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A Linear-Time Algorithm for 2-Step Domination in Block Graphs

Yancai ZHAO[1,3,*], **Lianying MIAO**[2], **Zuhua LIAO**[1]
1. *School of Science, Jiangnan University, Jiangsu* 214122*, P. R. China;*
2. *Department of Mathematics, China University of Mining and Technology, Jiangsu* 221116*, P. R. China;*
3. *Wuxi City College of Vocational Technology, Jiangsu* 214153*, P. R. China*

Abstract The 2-step domination problem is to find a minimum vertex set D of a graph such that every vertex of the graph is either in D or at distance two from some vertex of D. In the present paper, by using a labeling method, we provide an $O(m)$ time algorithm to solve the 2-step domination problem on block graphs, a superclass of trees.

Keywords 2-step domination; block graph; algorithm; labeling method

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1. Introduction

In this paper, all graphs considered are finite, undirected, loopless and without multiple edges. We refer the reader to the book [13] for graph theory notation and terminology not defined here. Specifically, let $G=(V,E)$ be a simple graph with vertex set V and edge set E. For any $v\in V$, the neighborhood $N(v)$ of v is the set of vertices adjacent to v, the closed neighborhood of v is $N[v]=N(v)\cup\{v\}$. The distance between two vertices x and y is the length of a shortest xy-path in G, denoted by $d_G(x,y)$. The 2-step neighborhood of v is $N_2(v)=\{u|d_G(v,u)=2\}$. The closed 2-step neighborhood of v is $N_2[v]=N_2(v)\cup\{v\}$.

Given a graph $G=(V,E)$, a subset $D\subseteq V$ is called a dominating set of G if every vertex in G is either in D or adjacent to a vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality among all dominating sets of G.

Given a graph $G=(V,E)$, a subset $D\subseteq V$ is called a 2-step dominating set of G if every vertex in G is either in D or at distance two from a vertex in D. The 2-step domination number $\gamma_2(G)$ of G is the minimum cardinality among all 2-step dominating sets of G. If $u\in N_2[v]$, then we usually say that v 2-step dominates u.

Domination theory has become an important part of graph theory, and various domination-related parameters have been widely studied. Among these parameters, the 2-step domina-

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* Corresponding author
E-mail address: zhaoyc69@126.com (Yancai ZHAO)

tion was introduced by Slater in [11], and has been extensively studied in papers such as [7,10,12,14,19,20]. However, these papers mainly concentrate on the characterizations of 2-step domination graphs. A graph G is called a 2-step domination graph if it contains a set $S \subseteq V(G)$ such that the sets $N_2(v), v \in S$ form a partition of $V(G)$. In the present paper, we focus on the computation of the 2-step domination numbers of graphs.

Since the domination problem is NP-complete for general graphs, chordal graphs, and bipartite graphs [1,3,8,18], it is easy to see that 2-step domination problem is also NP-complete for the above classes of graphs. The purpose of this paper is to initiate the study of efficient algorithms for solving 2-step domination problem on some graph classes. In this paper, by using labeling method, we provide a linear-time algorithm to solve 2-step domination problem in block graphs, a superclass of trees.

2. Definitions

A forest is a graph without cycles. A tree is a connected forest. A leaf in a graph is a vertex with degree one. In a graph G, a vertex x is a cut vertex if deleting x (together with all edges incident to it) increases the number of connected components. A block of G is a maximal connected subgraph without a cut vertex. If G has no cut vertex, G itself is a block. The intersection of two blocks contains at most one vertex and a vertex is a cut vertex if and only if it is the intersection of two or more blocks. In general, the blocks of a connected graph fit together in a treelike structure. A block B of G is called an end block if B contains at most one cut vertex of G. A block graph is a graph whose blocks are complete graphs. This name arises because a graph G is the intersection graph of the blocks of some graph if and only if every block of G is complete [12].

Given a block graph G, since its blocks fit together in a treelike structure, then we may give some similar terminology and definitions to that in a tree. Define the distance between two blocks B_1 and B_2 as $d_G(B_1, B_2) = \max\{d_G(v_1, v_2) | v_1 \in B_1, v_2 \in B_2\} - 1$. Define the distance between a vertex v and a block B as $d_G(v, B) = \max\{d_G(v, u) | u \in B\} - 1$. Now, we assume that the block graph G is rooted at any block, say B_0, of it. Then the height of G is the maximum distance from an end block to the root block B_0. If $G = B_0$, then the height of G is zero. Let h be the height of G and let the i-th level $A_i, 0 \le i \le h$, be the set of blocks of G which are at distance i from B_0.

3. Algorithm for 2-step domination in block graphs

Now, we work on an algorithm for finding a minimum 2-step dominating set of a block graph. For technical reasons, we actually consider a slightly more general problem. Suppose the vertex set of a graph G is partitioned into three sets, N, F, and R, where N consists of needed vertices, F consists of free vertices, and R consists of required vertices. An optional 2-step dominating set of G is any set $D \subseteq V$ which contains all required vertices, that is, $R \subseteq D$, and 2-step dominates all vertices in N. The optional 2-step domination number $\gamma^2_{\rm opt}(G)$ is the minimum cardinality among all optional 2-step dominating sets of G. An optional 2-step dominating set of G with cardinality $\gamma^2_{\mathrm{opt}}(G)$ is also called a γ^2_{opt}-set.

Note that the 2-step domination problem is just the optional 2-step domination problem with $F = R = \emptyset$ and $N = V$. This generalization can be viewed as a labeling algorithm in which a vertex has a label "needed" or "free" or "required" if it is in N or F or R, respectively. The labeling method was first used by Cockayne, Goodman, and Hedetniemi for solving the domination problem in trees [5], and then widely used by various authors in the literature for solving the domination-related problems [4,6,8,9,15-17,21,22]. It is a natural but powerful tool when we use an induction to treat a tree from leaves toward to the center.

As an optional 2-step dominating set of a graph $G = (V, E)$ is indeed a 2-step dominating set of G when $V = N$, in order to find a minimum 2-step dominating set, we only have to label all vertices "needed" and find a minimum optional 2-step dominating set. Now a linear-time algorithm for finding a minimum optional 2-step dominating set of a block graph is shown as follows.

Algorithm 2-StepDomBlock (finding a minimum optional 2-step dominating set of a block graph).

Input: A block graph $G = (V, E)$ rooted at B_0, with an arbitrary FNR assignments to its vertex set.

Output: A minimum optional 2-step dominating set D of G, consisting of the vertices with label R.

Method. In every step, the algorithm visits a non-cut vertex in an end block B, label or relabel some vertices, deletes this vertex from B and G. For a non-cut vertex $v \in B$, define a parameter $f_2(v)$ as follows. If the height of G is at least two, then let $f_2(v)$ be the cut vertex in $N_2(v)$ with the closest distance from B_0. If the height of G is one, then let $f_2(v)$ be an arbitrary non-cut vertex in $N_2(v)$.

Begin
$L(x) \leftarrow N$ for all $x \in N$;
$L(x) \leftarrow F$ for all $x \in F$;
$L(x) \leftarrow R$ for all $x \in R$;
$D \leftarrow \emptyset$;
do while the height of G is at least 1
do while B is an end block of G with the maximum level number
for any non-cut vertex $v \in B$ **do**
if $L(v) = F$, **then** $B \leftarrow B - v$, $G \leftarrow G - v$;
if $L(v) = N$, **then**
if there exists some vertex $x \in N_2(v)$ such that $L(x) = R$, **then**
$B \leftarrow B - v$, $G \leftarrow G - v$;
if all vertices in $N_2(v)$ are not labeled R, **then**
$B \leftarrow B - v$, $G \leftarrow G - v$, $L(f_2(v)) \leftarrow R$;
if $L(v) = R$, **then**

for every vertex $x \in N_2(v)$ **do**
if $L(x) = R$, **then** do nothing;
if $L(x) \neq R$, **then** $L(x) \leftarrow F$;
end for
$B \leftarrow B - v$, $G \leftarrow G - v$, $D \leftarrow D \cup \{v\}$;
end while
end while
do while the height of G is zero
Let $L(x) = R$ for every vertex with $L(x) = N$, and then $D \leftarrow D \cup \{u \in G |\ L(u) = R\}$;
end while
End

It is easy to see that the running time of the algorithm is $O(m)$, where m is the edge number of a graph. When we visit a vertex v, we should scan the labels of the vertices in $N_2(v)$, whose cardinality is less by one than the degree of the father of v. Then the amount of time for scanning is at most $\sum_{v \in V}(d(v) - 1) = 2m - 2$. Thus the running time of the algorithm is $O(m)$. The correctness of the algorithm is based on the following theorem.

Theorem 3.1 *2-StepDomBlock produces a minimum optional* 2*-step dominating set of a block graph* G.

Proof It is sufficient to consider G with height at least one, since the last step obviously produces a minimum optional 2-step dominating set of a complete graph correctly. We still assume that B is an end block with the biggest level number in G, v is a non-cut vertex in B, and $f_2(v)$ is defined as in Algorithm 2-StepDomBlock. Then, the proof of Theorem 3.1 is followed by a series of claims.

Claim 1 If $v \in F$, then $\gamma^2_{\rm opt}(G) = \gamma^2_{\rm opt}(G - v)$.

Let D be a $\gamma^2_{\rm opt}$-set of G. If $v \in D$, then $D \setminus \{v\} \cup \{f_2(v)\}$ is an optional 2-step dominating set of $G - v$. If $v \notin D$, then clearly D is also an optional 2-step dominating set of $G - v$. Hence $\gamma^2_{\rm opt}(G - v) \leq \gamma^2_{\rm opt}(G)$. Conversely, let D' be a $\gamma^2_{\rm opt}$-set of $G - v$. Since $v \in F$, D' is also an optional 2-dominating set of G. Therefore, $\gamma^2_{\rm opt}(G) \leq \gamma^2_{\rm opt}(G - v)$.

Claim 2 If $v \in N$ and there exists some vertex $x \in N_2(v)$ with label R, then we have $\gamma^2_{\rm opt}(G) = \gamma^2_{\rm opt}(G - v)$.

Let D be a $\gamma^2_{\rm opt}$-set of G. Since $x \in R$ in G, $x \in D$. If $v \in D$, then $D \setminus \{v\} \cup \{f_2(v)\}$ is an optional 2-step dominating set of $G - v$. If $v \notin D$, then clearly D is also an optional 2-step dominating set of $G - v$. Thus $\gamma^2_{\rm opt}(G - v) \leq \gamma^2_{\rm opt}(G)$. Conversely, let D' be a $\gamma^2_{\rm opt}$-set of $G - v$. Since $x \in R$ in $G - v$, $x \in D'$. Then it follows that D' is also an optional 2-step dominating set of G, since $v \in N$ is 2-step dominated by x in G. Hence, $\gamma^2_{\rm opt}(G) \leq \gamma^2_{\rm opt}(G - v)$.

Claim 3 If $v \in N$ and there exists no vertex in $N_2(v)$ with label R, and G' is the block graph which results from G by deleting v and relabeling $f_2(v)$ with R, then $\gamma^2_{\rm opt}(G) = \gamma^2_{\rm opt}(G')$.

Let D be a $\gamma^2_{\rm opt}$-set of G. If $v \in D$, then $D \setminus \{v\} \cup \{f_2(v)\}$ is an optional 2-step dominating set of $G - v$, in which $f_2(v)$ is considered as a required vertex. So suppose $v \notin D$. Since $v \in N$, there must exist some vertex $x \in N_2(v) \cap D$ to 2-step dominate v. If $x \neq f_2(v)$, then noting the fact that v is the farthest vertex from B_0, it is easy to see that all the vertices which are 2-step dominated by x can also be 2-step dominated by $f_2(v)$. So $D \setminus \{x\} \cup \{f_2(v)\}$ is an optional 2-step dominating set of $G - v$, in which $f_2(v)$ is considered as a required vertex. If $x = f_2(v)$, then D is obviously an optional 2-step dominating set of $G - v$, in which $f_2(v)$ is also considered as a required vertex. In either case, $\gamma^2_{\rm opt}(G') \le \gamma^2_{\rm opt}(G)$. Conversely, let D' be a $\gamma^2_{\rm opt}$-set of G'. Since $f_2(v) \in R$ in G', $f_2(v) \in D'$. Then it follows that D' is also an optional 2-step dominating set of G, since $v \in N$ is 2-step dominated by $f_2(v)$ in G. Hence, $\gamma^2_{\rm opt}(G) \le \gamma^2_{\rm opt}(G')$.

Claim 4 If $v \in R$ and G' is the block graph which results from G by deleting v and relabeling the vertices in $N_2(v)$ as the corresponding statements in Algorithm 2-StepDomBlock, then $\gamma^2_{\rm opt}(G) = \gamma^2_{\rm opt}(G') + 1$.

Let D be a $\gamma^2_{\rm opt}$-set of G. Since $v \in R$, we have $v \in D$. Then it follows that $D \setminus \{v\}$ is an optional 2-step dominating set of G', since all the vertices in $N_2(v)$ are labeled R or F in G'. Hence, $\gamma^2_{\rm opt}(G') \le \gamma^2_{\rm opt}(G) - 1$. Conversely, let D' be a $\gamma^2_{\rm opt}$-set of G'. Obviously $D' \cup \{v\}$ is an optional 2-step dominating set of G. This means that $\gamma^2_{\rm opt}(G) \le \gamma^2_{\rm opt}(G') + 1$. $\Box$

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