Journal of Mathematical Research with Applications
May, 2015, Vol. 35, No. 3, pp. 279–284
DOI:10.3770/j.issn:2095-2651.2015.03.005
Http://jmre.dlut.edu.cn

Fractional Domination of the Cartesian Products in Graphs

Baogen XU
Department of Mathematics, East China Jiaotong University, Jiangxi 330013*, P. R. China*

Abstract Let $G=(V,E)$ be a simple graph. For any real function $g:V\longrightarrow R$ and a subset $S\subseteq V$, we write $g(S)=\sum_{v\in S}g(v)$. A function $f:V\longrightarrow[0,1]$ is said to be a fractional dominating function (FDF) of G if $f(N[v])\geq 1$ holds for every vertex $v\in V(G)$. The fractional domination number $\gamma_f(G)$ of G is defined as $\gamma_f(G)=\min\{f(V)|f \text{ is an } FDF \text{ of } G\}$. The fractional total dominating function f is defined just as the fractional dominating function, the difference being that $f(N(v))\geq 1$ instead of $f(N[v])\geq 1$. The fractional total domination number $\gamma_f^0(G)$ of G is analogous. In this note we give the exact values of $\gamma_f(C_m\times P_n)$ and $\gamma_f^0(C_m\times P_n)$ for all integers $m\geq 3$ and $n\geq 2$.

Keywords Cartesian products; fractional domination number; fractional total domination number

MR(2010) Subject Classification 05C69

1. Introduction

We use Bondy and Murty [1] for terminology and notation not defined here and consider finite simple graph only.

Let $G=(V,E)$ be a graph. The open neighborhood of a vertex v in G is $N(v)=\{u\in V|uv\in E(G)\}$, while $N[v]=N(v)\cup\{v\}$ is the closed neighborhood of v. C_n and P_n denote the cycle and the path of order n, respectively. If $u,v\in V(G)$, then $u\sim v$ denotes u is adjacent to v in G.

For any two disjoint graphs G and H, the Cartesian product $G\times H$ is defined as follows:

$$V(G\times H)=V(G)\times V(H),$$

$$E(G\times H)=\{(u_1,v_1)(u_2,v_2)|(u_1=u_2 \text{ and } v_1\sim v_2) \text{ or } (v_1=v_2 \text{ and } u_1\sim u_2)\}.$$

Let $G=(V,E)$ be a graph. For any real function $g:V\longrightarrow R$ and a subset $S\subseteq V$, we write $g(S)=\sum_{v\in S}g(v)$.

Hare [3] and Stewart [4] introduced the following concept of the fractional domination and the fractional total domination in graphs.

Received March 1, 2014; Accepted January 16, 2015
Supported by the National Natural Science Foundation of China (Grant Nos. 11361024; 11061014) and the Jiangxi Provincial Science and Technology Project (Grant No. KJLD12067).
E-mail address: Baogenxu@163.com

Let $G=(V,E)$ be a graph. A function $f:V\longrightarrow[0,1]$ is said to be a fractional dominating function (FDF) of G if $f(N[v])\geq 1$ holds for every vertex $v\in V(G)$. The fractional domination number $\gamma_f(G)$ of G is defined as $\gamma_f(G)=\min\{f(V)|f$ is an FDF of $G\}$.

A fractional total dominating function ($FTDF$) f of G is defined similarly, the difference being that $f(N(v))\geq 1$ instead of $f(N[v])\geq 1$. The fractional total domination number $\gamma_f^0(G)$ of G is defined as $\gamma_f^0(G)=\min\{f(V)|f$ is an $FTDF$ of $G\}$.

Fractional packing numbers are defined analogously; a real function $f:V(G)\longrightarrow[0,1]$ is a fractional packing function of G if $f(N[v])\leq 1$ holds for every vertex $v\in V(G)$. A fractional packing function f is maximal if for every $u\in V(G)$ with $f(u)<1$, there exists a vertex $v\in N[u]$ such that $f(N[v])=1$. The upper fractional packing number $P_f(G)$ of G is defined as $P_f(G)=\max\{f(V)|f$ is a maximal packing function of $G\}$.

Lemma 1.1 ([2]) *For any graph G, $P_f(G)=\gamma_f(G)$.*

Lemma 1.2 ([2]) *For any r-regular graph G ($r\geq 1$), then*

(1) $\gamma_f(G)=\frac{n}{r+1}$; (2) $\gamma_f^0(G)=\frac{n}{r}$.

For the Cartesian product $P_m\times P_n$, Hare [3] and Stewart [4] gave an exact formula for $\gamma_f(P_2\times P_n)$ and some bounds of $\gamma_f(P_m\times P_n)$ for $3\leq m\leq n$.

Lemma 1.3 ([2]) *For all integers $n\geq 1$, then*

(1) when $n\equiv 1 \pmod 2, \gamma_f(P_2\times P_n)=\frac{n+1}{2}$;

(2) when $n\equiv 0 \pmod 2, \gamma_f(P_2\times P_n)=\frac{n^2+2n}{2(n+1)}$.

However, there is no known formula of $\gamma_f(P_m\times P_n)$ for $3\leq m\leq n$. It is very difficult to give the exact value of $\gamma_f(P_m\times P_n)$. Fisher [5] has tried without success to find such a formula for $\gamma_f(P_3\times P_n)$. Up to now, few exact value of $\gamma_f(P_m\times P_n)$ is known when $3\leq m\leq n$.

We are interested in the Cartesian products $C_m\times P_n$. In this note we give exact formulas of $\gamma_f(C_m\times P_n)$ and $\gamma_f^0(C_m\times P_n)$ for all integers $m\geq 3$ and $n\geq 2$.

2. Fractional total domination number for $C_m\times P_n$

Theorem 2.1 *For all integers $m\geq 3$ and $n\geq 2$, we have $\gamma_f^0(C_m\times P_n)=\frac{m}{4(n+1)}(n^2+n+2\lceil\frac{n}{2}\rceil)$.*

Proof Let $G=C_m\times P_n, V(G)=\{(i,j)|1\leq i\leq m, 1\leq j\leq n\}$, and

$$E(G)=\{(i,j)(i,j+1)|1\leq i\leq m, 1\leq j\leq n-1\}\cup\{(i,j)(i+1,j)|1\leq i\leq m, 1\leq j\leq n\},$$

where $(m+1,j)=(1,j)$ for every integer j ($1\leq j\leq n$).

Define an $FTDF$ f of G as follows:

Let $f((i,j))=x_j$ ($i=1,2,\ldots,m$) for every integer j ($1\leq j\leq n$).

Case 1 $n=2k+1$; for some $k\in N^+$.

Let $x_{2j}=0$ ($1\leq j\leq k$) and $x_{2j-1}=\frac{1}{2}$ for every integer j ($1\leq j\leq k+1$);

It is easy to check that $f(N(i,j))=1$ holds for all vertices $(i,j)\in V(G)$, and hence f is an

$FTDF$ of G, which means

$$\gamma_f^0(G) \leq f(V(G)) = \frac{m(n+1)}{4}. \tag{1}$$

On the other hand, let g be an $FTDF$ of G such that $\gamma_f^0(G) = g(V(G))$. By the definition, for every vertex $(i, 2j-1) \in V(G)$ $(1 \leq i \leq m, 1 \leq j \leq k+1)$, we have $g(N(i, 2j-1)) \geq 1$, and hence $2g(V(G)) = \sum_{i=1}^{m}\sum_{j=1}^{k+1} g(N(i, 2j-1)) \geq m(k+1)$, i.e.,

$$\gamma_f^0(G) = g(V(G)) \geq \frac{m(k+1)}{2} = \frac{m(n+1)}{4}.$$

Combining with (1), we have $\gamma_f^0(G) = \frac{m(n+1)}{4}$, and the theorem holds for all odd $n \geq 3$.

Case 2 $n = 2k$; for some $k \in N^+$.

Let $x_{2j} = \dfrac{j}{n+1}$ and $x_{2j-1} = \frac{n-2j+2}{2(n+1)}$ for every integer j $(1 \leq j \leq k)$.

It is easy to see that $f(N(i,j)) = 1$ holds for all vertices $(i,j) \in V(G)$, and hence f is an $FTDF$ of G, which means

$$\gamma_f^0(G) \leq f(V(G)) = m\sum_{j=1}^{k}(\frac{j}{n+1} + \frac{n-2j+2}{2(n+1)}) = \frac{mk(n+2)}{2(n+1)} = \frac{m(n^2+2n)}{4(n+1)}.$$

Next we prove that $\gamma_f^0(G) \geq \dfrac{m(n^2+2n)}{4(n+1)}$.

When $n = 2$, G is a 3-regular graph. By Lemma 1.2, Theorem 2.1 holds. Next suppose that $n \geq 4$ and $n = 2k$ is even.

Assume, to the contrary, that

$$\gamma_f^0(G) < \frac{m(n^2+2n)}{4(n+1)}. \tag{2}$$

Let g be such an $FTDF$ of G that $\gamma_f^0(G) = g(V(G))$, and for each $j = 1, 2, \ldots, n$, let $C(j) = \{(i,j) | 1 \leq i \leq m\} \subseteq V(G)$. Clearly, $V(G) = \bigcup_{i=1}^{\frac{n}{2}}(C(2i-1) \cup C(2i))$, thus, there exists an odd integer r $(1 \leq r \leq n)$, so that

$$g(C(r)) + g(C(r+1)) \leq \frac{2}{n}g(V(G)) = \frac{2}{n}\gamma_f^0(G) < \frac{m(n+2)}{2(n+1)}.$$

Let $g(N(j)) = \sum_{i=1}^{m} g(N(i,j))$ for every integer $j \in \{1, 2, \ldots, n\}$. Since $g(N(i,j)) \geq 1$ holds for all vertices $(i,j) \in V(G)$, we have $g(N(j)) \geq m$ holds for all integers $j \in \{1, 2, \ldots, n\}$. Note that r is odd and $n = 2k$ is even. We have

$$\begin{aligned}
&2g(V(G)) + g(C(r)) + g(C(r+1)) \\
&\quad = g(N(1)) + g(N(3)) + \cdots + g(N(r)) + g(N(r+1)) + g(N(r+3)) + \cdots + g(N(n)) \\
&\quad \geq (\frac{n}{2} + 1)m.
\end{aligned}$$

And hence, we have

$$\begin{aligned}
2g(V(G)) &\geq (\frac{n}{2} + 1)m - (g(C(r)) + g(C(r+1)) \\
&\geq (\frac{n}{2} + 1)m - \frac{m(n+2)}{2(n+1)} = \frac{m(n^2+2n)}{2(n+1)},
\end{aligned}$$

$$\gamma_f^0(G) = g(V(G)) \geq \frac{m(n^2+2n)}{4(n+1)}.$$

This contradicts (3). Combining with (2), we have proved that $\gamma_f^0(G) = \frac{m(n^2+2n)}{4(n+1)}$ holds for all even $n \geq 2$. The proof of Theorem 2.1 is completed. □

3. Fractional domination number for $C_m \times P_n$

The following two lemmas are useful to obtain our main results.

Lemma 3.1 *Let A and B be both matrices of order $n \geq 2$, and*

$$A = \begin{pmatrix} 3 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 3 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 3 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 3 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 3 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 1 & 3 \end{pmatrix}.$$

Then (1) $A_n = \det A = \frac{a^{n+1}-b^{n+1}}{a-b}$, where $a = \frac{3+\sqrt{5}}{2}$ and $b = \frac{3-\sqrt{5}}{2}$;

(2) $B_n = \det B = \frac{1}{5}(A_n + A_{n-1} + (-1)^{n-1})$, where let $A_0 = 1$.

Proof We use the induction on $n \geq 1$.

When $n = 1$, clearly, $A_1 = 3 = a + b$, and $B_1 = 1$, and the result follows.

We suppose that Lemma 3.1 is true for all matrices with determinants of order $k \leq n-1$. Now we consider the two $n \times n$ matrices A and B. Note that $a + b = 3$ and $ab = 1$. By the induction hypothesis, we have

$$A_n = 3A_{n-1} - A_{n-2} = (a+b)\frac{a^n - b^n}{a-b} - ab\frac{a^{n-1}-b^{n-1}}{a-b} = \frac{a^{n+1}-b^{n+1}}{a-b}$$

$$\begin{aligned} B_n &= A_{n-1} - B_{n-1} = A_{n-1} - \frac{1}{5}(A_{n-1} + A_{n-2} + (-1)^{n-2}) \\ &= \frac{1}{5}(4A_{n-1} - A_{n-2} + (-1)^{n-1}) = \frac{1}{5}(A_n + A_{n-1} + (-1)^{n-1}). \end{aligned}$$

So, Lemma 3.1 is true for all determinants of order n, this proof is completed. □

Lemma 3.2 *Let $X^T = (x_1, x_2, \ldots, x_n)$, and $C^T = (1, 1, 1, \ldots, 1)$ be an n-dimensional vector ($n \geq 2$). Then the linear equation*

$$AX = C \qquad (*)$$

has the unique solution $(x_1, x_2, \ldots, x_n)$ which satisfies the following two conditions:

(1) $x_1 = x_n = \frac{B_n}{A_n}$, and $x_i = x_{n+1-i}$ ($1 \leq i \leq \lceil \frac{n}{2} \rceil$);

(2) $0 \leq x_i \leq 1$ ($1 \leq i \leq n$),

where A, A_n and B_n are defined as in Lemma 3.1.

Proof (1) Since $A_n \neq 0$, the linear equation $(*)$ has the unique solution $(x_1, x_2, \ldots, x_n)$, from the uniqueness of the solution and the symmetry of A, and by Cramer' Rule, we have $x_1 = x_n = \frac{B_n}{A_n}$, and $x_i = x_{n+1-i}$ ($1 \leq i \leq \lceil \frac{n}{2} \rceil$).

(2) When $2 \le n \le 6$, it is easy to check that $0 \le x_i \le 1$ $(1 \le i \le n)$. The solution $(x_1, x_2, \ldots, x_n)$ is listed in the proof of Theorem 3.3 (1) for every $n \in \{2,3,4,5,6\}$.

Next we suppose $n \ge 7$.

Now we prove that $x_i \ge 0$ holds for every integer i $(1 \le i \le n)$.

Assume, to the contrary, that there exists an integer i such that $x_i < 0$.

Let $r = x_j = \min\{x_i | 1 \le i \le n\}$. Note that $a = \frac{3+\sqrt{5}}{2}$, $b = \frac{3-\sqrt{5}}{2}$, $ab = 1$, we have $A_n = aA_{n-1} + b^n$, $A_{n-1} = bA_n - b^{n+1}$. By Lemma 3.1, we have

$$x_1 = x_n = \frac{B_n}{A_n} = \frac{1}{5A_n}(A_n + A_{n-1} + (-1)^{n-1}) = \frac{1+b}{5} + \frac{(-1)^{n-1} - b^{n+1}}{5A_n}.$$

Note that $\frac{1}{3} \le b = \frac{3-\sqrt{5}}{2} \le \frac{2}{5}$ and $A_n \ge 6$, we have $0 \le x_1 \le \frac{1}{3}$. It is easy to see from the linear equation $AX = C$ that $x_2 = x_{n-1} = 1 - 3x_1 \ge 0$, and hence $3 \le j \le n-2$. Since $x_{j-1} + 3x_j + x_{j+1} = 1$, we have $x_{j-1} \ge \frac{1-3r}{2}$ or $x_{j+1} \ge \frac{1-3r}{2}$.

If $x_{j-1} \ge \frac{1-3r}{2}$, since $x_{j-2} + 3x_{j-1} + x_j = 1$, and note that $r \le 0$, we have $x_{j-2} = 1 - 3x_{j-1} - r \le 1 - \frac{3}{2}(1-3r) - r = \frac{7}{2}r - \frac{1}{2} < r$, this contradicts the choice of r.

If $x_{j+1} \ge \frac{1-3r}{2}$, similarly, since $x_j + 3x_{j+1} + x_{j+2} = 1$, we have $x_{j+2} = 1 - 3x_{j+1} - r \le 1 - \frac{3}{2}(1-3r) - r = \frac{7}{2}r - \frac{1}{2} < r$, this contradicts the choice of r as well.

Thus, $x_i \ge 0$ holds for every integer i $(1 \le i \le n)$, implying that $x_i \le 1$ holds for every integer i $(1 \le i \le n)$. We have completed the proof of Lemma 3.2. □

Theorem 3.3 *For all integers $m \ge 3$ and $n \ge 2$, then*

(1) $\gamma_f(C_m \times P_2) = \frac{1}{2}m$, $\gamma_f(C_m \times P_3) = \frac{5}{7}m$, $\gamma_f(C_m \times P_4) = \frac{10}{11}m$, $\gamma_f(C_m \times P_5) = \frac{10}{9}m$, $\gamma_f(C_m \times P_6) = \frac{38}{29}m$;

(2) When $n \ge 7$, $\gamma_f(C_m \times P_n) = \frac{(5n+2)A_n + 2A_{n-1} + 2(-1)^{n-1}}{25A_n}m$,
where $A_n = \frac{(3+\sqrt{5})^{n+1} - (3-\sqrt{5})^{n+1}}{2^{n+1}\cdot\sqrt{5}}$ *for each integer* $n \ge 1$.

Proof Let $G = C_m \times P_n$, and $V(G)$ and $E(G)$ be the same as in the proof of Theorem 2.1.

Next we define a maximal packing function f of G such that $f(N[v]) = 1$ holds for every vertex $v \in V(G)$.

For every vertex $(i,j) \in V(G)$, define $f((i,j)) = x_i$ $(i = 1,2,\ldots,n; j = 1,2,\ldots,m)$. $S(n) = \sum_{i=1}^{n} x_i$, clearly, $f(V(G)) = mS(n)$.

(1) When $n = 2$; let $(x_1, x_2) = (\frac{1}{4}, \frac{1}{4}), S(2) = \frac{1}{2}$;

when $n = 3$; let $(x_1, x_2, x_3) = (\frac{2}{7}, \frac{1}{7}, \frac{2}{7}), S(3) = \frac{5}{7}$;

when $n = 4$; let $(x_1, x_2, x_3, x_4) = (\frac{3}{11}, \frac{2}{11}, \frac{2}{11}, \frac{3}{11}), S(4) = \frac{10}{11}$;

when $n = 5$; let $(x_1, x_2, x_3, x_4, x_5) = (\frac{5}{18}, \frac{3}{18}, \frac{4}{18}, \frac{3}{18}, \frac{5}{18}), S(5) = \frac{10}{9}$;

when $n = 6$; let $(x_1, x_2, x_3, x_4, x_5, x_6) = (\frac{8}{29}, \frac{5}{29}, \frac{6}{29}, \frac{6}{29}, \frac{5}{29}, \frac{8}{29}), S(6) = \frac{38}{29}$.

It is easy to see that $f(N[v]) = 1$ holds for every vertex $v \in V(G)$, and hence f is a maximum packing function. By Lemma 1.1, these five equalities in Theorem 3.3 hold.

(2) When $n \ge 7$, let $(x_1, x_2, \ldots, x_n)$ be the unique solution of the linear equation $(*)$. It is easy to see from Lemma 3.2 that f is a maximum packing function of G. And

$$4(x_1 + x_n) + 5(x_2 + x_3 + \cdots + x_{n-1}) = C^T AX = C^T C = n.$$

By Lemmas 3.1 and 3.2, we have

$$S(n)=\sum_{i=1}^{n}x_i=\frac{n+x_1+x_n}{5}=\frac{n}{5}+\frac{2B_n}{5A_n}=\frac{(5n+2)A_n+2A_{n-1}+2(-1)^{n-1}}{25A_n}.$$

By Lemma 1.1,

$$\gamma_f(G)=P_f(G)=f(V(G))=mS(n)=\frac{(5n+2)A_n+2A_{n-1}+2(-1)^{n-1}}{25A_n}m,$$

where $A_n=\frac{a^{n+1}-b^{n+1}}{a-b}=\frac{(3+\sqrt{5})^{n+1}-(3-\sqrt{5})^{n+1}}{2^{n+1}\cdot\sqrt{5}}$.

We have completed the proof of Theorem 3.3. □

Acknowledgements I am very grateful to the referees for their careful reading with corrections.

References

[1] J. A. BONDY, V. S. R. MURTY. *Graph Theory with Applications.* Elsevier, Amsterdam, 1976.
[2] T. W. HAYNES, S. T. HEDETNIEMI, P. J. SLATER. *Domination in graphs.* Marcel Dekker INC, New York, 1998.
[3] E. O. HARE. *K-weight domination and fractional domination of* $P_m\times P_n$. Congr. Numer., 1990, **78**: 71–80.
[4] L. S. STEWART, E. O. HARE. *Fractional domination of* $P_m\times P_n$. Congr. Numer., 1992, **91**: 35–42.
[5] D. C. FISHER. *Domination, fractional domination, 2-packing and graph products.* SIAM Discrete Math., 1994, **7**(3): 493–498.