contained together are compatible. If we superimpose these arrays, then an $\mathrm{SOA}_{\lambda\mu}(t,k,v)$ can be obtained. To be more precise, we have the following result.

Lemma 2.6 *If an $SSOA_\lambda(t,k,v)$ with $k \geq t+2$ over G exists, then μ compatible $SOA_\lambda(t,k,v)'s$ over G exists, where $2 \leq \mu \leq v^{k-t-1}$.*

Construction 2.7 *Let A_i be a μ_i-row-divisible $OA_{\lambda_i}(t,k,v)$, $i \in I_m$. Suppose that $B_{1,1}, B_{1,2}, \ldots, B_{1,\mu_1}, B_{2,1}, \ldots, B_{2,\mu_2}, \ldots, B_{m,1}, \ldots, B_{m,\mu_m}$ are compatible $SOA_{u_i}(t,k,h)'s$, where $1 \leq i \leq \sum_{i=1}^m \mu_i$ and $u_1 = \cdots = u_{\mu_1} = r_1, u_{\mu_1+1} = \cdots = u_{\mu_1+\mu_2} = r_2, \ldots, u_{\sum_{i=1}^{m-1}+1} = \cdots = u_{\sum_{i=1}^m} = r_m$. Then there exists an $SOA_\rho(t,k,vh)$ with $\rho = \sum_{i=1}^m \lambda_i r_i$.*

Proof By assumption, let A_i be a μ_i-row-divisible $\mathrm{OA}_{\lambda_i}(t,k,v)$ with the partition: $A_{i,1}, A_{i,2}, \ldots, A_{i,\mu_i}$, where $1 \leq i \leq m$. For each $i \in I_m$, consider the array $C_{ij} = A_{i,j} \otimes B_{i,j}$ $(j \in I_{\mu_i})$ and denote their union by C_i. From the usual weighting method in design theory, it is easily verified that each C_i is an $\mathrm{OA}_{\lambda_i r_i}(t,k,vh)$.

Write $C = (C_1^T|C_2^T|\cdots|C_m^T)^T$. Clearly, it is an $\mathrm{OA}_\rho(t,k,vh)$, where $\rho = \sum_{i=1}^m \lambda_i r_i$. It is easy to check that C is simple, and we omit the proof here. $\square$

By taking $m = 1$, $\mu_1 = \mu$ and $r_1 = \lambda_2$ in Construction 2.7, we obtain the following construction.

Construction 2.8 *Let v, k and t be all integers satisfying $k \geq t \geq 2$. If a μ-row-divisible $OA_{\lambda_1}(t,k,v)$ and μ compatible $SOA_{\lambda_2}(t,k,h)$'s all exist, then so does an $SOA_{\lambda_1\lambda_2}(t,k,hv)$. In particular, if an $SOA_{\lambda_1}(t,k,v)$ and an $SOA_{\lambda_2}(t,k,h)$ both exist, then so does an $SOA_{\lambda_1\lambda_2}(t,k,hv)$.*

By taking $\lambda_1 = \lambda$ and $\lambda_2 = 1$ in Construction 2.8, we obtain the following construction.

Construction 2.9 *Suppose that both an $SOA_\lambda(t,k,v)$ and an $OA(t,k,h)$ exist. Then there exists an $SOA_\lambda(t,k,vh)$.*

We will mainly use the following working corollary of Construction 2.7.

Corollary 2.10 *Let v_2 be a positive integer such that v_2 compatible $OA(3,5,v_2)'s$ exist. Let $v = v_1v_2$ and λ be an arbitrary integer satisfying $2 \leq \lambda \leq v^2$. Suppose that there are two non-negative integers m_1 and m_2 and two positive integers λ_1 and λ_2 such that the following conditions are all satisfied:*

(1) $1 \leq m_1\mu_1 + m_2\mu_2 \leq v_2$;

(2) $m_1\lambda_1 + m_2\lambda_2 = \lambda$;

(3) a μ_1-row-divisible $OA_{\lambda_1}(3,5,v_1)$ and a μ_2-row-divisible $OA_{\lambda_2}(3,5,v_1)$ both exist.

Then there exists an $SOA_\lambda(3,5,v)$.

Proof First, we form an $m_i\mu_i$-row-divisible $\mathrm{OA}_{m_i\lambda_i}(3,5,v_1)$ by taking m_i copies of a μ_i-row-divisible $\mathrm{OA}_{\lambda_i}(3,5,v_1)$, where $i=1,2$. The superimposition of the resulted new row-divisible OAs is then to produce an $(m_1\mu_1+m_2\mu_2)$-row-divisible $\mathrm{OA}_{m_1\lambda_1+m_2\lambda_2}(t,k,v_1)$. Set $\mu=m_1\mu_1+m_2\mu_2$. Then $\mu\le v_2$. The existence of v_2 compatible $\mathrm{OA}(3,5,v_2)'$s implies the existence of μ compatible $\mathrm{OA}(3,5,v_2)'$s. The conclusion then follows from Construction 2.8. $\Box$

3. Main results

We are now in the position to establish our results on the existence of simple orthogonal arrays of degree 5, strength 3, index $\lambda\ge 2$. We will construct some μ-row-divisible $\mathrm{OA}_\lambda(3,5,v)'$s of small orders, which is crucial in applying Corollary 2.10. First of all, we record some of the known results.

Lemma 3.1 ([10,14]) *Let v and λ be two positive integers with $\lambda\le v$. Then*

(1) An $\mathrm{SSOA}_3(3,5,v)$ exists if and only if $v\ge 3$ except possibly when $v=6$;

(2) An $\mathrm{SSOA}_2(3,5,v)$ exists if and only if $v\ge 2$ except definitely $v=3$.

Let C be a $v^k\times k$ array consisting of all the k-tuples over G of order v. It is obvious that C is an $\mathrm{OA}(k,k,v)$ and hence, it is an $\mathrm{OA}_{v^{k-t}}(t,k,v)$ for $t\le k$. Furthermore, suppose that C can be partitioned into two subarrays A and B. If A is an $\mathrm{OA}_\lambda(t,k,v)$, then B is an $\mathrm{OA}_{v^{k-t}-\lambda}(t,k,v)$. The conclusion also holds for the simple orthogonal arrays, which state it as follows.

Lemma 3.2 *If an $\mathrm{SOA}_\lambda(t,k,v)$ with $\lambda<v^{k-t}$ exists, then an $\mathrm{SOA}_{v^{k-t}-\lambda}(t,k,v)$ also exists.*

The next results are about the existence of an $\mathrm{SOA}_\lambda(3,5,v)$ for $v\in\{2,3,6\}$.

Lemma 3.3 *An $\mathrm{SOA}_\lambda(3,5,2)$ exists if and only if $\lambda\le 4$ except definitely $\lambda=1,3$.*

Proof The non-existence of an $\mathrm{OA}(3,5,2)$ was given in [5]. Thus, by Lemma 3.2, there is no $\mathrm{SOA}_3(3,5,2)$. An $\mathrm{SSOA}_2(3,5,2)$ was given in Lemma 3.1. Hence, an $\mathrm{SOA}_2(3,5,2)$ exists since the super-simple property of an OA yields the simple property. An $\mathrm{SOA}_4(3,5,2)$ was given in Example 2.5. $\Box$

Lemma 3.4 *A 2-row-divisible $\mathrm{OA}_3(3,5,2)$ over $\mathbb{Z}_2$ exists.*

Proof A 2-row-divisible $\mathrm{OA}_3(3,5,2)$ over $\mathbb{Z}_2$ with the partition can be formed by transposing the following two arrays:

$$\begin{pmatrix} 1&0&1&1&1&0&0&0&1&0&1&1\\ 0&0&1&1&0&0&1&1&0&0&0&1\\ 0&0&1&0&1&1&0&1&0&0&1&0\\ 1&0&1&0&0&1&1&0&0&1&1&1\\ 0&0&0&0&0&0&0&1&1&1&1&1 \end{pmatrix},\quad \begin{pmatrix} 0&1&0&1&0&1&0&0&0&1&0&1\\ 1&1&1&0&0&1&0&0&1&1&1&0\\ 1&1&0&0&1&1&0&1&0&0&1&1\\ 1&0&0&1&0&1&0&1&1&0&0&0\\ 0&0&0&0&0&1&1&1&1&1&1&1 \end{pmatrix}.\ \Box$$

Lemma 3.5 *There exists an $\mathrm{SOA}_\lambda(3,5,3)$ for $\lambda=2$ or 4 over $\mathbb{Z}_3$.*

Proof In view of Construction 2.2, we only need to construct a simple $(3,4,2)$-DM over $\mathbb{Z}_3$ and $(3,4,4)$-DM over $\mathbb{Z}_3$. A simple $(3,4,2)$-DM was given in Example 2.1. A simple $(3,4,4)$-DM over $\mathbb{Z}_3$ is constructed as follows:

$$A = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&1&2&2&0&0&1&1&2&2 \\ 0&1&2&0&1&2&0&1&2&0&1&2 \\ 0&2&2&1&0&1&1&0&0&2&1&2 \end{pmatrix} \quad \square$$

Lemma 3.6 *An $SOA_\lambda(3,5,3)$ exists if and only if $\lambda \le 9$ except definitely $\lambda = 1, 8$.*

Proof An $\mathrm{SSOA}_3(3,5,3)$ was given in Lemma 3.1, it is clearly an $\mathrm{SOA}_3(3,5,3)$. By Lemma 3.5, an $\mathrm{SOA}_\lambda(3,5,3)$ exists for $\lambda = 2, 4$. Applying Lemma 3.2 yields an $\mathrm{SOA}_{9-\lambda}(3,5,3)$ for $\lambda = 2,3,4$. The non-existence of an $\mathrm{OA}(3,5,3)$ was given in [2]. This result implies the non-existence of an $\mathrm{SOA}_8(3,5,3)$. An $\mathrm{SOA}_9(3,5,3)$ is as well as an $\mathrm{OA}(5,5,3)$, which is formed by listing all the 5-tuple over $\mathbb{Z}_3$. $\square$

Lemma 3.7 *An $SOA_\lambda(3,5,6)$ exists if and only if $\lambda \le 36$ except definitely $\lambda = 1, 35$ and possibly when $\lambda \in \{3,7,11,13,15,17,19,21,23,25,29,33\}$.*

Proof The well-known non-existence of a pair of MOLSs of size 6 implies that neither an OA(3,5,6) can exist. Thus, an $\mathrm{SOA}_{35}(3,5,6)$ does not exist. Clearly, an $\mathrm{OA}(5,5,6)$ is exactly an $\mathrm{SOA}_{36}(3,5,6)$. Next, we consider the case where $\lambda \notin \{3,7,11,13,15,17,19,21,23,25,29,33\}$. As mentioned early, it remains to consider the indices $2 \le \lambda \le 18$ and $\lambda \notin \{3,7,11,13,15,17\}$. For $\lambda \in \{4,6,8,10,12,14,16,18\}$, an $\mathrm{SOA}_\lambda(3,5,6)$ is obtainable by applying Construction 2.8 with $(\lambda_1,\lambda_2) \in \{(2,2),(2,3),(2,4),(2,5),(2,6),(2,7),(4,4),(2,9)\}$. The required ingredient SOAs all exist from Lemmas 3.3 and 3.6. An $\mathrm{SSOA}_2(3,5,6)$ was given in Lemma 3.1, it is also an $\mathrm{SOA}_2(3,5,6)$. An $\mathrm{OA}_5(3,5,6)$ was given in [10], it is easy to check that it is simple. For an $\mathrm{SOA}_9(3,5,6)$, the conclusion follows from applying Construction 2.8 with a 2-row-divisible $\mathrm{OA}_3(3,5,2)$ and 2 compatible $\mathrm{SOA}_3(3,5,3)'$s. A 2-row-divisible $\mathrm{OA}_3(3,5,2)$ was given in Lemma 3.4 while the existence of 2 compatible $\mathrm{SOA}_3(3,5,3)'$s are constructed from Lemmas 2.6 and 3.1. $\square$

Now, we will turn to determine the existence spectrum of an $\mathrm{SOA}_\lambda(3,5,v)$ with $\lambda \ge 2$. First, combining Lemma 1.3 and Construction 2.3, we can prove the following existence result.

Lemma 3.8 *Let $v \ge 4$ be an integer. If $v \not\equiv 2 \pmod 4$, then an $SOA_\lambda(3,5,v)$ exists if and only if $\lambda \le v^2$.*

Next, it remains to solve the existence of an $\mathrm{SOA}_\lambda(3,5,v)$ with $\lambda \ge 2$ and $v \equiv 2 \pmod 4$.

Lemma 3.9 *Let $v \ge 10$ be an integer. If $v \equiv 2 \pmod 4$, then an $SOA_\lambda(3,5,v)$ with $\lambda \ge 2$ exists if and only if $\lambda \le v^2$.*

Proof The necessity follows from Lemma 1.4. For the sufficiency, let $v = 4t+2 = 2(2t+1)$, where $2t+1 \ge 5$. Since $2t+1$ is odd and $2t+1 \ge 4$, an $\mathrm{OA}(3,5,2t+1)$ exists by Lemma 1.3. Hence, applying Construction 2.3 with an $\mathrm{OA}(3,5,2t+1)$ gives rise to a completely reducible simple $\mathrm{OA}_{(2t+1)^2}(3,5,2t+1)$. This means that $(2t+1)^2$ compatible $\mathrm{OA}(3,5,2t+1)'$s exist. Furthermore, superimposing the λ compatible $\mathrm{OA}(3,5,2t+1)'$s forms an $\mathrm{SOA}_\lambda(3,5,2t+1)$, where $1 \le \lambda \le (2t+1)^2$.

By Lemma 3.2, it only needs to consider the case where $2 \le \lambda \le v^2/2 = 2(2t+1)^2$. If λ is even, then an $\mathrm{SOA}_\lambda(3,5,v)$ exists by employing Construction 2.8 with the ingredients $\mathrm{SOA}_2(3,5,2)$ and $\mathrm{SOA}_{\lambda/2}(3,5,2t+1)$. Otherwise, assume that λ is odd. Apply Corollary 2.10 with $v_1 = 2$, $v_2 = 2t+1$ and $(\lambda_1, \lambda_2) = (2,3)$. From Lemmas 3.3 and 3.4, we know that an $\mathrm{SOA}_2(3,5,2)$ and 2-row-divisible $\mathrm{OA}_3(3,5,2)$ both exist. It is left to show that the system of equations

$$\begin{cases} 2m_1 + 3m_2 = \lambda, \\ m_1 + 2m_2 \le (2t+1)^2, \end{cases}$$

is solvable in non-negative integers m_1 and m_2 for any given λ with $3 \le \lambda \le 2(2t+1)^2 - 1$. It now turns out that

$$(m_1, m_2) = (\frac{\lambda-3}{2}, 1)$$

is one solution of the above system of equations. $\square$

Summarizing the results in Lemmas 3.3 and 3.6–3.9, we will establish our main result of this paper.

Theorem 3.10 *The necessary conditions for the existence of an* $\mathrm{SOA}_\lambda(3,5,v)$ *with* $\lambda \ge 2$ *are also sufficient with definite exceptions:*

(1) $v = 2$ *and* $\lambda = 3$*; (2)* $v = 3$ *and* $\lambda = 8$*; (3)* $v = 6$ *and* $\lambda = 35$*;*

and possible exceptions where $v = 6$ *and* $\lambda \in \{3,7,11,13,15,17,19,21,23,25,29,33\}$.

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