Journal of Mathematical Research with Applications
May, 2015, Vol. 35, No. 3, pp. 291–302
DOI:10.3770/j.issn:2095-2651.2015.03.007
Http://jmre.dlut.edu.cn

Lie Algebras for Constructing Nonlinear Integrable and Bi-Integrable Hamiltonian Couplings

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Abstract With the help of a Lie algebra, two kinds of Lie algebras with the forms of blocks are introduced for generating nonlinear integrable and bi-integrable couplings. For illustrating the application of the Lie algebras, an integrable Hamiltonian system is obtained, from which some reduced evolution equations are presented. Finally, Hamiltonian structures of nonlinear integrable and bi-integrable couplings of the integrable Hamiltonian system are furnished by applying the variational identity. The approach presented in the paper can also provide nonlinear integrable and bi-integrable couplings of other integrable system.

Keywords Lie algebra; integrable couplings; bi-integrable couplings; Hamiltonian structure

MR(2010) Subject Classification 35Q53; 37K05; 37K10

1. Introduction

Integrable couplings is an important and attractive topic in the soliton theory. On the one hand, coupled equations have many applications in various scientific contexts ranging from physics, biochemistry to mechanics. On the other hand, there are much richer mathematical structures behind integrable couplings than scalar integrable equations. Moreover, the study of integrable couplings generalizes the symmetry problem and provides clues toward complete classification of integrable equations.

Definition 1.1 ([1,2]) *For a given integrable system*

$$u_t = K(u) = K(x, t, u, u_x, u_{xx}, \ldots), \tag{1.1}$$

the following bigger triangular system

$$\overline{u}_t = \overline{K}(\overline{u}) = \begin{pmatrix} K(u) \\ S(u,v) \end{pmatrix}, \overline{u} = \begin{pmatrix} u \\ v \end{pmatrix} \tag{1.2}$$

is called an integrable couplings of the system (1.1), if (1.2) is integrable and $S(u,v)$ explicitly contains u or u-derivatives with respect to x, where the x is the space variable. Especially, $S(u,v)$ is nonlinear with respect to the sub-vector v, then the system (1.2) is called a nonlinear

Received March 24, 2014; Accepted January 16, 2015
Supported by the National Natural Science Foundation of China (Grant Nos. 61273011; 11401392).
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integrable couplings of the system (1.1).

Definition 1.2 ([3]) *A bi-integrable coupling of a given integrable system (1.1) is an enlarged triangular integrable system of the following form*

$$\widetilde{u}_t = \widetilde{K}(\widetilde{u}) = \begin{pmatrix} K(u) \\ S_1(u,v) \\ S_2(u,v,w) \end{pmatrix}, \widetilde{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \tag{1.3}$$

It is notable that S_2 depends on the sub-vector w but S_1 does not. Further, if $S_2(u,v,w)$ is nonlinear for w, then the system (1.3) is called a nonlinear bi-integrable couplings of the system (1.1). The semi-direct sums of Lie algebras lay a foundation for constructing integrable couplings.

Definition 1.3 *The semi-direct sums $\overline{g} = g \uplus g_c$, means that the two Lie sub-algebras g and g_c satisfy $[g, g_c] \subseteq g_c$, where $[g, g_c] = [A, B] | A \in g, B \in g_c$, with $[\cdot,\cdot]$ denoting the Lie bracket of $\overline{g}$ and $\uplus$ standing for semi-direct sum.*

Obviously, g_c is an ideal Lie sub-algebra of $\overline{g}$. The subscript c indicates a contribution to the construction of coupling systems. We also require the closure property between g and g_c under the matrix multiplication.

Many ways to construct linear and nonlinear integrable couplings are presented, for example, the perturbation method [1,2], the enlarged spectral problems [4–10], the block type matrix algebra [11–17]. Recently, Ma [11,12] proposed a general scheme to generate nonlinear integrable couplings and Zhang [13,14] extended this method.

In this paper, inspired by the previous work, we present two kinds of explicit Lie algebras for constructing nonlinear integrable and bi-integrable couplings of the integrable hierarchy. We illustrate the applications of the new Lie algebras by means of an integrable hierarchy, which can reduce to some evolution equations, including the well-known mKdV equation. Then we also obtain the Hamiltonian structures of nonlinear integrable and bi-integrable couplings of the integrable hierarchy by applying the variational identity [18,19], respectively.

2. Lie algebras

We have known the simple Lie algebra A_1 with the basis as the following is frequently used to construct the spectral problems by Tu scheme [20–22].

$$e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$[e,h] = 2f, [e,f] = 2h, [h,f] = 2e.$$

In this paper we take the linear combination of A_1

$$G = \{e_1, e_2, e_3 | e_1 = -e, e_2 = h, e_3 = h - f\},$$
$$[e_1, e_2] = 2(e_3 - e_2), [e_1, e_3] = 2e_3, [e_2, e_3] = 2e_1, \tag{2.1}$$

and its corresponding loop algebra

$$\widetilde{G} = \{e_1(n), e_2(n), e_3(n) | e_i(n) = e_i\lambda^n, i = 1, 2, 3\}, \ [e_i(m), e_j(n)] = [e_i, e_j]\lambda^{m+n}. \tag{2.2}$$

We start from the Lie algebra G to generate the higher-dimensional Lie algebra G_h.

$$G_h = \{w_1, w_2, \ldots, w_6\}, \tag{2.3}$$

where

$$w_1 = \begin{pmatrix} e_1 & 0 \\ 0 & e_1 \end{pmatrix}, w_2 = \begin{pmatrix} e_2 & 0 \\ 0 & e_2 \end{pmatrix}, w_3 = \begin{pmatrix} e_3 & 0 \\ 0 & e_3 \end{pmatrix},$$
$$w_4 = \begin{pmatrix} 0 & e_1 \\ 0 & e_1 \end{pmatrix}, w_5 = \begin{pmatrix} 0 & e_2 \\ 0 & e_2 \end{pmatrix}, w_6 = \begin{pmatrix} 0 & e_3 \\ 0 & e_3 \end{pmatrix}.$$

A direct verification exhibits that

$$\begin{aligned}
&[w_1, w_2] = 2(w_3 - w_2), [w_1, w_3] = 2w_3, [w_2, w_3] = 2w_1, \\
&[w_4, w_5] = 2(w_6 - w_5), [w_4, w_6] = 2w_6, [w_5, w_6] = 2w_4, \\
&[w_2, w_4] = 2(w_5 - w_6), [w_3, w_4] = -2w_6, [w_1, w_5] = 2(w_6 - w_5), [w_1, w_2] = 2(w_3 - w_2), \\
&[w_3, w_5] = -2w_4, [w_1, w_6] = 2w_6, [w_2, w_6] = 2w_4, [w_1, w_4] = [w_2, w_5] = [w_3, w_6] = 0.
\end{aligned}$$

Let $G_{h1} = \{w_1, w_2, w_3\}$, $G_{h2} = \{w_4, w_5, w_6\}$. We find that $G_h = G_{h1} \uplus G_{h2}$, G_{h1} is isomorphic to the Lie algebra G, and is simple, which is a key fact for generating nonlinear integrable couplings. It is easy to see that again

$$[G_{h1}, G_{h1}] = G_{h1}, [G_{h2}, G_{h2}] = G_{h2}, [G_{h1}, G_{h2}] \subseteq G_{h2}. \tag{2.4}$$

So, the Lie sub-algebra G_{h1} and G_{h2} are simple. Define a loop algebra $\widetilde{G_h}$ corresponding to the Lie algebra G_h

$$\widetilde{G_h} = \{w_1(n), w_2(n), \ldots, w_6(n)\}, w_i(n) = w_i\lambda^n, [w_i(m), w_j(n)] = [w_i, w_j]\lambda^{m+n}. \tag{2.5}$$

By using the Lie algebra G and G_h, we can construct another new higher-dimensional Lie algebra G_a.

$$G_a = \{g_1, g_2, \ldots, g_9\}, \tag{2.6}$$

where

$$g_1 = \begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_1 & 0 \\ 0 & 0 & e_1 \end{pmatrix}, g_2 = \begin{pmatrix} e_2 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_2 \end{pmatrix}, g_3 = \begin{pmatrix} e_3 & 0 & 0 \\ 0 & e_3 & 0 \\ 0 & 0 & e_3 \end{pmatrix},$$

$$g_4 = \begin{pmatrix} 0 & e_1 & 0 \\ 0 & e_1 & 0 \\ 0 & 0 & e_1 \end{pmatrix}, g_5 = \begin{pmatrix} 0 & e_2 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_2 \end{pmatrix}, g_6 = \begin{pmatrix} 0 & e_3 & 0 \\ 0 & e_3 & 0 \\ 0 & 0 & e_3 \end{pmatrix},$$

$$g_7 = \begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_1 \\ 0 & 0 & e_1 \end{pmatrix}, g_8 = \begin{pmatrix} 0 & 0 & e_2 \\ 0 & 0 & e_2 \\ 0 & 0 & e_2 \end{pmatrix}, g_9 = \begin{pmatrix} 0 & 0 & e_3 \\ 0 & 0 & e_3 \\ 0 & 0 & e_3 \end{pmatrix},$$

which has the following operation relations

$$[g_1, g_2] = 2(g_3 - g_2), [g_1, g_3] = 2g_3, [g_2, g_3] = 2g_1,$$
$$[g_4, g_5] = [g_1, g_5] = [g_4, g_2] = 2(g_6 - g_5), [g_7, g_9] = [g_1, g_9] = [g_7, g_3] = [g_4, g_9] = [g_7, g_6] = 2g_9,$$
$$[g_4, g_6] = [g_1, g_6] = [g_4, g_3] = 2g_6, [g_7, g_8] = [g_1, g_8] = [g_7, g_2] = [g_4, g_8] = [g_7, g_5] = 2(g_9 - g_8),$$
$$[g_5, g_6] = [g_2, g_6] = [g_5, g_3] = 2g_4, [g_8, g_9] = [g_2, g_9] = [g_8, g_3] = [g_5, g_9] = [g_8, g_6] = 2g_7,$$
$$[g_1, g_4] = [g_2, g_5] = [g_3, g_6] = [g_1, g_7] = [g_4, g_7] = [g_2, g_8] = [g_5, g_8] = [g_3, g_9] = [g_6, g_9] = 0.$$

Let $G_{a1} = \{g_1, g_2, g_3\}$, $G_{a2} = \{g_4, g_5, g_6\}$, and $G_{a3} = \{g_7, g_8, g_9\}$. We find that they satisfy the condition

$$G_a = G_{a1} \biguplus G_{a2} \biguplus G_{a3}, [G_{a1}, G_{a2}] \subseteq G_{a2}, [G_{a2}, G_{a3}] \subseteq G_{a3}, [G_{a1}, G_{a3}] \subseteq G_{a3}, \tag{2.7}$$

and the Lie sub-algebra G_{a1}, G_{a2} and G_{a3} are all simple, which is a key fact for generating nonlinear bi-integrable couplings. A loop algebra $\widetilde{G_a}$ is defined as

$$\widetilde{G_a} = \{g_1(n), g_2(n), \ldots, g_9(n)\}, g_i(n) = g_i\lambda^n, [g_i(m), g_j(n)] = [g_i, g_j]\lambda^{(m+n)}. \tag{2.8}$$

3. The hierarchy

In this section, we make use of the Lie algebra G and Tu scheme to get an integrable hierarchy along with Hamiltonian structure, which can reduce to some evolution equation including the well-known mKdV equation.

Considering the spectral problem

$$\begin{cases} \varphi_x = U\varphi, U = -e_1(1) + qe_2(0) + re_3(0), \\ \varphi_t = V\varphi, V = ae_1(0) + be_2(0) + ce_3(0), \end{cases} \tag{3.1}$$

and solving the stationary equation

$$V_x = [U, V], \tag{3.2}$$

we obtain the recursive relations

$$\begin{cases} a_{mx} = 2qc_m - 2rb_m, \\ b_{m+1} = \frac{1}{2}b_{mx} - qa_m, \\ c_{m+1} = -\frac{1}{2}(b_{mx} + c_{mx}) - ra_m. \end{cases} \tag{3.3}$$

Setting $a_0 = \alpha, b_0 = 0, c_0 = 0$, one infers from (3.3)

$$b_1 = -\alpha q, c_1 = \alpha r, a_1 = 0, b_2 = -\frac{\alpha}{2}q_x, c_2 = \frac{\alpha}{2}(q_x + r_x),$$
$$a_2 = \frac{\alpha}{2}q^2 + qr, b_3 = -\frac{\alpha}{4}q_{xx} - \frac{\alpha}{2}q^3 + \alpha q^2 r, \ldots. \tag{3.4}$$

Denote

$$V_+^{(n)} = \sum_{m=0}^{n} a_m e_1(n-m) + b_m e_2(n-m) + c_m e_3(n-m) = V^{(n)},$$

we can obtain the following the integrable hierarchy

$$u_{t_n} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -2(b_{n+1} + c_{n+1}) \\ -2b_{n+1} \end{pmatrix} = JG_{m+1}, \tag{3.5}$$

which satisfy the zero curvature equation

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0. \tag{3.6}$$

Taking $n = 2, \alpha = 2$ in (3.5) gives rise to

$$\begin{cases} q_{t_2} = -q_{xx} - 2q^3 + 4rq^2, \\ r_{t_2} = q_{xx} + r_{xx} - 2q^3 + 4qr^2 - 2q^2 r; \end{cases} \tag{3.7}$$

taking $n = 3, \alpha = 2$ in (3.5) gives rise to

$$\begin{cases} q_{t_3} = -\frac{1}{2} q_{xxx} - 3q^2 q_x + 2qrq_x + 4q^2 r_x, \\ r_{t_3} = \frac{1}{2} r_{xxx} + 3q^2 r_x + 6qrr_x. \end{cases} \tag{3.8}$$

If set $q = r$, the second equation of (3.8) is just the well-known mKdV equation, so (3.8) can be called one coupled equation of mKdV.

In the following, we construct the Hamiltonian structure of (3.5) by means of the trace identity [20]. A direct calculation reads

$$\langle V, \frac{\partial U}{\partial q} \rangle = -2b - 2c, \ \langle V, \frac{\partial U}{\partial r} \rangle = -2b, \ \langle V, \frac{\partial U}{\partial \lambda} \rangle = -2a. \tag{3.9}$$

Substituting above formulae into trace identity and comparing the coefficients of λ^{-n-1} yields

$$\begin{pmatrix} \frac{\delta}{\delta q} \\ \frac{\delta}{\delta r} \end{pmatrix} (-2a_{n+1}) = (-n + \gamma) \begin{pmatrix} -2b_n - 2c_n \\ -2b_n \end{pmatrix}. \tag{3.10}$$

Setting $n = 1$ leads to $\gamma = 0$. Therefore, we obtain the Hamiltonian structure of (3.5)

$$u_{t_n} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = J \frac{\delta H_{n+1}}{\delta u}, \quad H_n = \frac{2a_{n+1}}{n}. \tag{3.11}$$

Obviously, J is a Hamiltonian operator. We can verify that the integrable hierarchy (3.5) is integrable in the sense of Liouville.

4. Nonlinear integrable couplings

As for nonlinear integrable couplings, many interesting results have been obtained [11–16,23–25]. In this section, we start from the Lie algebra G_h to generate the nonlinear integrable coupling of (3.5).

Apply the loop algebra $\widetilde{G_h}$ and take the linear forms as follows

$$\begin{cases} \varphi_x = U\varphi, U = -w_1(1) + qw_2(0) + rw_3(0) + u_1 w_5(0) + u_2 w_6(0), \\ \varphi_t = V\varphi, V = aw_1(0) + bw_2(0) + cw_3(0) + dw_4(0) + fw_5(0) + hw_6(0). \end{cases} \tag{4.1}$$

Solving the auxiliary equation (3.2) yields

$$\begin{cases} a_{mx} = 2qc_m - 2rb_m, \\ b_{m+1} = \frac{1}{2}b_{mx} - qa_m, \\ c_{m+1} = -\frac{1}{2}(b_{mx} + c_{mx}) - ra_m, \\ d_{mx} = 2(q+u_1)h_m - 2(v+u_2)f_m + 2u_1c_m - 2u_2b_m, \\ f_{m+1} = \frac{1}{2}f_{mx} - (q+r)d_m - u_1a_m, \\ h_{m+1} = -\frac{1}{2}(f_{mx} + h_{mx}) - (r+u_2)d_m - u_2a_m, \end{cases} \tag{4.2}$$

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = -2b_{n+1}w_2(0) + 2(b_{n+1} + c_{n+1})w_3(0) - 2f_{n+1}w_5(0) + 2(f_{n+1} + h_{n+1})w_6(0), \tag{4.3}$$

and if set $a_0 = \alpha, d_0 = \beta, b_0 = c_0 = f_0 = h_0 = 0$, one infers from (4.2)

$$b_1 = -\alpha q, c_1 = \alpha r, a_1 = 0, b_2 = -\frac{\alpha}{2}q_x, c_2 = \frac{\alpha}{2}(q_x + r_x), a_2 = \frac{\alpha}{2}q^2 + qr, f_1 = -\beta(q+u_1) - \alpha u_1,$$

$$h_1 = -\beta(r+u_2) - \alpha u_2, f_2 = -\frac{\beta}{2}(q_x + u_{1x}) - \frac{\alpha}{2}u_{1x}, h_2 = \frac{\beta}{2}(r_x + u_{2x} + q_x + u_{1x}) + \frac{\alpha}{2}(u_{2x} + u_{1x}),$$

$$d_2 = \beta((r+u_2)(u+u_1) + \frac{1}{2}(u^2 + u_1^2) + uu_1) + \alpha(u_1(q+r) + u_2(u+u_1) + \frac{1}{2}u_1^2), \ldots.$$

Set $V^{(n)} = V_+^{(n)}$. Then the zero curvature equation (3.6) determines the Lax integrable system as follows

$$\overline{u}_{t_n} = \begin{pmatrix} u_{t_n} \\ v_{t_n} \end{pmatrix} = \begin{pmatrix} K(u) \\ S(u,v) \end{pmatrix}, \tag{4.5}$$

where

$$u_{t_n} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = \begin{pmatrix} b_{nx} - 2qa_n \\ c_{nx} + 2(q+r)a_n \end{pmatrix}, \tag{4.6}$$

$$v_{t_n} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{t_n} = \begin{pmatrix} f_{nx} - 2(q+u_1)d_n - 2u_1a_n \\ h_{nx} + 2(q+r+u_1+u_2)d_n + 2(u_1+u_2)a_n \end{pmatrix}. \tag{4.7}$$

Taking $n = 2$ in (4.7) gives rise to

$$\begin{cases} u_{1t_2} = \frac{-\beta}{2}(q_{xx} + u_{1xx}) - \frac{\alpha}{2}u_{1xx} - 2\beta((r+u_2)(q+u_1)^2 + \frac{1}{2}(q+u_1)(q^2+u_1^2) + qu_1(q+u_1)) - \\ \quad 2\alpha(u_1(q+r)(q+u_1) + u_2(q+u_1)^2 + \frac{1}{2}u_1^2(q+u_1) + u_1(\frac{1}{2}q^2 + qr)), \\ u_{2t_2} = \frac{\beta}{2}(r_{xx} + u_{2xx} + q_{xx} + u_{1xx}) + \frac{\alpha}{2}(u_{1xx} + u_{2xx}) + 2\beta(q+r+u_1+u_2)((r+u_2)(q+u_1) + \\ \quad \frac{1}{2}(q^2+u_1^2) + qu_1) + 2\alpha(q+r+u_1+u_2)(u_1(q+r) + u_2(q+u_1) + \frac{1}{2}u_1^2) + \\ \quad 2\alpha(u_1+u_2)(\frac{1}{2}q^2 + qr). \end{cases} \tag{4.8}$$

Comparing the structures of (4.5) with (3.5) and according to the definition of integrable couplings, we can conclude that (4.5) are integrable couplings of the system (3.5). Moreover, (4.8) is a nonlinear coupled system in u_1, u_2 and along with the variable coefficient functions q, r which satisfy (3.7). Hence (4.5) is a nonlinear integrable coupled of the (3.5).

If set $\alpha = 0, \beta = 2, u_{1t_2} = u_{2t_2} = 0$. (4.8) can be reduced to

$$\begin{cases} q_{xx} + u_{1xx} + 4(q+u_1)((r+u_2)(u+u_1) + \frac{1}{2}(q+u_1)^2) = 0, \\ r_{xx} + u_{2xx} + 4(r+u_2)((r+u_2)(u+u_1) + \frac{1}{2}(q+u_1)^2) = 0, \end{cases} \tag{4.9}$$

which is a new nonlinear coupled partial equation with variable coefficients $q(x,t)$, and $r(x,t)$.

In what follows, we investigate the Hamiltonian structure of the nonlinear integrable coupling (4.5) by applying the variational identity. For $a = \Sigma_{i=1}^{6} a_i w_i, b = \Sigma_{j=1}^{6} b_j w_j \in G_2$, the commutator $[a,b]^T$ can be obtained

$$[a,b]^T = (a_1, a_2, \ldots, a_6) R_h(b), \tag{4.10}$$

where

$$R_h(b) = \begin{pmatrix} 0 & -2b_2 & 2(b_2+b_3) & 0 & -2b_5 & 2(b_5+b_6) \\ 2b_3 & 2b_1 & -2b_1 & 2b_6 & 2b_4 & -2b_4 \\ -2b_2 & 0 & -2b_1 & -2b_5 & 0 & -2b_4 \\ 0 & 0 & 0 & 0 & -2(b_2+b_5) & 2(b_2+b_3+b_5+b_6) \\ 0 & 0 & 0 & 2(b_3+b_6) & 2(b_1+b_4) & -2(b_1+b_4) \\ 0 & 0 & 0 & -2(b_2+b_5) & 0 & -2(b_1+b_4) \end{pmatrix}.$$

Solving the matrix equation

$$R(b)F = -(R(b)F)^T, \quad F^T = F, \tag{4.11}$$

yields that

$$F_h = \begin{pmatrix} \eta_1 & 0 & 0 & \eta_2 & 0 & 0 \\ 0 & -\eta_1 & -\eta_1 & 0 & -\eta_2 & -\eta_2 \\ 0 & -\eta_1 & 0 & 0 & -\eta_2 & 0 \\ \eta_2 & 0 & 0 & \eta_2 & 0 & 0 \\ 0 & -\eta_2 & -\eta_2 & 0 & -\eta_2 & -\eta_2 \\ 0 & -\eta_2 & 0 & 0 & -\eta_2 & 0 \end{pmatrix},$$

A direct calculation reads

$$\begin{aligned} &\langle V, \frac{\partial U}{\partial q}\rangle = -\eta_1(b+c) - \eta_2(f+h), \langle V, \frac{\partial U}{\partial r}\rangle = -\eta_1 b - \eta_2 f, \langle V, \frac{\partial U}{\partial \lambda}\rangle = -\eta_1 a - \eta_2 d, \\ &\langle V, \frac{\partial U}{\partial u_1}\rangle = -\eta_2(b+c+f+h), \langle V, \frac{\partial U}{\partial u_2}\rangle = -\eta_2(b+f). \end{aligned} \tag{4.12}$$

Inserting the above formulas into the variational identity and comparing the coefficients of λ^{-n-1} yields

$$\begin{pmatrix} \frac{\delta}{\delta q} \\ \frac{\delta}{\delta r} \\ \frac{\delta}{\delta u_1} \\ \frac{\delta}{\delta u_2} \end{pmatrix} \int^x (-\eta_1 a_{n+1} - \eta_2 d_{n+1}) dx = (-n+\gamma) \begin{pmatrix} -\eta_1(b_n + c_n) - \eta_2(f_n + h_n) \\ -\eta_1 b_n - \eta_2 f_n \\ -\eta_2(b_n + c_n + f_n + h_n) \\ -\eta_2(b_n + f_n) \end{pmatrix}. \tag{4.13}$$

Setting $n=1$ leads to $\gamma=0$. Therefore, we obtain the Hamiltonian structure of (4.5)

$$\begin{pmatrix} q \\ r \\ u_1 \\ u_2 \end{pmatrix}_{t_n} = \begin{pmatrix} 0 & \frac{-2}{\eta_1-\eta_2} & 0 & \frac{-2}{\eta_2^2-\eta_1\eta_2} \\ \frac{2}{\eta_1-\eta_2} & 0 & \frac{2}{\eta_2^2-\eta_1\eta_2} & 0 \\ 0 & \frac{2}{\eta_1-\eta_2} & 0 & \frac{2}{\eta_2^2-\eta_1\eta_2} \\ \frac{-2}{\eta_1-\eta_2} & 0 & \frac{-2}{\eta_2^2-\eta_1\eta_2} & 0 \end{pmatrix} \frac{\delta \overline{H}_{n+1}}{\delta \overline{u}},\ \overline{H}_n = \int^x \frac{\eta_1 a_{n+1} + \eta_2 d_{n+1}}{n} \mathrm{d}x. \tag{4.14}$$

5. Nonlinear Bi-integrable couplings

Bi-integrable couplings were proposed by Ma [3]. In this section, with the help of the Lie algebra G_a, we can derive a nonlinear bi-integrable coupling of (3.5).

Taking the linear forms as follows

$$\begin{cases} \varphi_x = U\varphi, U = -g_1(1) + qg_2(0) + rg_3(0) + u_1g_5(0) + u_2g_6(0) + u_3g_8(0) + u_4g_9(0), \\ \varphi_t = V\varphi, V = \sum\limits_{m=0}^{\infty}(\sum\limits_{i=1}^{9} V_{jm}g_j(-m)), \end{cases} \tag{5.1}$$

and solving the auxiliary equation (3.2) yields

$$\begin{cases} V_{7mx} = 2(u+u_1+u_3)V_{9m} - 2(v+u_2+u_4)V_{8m} + 2u_3(V_{3m}+V_{6m}) - 2u_4(V_{2m}+V_{5m}), \\ V_{8,m+1} = \frac{1}{2}V_{8mx} - (u+u_1+u_3)V_{7m} - u_3(V_{1m}+V_{4m}), \\ V_{9,m+1} = \frac{1}{2}(V_{8mx} + V_{9mx} - (v+u_2+u_4)V_{7m} - u_4(V_{1m}+V_{4m}), \end{cases} \tag{5.2}$$

$$\begin{aligned} -V_{+x}^{(n)} + [U, V_+^{(n)}] = & -2V_{2,n+1}g_2(0) + 2(V_{2,n+1}+V_{3,n+1})g_3(0) - 2V_{5,n+1}g_5(0) + 2(V_{5,n+1}+ \\ & V_{6,n+1})g_6(0) - 2V_{8,n+1}g_8(0) + 2(V_{8,n+1}+V_{9,n+1})g_9(0). \end{aligned} \tag{5.3}$$

If set $V_{10}=\alpha_1, V_{40}=\alpha_2, V_{70}=\alpha_3, V_{20}=V_{30}=V_{50}=V_{60}=V_{40}=V_{80}=V_{90}=0$, one infers from (5.2)

$$\begin{aligned} & V_{81} = -\alpha_3(q+u_1+u_3) - (\alpha_1+\alpha_2)u_3, V_{91} = -\alpha_3(r+u_2+u_4) - (\alpha_1+\alpha_2)u_4, \\ & V_{82} = \frac{-\alpha_3}{2}(q_x+u_{1x}+u_{3x}) - \frac{(\alpha_1+\alpha_2)}{2}u_{3x}, \\ & V_{72} = \alpha_3((q+u_1+u_3)(q+u_1+u_3+r+u_2+u_4) - \frac{1}{2}(q+u_1+u_3)^2), \\ & V_{92} = \frac{\alpha_3}{2}(q_x+u_{1x}+u_{3x}+r_x+u_{2x}+u_{4x}) + \frac{(\alpha_1+\alpha_2)}{2}(u_{3x}+u_{4x}). \end{aligned} \tag{5.4}$$

Set $V^{(n)} = V_+^{(n)}$. Then the zero curvature equation determines the Lax integrable system as follows

$$\widetilde{u}_{t_n} = \begin{pmatrix} u_{t_n} \\ v_{t_n} \\ w_{t_n} \end{pmatrix} = \begin{pmatrix} K(u) \\ S_1(u,v) \\ S_2(u,v,w) \end{pmatrix}, \tag{5.5}$$

where

$$u_{t_n} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = \begin{pmatrix} V_{2nx} - 2qV_{1n} \\ V_{3nx} + 2(q+r)V_{1n} \end{pmatrix}, \tag{5.6}$$

$$v_{t_n} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{t_n} = \begin{pmatrix} V_{5nx} - 2(q+u_1)V_{2n} - 2u_1V_{1n} \\ V_{6nx} + 2(q+r+u_1+u_2)V_{4n} + 2(u_1+u_2)V_{1n} \end{pmatrix}, \tag{5.7}$$

$$w_{t_n} = \begin{pmatrix} u_3 \\ u_4 \end{pmatrix}_{t_n} = \begin{pmatrix} V_{8nx} - 2(q+u_1+u_3)V_{7n} - 2u_3(V_{1n}+V_{4n}) \\ V_{9nx} + 2(q+r+u_1+u_2+u_3+u_4)V_{7n} + 2(u_3+u_4)(V_{1n}+V_{4n}) \end{pmatrix}. \tag{5.8}$$

When $n = 2$, (5.8) reduces to the following evolution equation

$$\begin{cases} u_{3t_2} = V_{82x} - 2(q+u_1+u_3)V_{72} - 2u_3(\alpha_2((r+u_2)(q+u_1) + \frac{1}{2}(q^2+u_1^2) + qu_1) + \\ \qquad \alpha_1(\frac{1}{2}q^2 + qr + u_1(q+r) + u_2(q+u_1) + \frac{1}{2}u_1^2)), \\ u_{4t_2} = V_{92x} + 2(q+r+u_1+u_2+u_3+u_4)V_{72} + 2(u_3+u_4)(\alpha_2((r+u_2)(q+u_1) + \\ \qquad \frac{1}{2}(q^2+u_1^2) + qu_1) + \alpha_1(\frac{1}{2}q^2 + qr + u_1(q+r) + u_2(q+u_1) + \frac{1}{2}u_1^2)), \end{cases} \tag{5.9}$$

where V_{72}, V_{82}, V_{92} are presented in (5.4). Comparing the structures of (5.5) with (4.5) and (3.5) and according to the definition of bi-integrable couplings, we can conclude that (5.5) is bi-integrable couplings of the system (3.5). Of course, (5.5) is also integrable couplings of the system (4.5) and (3.5). Especially, let $\alpha_2 = \alpha_3 = 0, \alpha_1 = 2$. We get the reduced bi-integrable coupling of (3.7).

$$\begin{cases} q_{t_2} = -q_{xx} - 2q^3 + 4rq^2, \\ r_{t_2} = q_{xx} + r_{xx} - 2q^3 + 4qr^2 - 2q^2r; \\ u_{1t_2} = -u_{1xx} - 4(u_1(q+r)(q+u_1) + u_2(q+u_1)^2 + \frac{1}{2}u_1^2(q+u_1) + u_1(\frac{1}{2}q^2+qr)), \\ u_{2t_2} = u_{1xx} + u_{2xx}4(q+r+u_1+u_2)(u_1(q+r) + u_2(q+u_1) + \frac{1}{2}u_1^2) + 4(u_1+u_2)(\frac{1}{2}q^2+qr); \\ u_{3t_2} = -u_{3xx} - 4(q+u_+u_3)((u_3+u_4)(q+r+u_1+u_2+u_3+u_4) - u_4(r+u_2+u_3+u_4) - \\ \qquad \frac{1}{2}u_3^2) - 2u_3(q^2+2qr+2u_1(q+r)+2u_2(q+u_1)+u_1^2), \\ u_{4t_2} = u_{3xx} + u_{4xx} + 4(q+r+u_1+u_2+u_3+u_4)((u_3+u_4)(q+r+u_1+u_2+u_3+u_4) - \\ \qquad u_4(r+u_2+u_3+u_4) - \frac{1}{2}u_3^2) + 2(u_3+u_4)(q^2+2qr+2u_1(q+r)+2u_2(q+u_1)+u_1^2). \end{cases} \tag{5.10}$$

It is easy to see that (5.10) is a nonlinear coupled system in u_3, u_4 and along with the variable coefficient functions q, r, u_1, u_2 which satisfy (3.7) and (4.8). Hence (5.5) is a nonlinear bi-integrable coupling of the (3.5).

In what follows, we investigate the Hamiltonian structure of the nonlinear integrable coupling (5.5) by applying the variational identity. For $a = \Sigma_{i=1}^9 a_i g_i, b = \Sigma_{j=1}^9 g_j w_j \in G_a$, the commutator $[a,b]^T$ can be obtained

$$[a,b]^T = (a_1, a_2, \ldots, a_9)R_a(b), \tag{5.11}$$

where

$$R_a(b)=\begin{pmatrix} R_{11} & R_{12} & R_{13}\\ 0 & R_{22} & R_{23}\\ 0 & 0 & R_{33}\end{pmatrix}, R_{11}=\begin{pmatrix} 0 & -2b_2 & 2(b_2+b_3)\\ 2b_3 & 2b_1 & -2b_1\\ -2b_2 & 0 & -2b_1\end{pmatrix},$$

$$R_{12}=\begin{pmatrix} 0 & -2b_5 & 2(b_5+b_6)\\ 2b_6 & 2b_4 & -2b_4\\ -2b_5 & 0 & -2b_4\end{pmatrix}, R_{13}=\begin{pmatrix} 0 & -2b_8 & 2(b_8+b_9)\\ 2b_9 & 2b_7 & -2b_7\\ -2b_8 & 0 & -2b_7\end{pmatrix},$$

$$R_{22}=\begin{pmatrix} 0 & -2(b_2+b_5) & 2(b_2+b_3+b_5+b_6)\\ 2(b_3+b_6) & 2(b_1+b_4) & -2(b_1+b_4)\\ -2(b_2+b_5) & 0 & -2(b_1+b_4)\end{pmatrix}, R_{23}=\begin{pmatrix} 0 & -2b_8 & 2(b_8+b_9)\\ 2b_9 & 2b_7 & -2b_7\\ -2b_8 & 0 & -2b_7\end{pmatrix},$$

$$R_{33}=\begin{pmatrix} 0 & -2(b_2+b_5+b_8) & 2(b_2+b_3+b_5+b_6+b_8+b_9)\\ 2(b_3+b_6+b_9) & 2(b_1+b_4+b_7) & -2(b_1+b_4+b_7)\\ -2(b_2+b_5+b_8) & 0 & -2(b_1+b_4+b_7)\end{pmatrix}.$$

Solving the matrix equation (4.11), yields that

$$F_a=\begin{pmatrix} \eta_1 & 0 & 0 & \eta_2 & 0 & 0 & \eta_3 & 0 & 0\\ 0 & -\eta_1 & -\eta_1 & 0 & -\eta_2 & -\eta_2 & 0 & -\eta_3 & -\eta_3\\ 0 & -\eta_1 & 0 & 0 & -\eta_2 & 0 & 0 & -\eta_3 & 0\\ \eta_2 & 0 & 0 & \eta_2 & 0 & 0 & \eta_3 & 0 & 0\\ 0 & -\eta_2 & -\eta_2 & 0 & -\eta_2 & -\eta_2 & 0 & -\eta_3 & -\eta_3\\ 0 & -\eta_2 & 0 & 0 & -\eta_2 & 0 & 0 & -\eta_3 & 0\\ \eta_3 & 0 & 0 & \eta_3 & 0 & 0 & \eta_3 & 0 & 0\\ 0 & -\eta_3 & -\eta_3 & 0 & -\eta_3 & -\eta_3 & 0 & -\eta_3 & -\eta_3\\ 0 & -\eta_3 & 0 & 0 & -\eta_3 & 0 & 0 & -\eta_3 & 0\end{pmatrix}.$$

A direct calculation reads

$$\begin{aligned}
&\langle V,\frac{\partial U}{\partial q}\rangle=-\eta_1(V_2+V_3)-\eta_2(V_5+V_6)-\eta_3(V_8+V_9), \langle V,\frac{\partial U}{\partial r}\rangle=-\eta_1V_2-\eta_2V_5-\eta_3V_8,\\
&\langle V,\frac{\partial U}{\partial \lambda}\rangle=-\eta_1V_1-\eta_2V_4-\eta_3V_7, \langle V,\frac{\partial U}{\partial u_1}\rangle=-\eta_2(V_2+V_3+V_5+V_6)-\eta_3(V_8+V_9),\\
&\langle V,\frac{\partial U}{\partial u_2}\rangle=-\eta_2(V_2+V_5)-\eta_3V_8, \langle V,\frac{\partial U}{\partial u_3}\rangle=-\eta_3(V_2+V_3+V_5+V_6+V_8+V_9),\\
&\langle V,\frac{\partial U}{\partial u_4}\rangle=-\eta_3(V_2+V_5+V_8).
\end{aligned}\tag{5.12}$$

Inserting the above formulas into the variational identity and comparing the coefficients of λ^{-n-1} yields

$$\frac{\delta}{\delta\widetilde{u}}\int^x(-\eta_1V_{1,n+1}-\eta_2V_{4,n+1}-\eta_3V_{7,n+1})\mathrm{d}x$$

$$= (-n+\gamma)\begin{pmatrix} -\eta_1(V_{2n}+V_{3n})-\eta_2(V_{5n}+V_{6n})-\eta_3(V_{8n}+V_{9n}) \\ -\eta_1 V_{2n}-\eta_2 V_{5n}-\eta_3 V_{8n} \\ -\eta_2(V_{2n}+V_{3n}+V_{5n}+V_{6n})-\eta_3(V_{8n}+V_{9n}) \\ -\eta_2(V_{2n}+V_{5n})-\eta_3 V_{8n} \\ -\eta_3(V_{2n}+V_{3n}+V_{5n}+V_{6n}+V_{8n}+V_{9n}) \\ -\eta_3(V_{2n}+V_{5n}+V_{8n}) \end{pmatrix}. \tag{5.13}$$

Employing the initial values gives $\gamma = 0$. Thus, the Hamiltonian structure of the bi-integrable coupling (5.5) can be written

$$\widetilde{u}_{t_n} = \begin{pmatrix} q \\ r \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_{t_n} = \widetilde{J}\frac{\delta \widetilde{H}_{n+1}}{\delta \widetilde{u}},\ \widetilde{H}_n = \int^x \frac{\eta_1 V_{1,n+1}+\eta_2 V_{4,n+1}+\eta_3 V_{7,n+1}}{n}\mathrm{d}x, \tag{5.14}$$

where

$$\widetilde{J} = \begin{pmatrix} 0 & \frac{-2}{\eta_1-\eta_2} & 0 & \frac{2}{\eta_1-\eta_2} & 0 & 0 \\ \frac{2}{\eta_1-\eta_2} & 0 & \frac{-2}{\eta_1-\eta_2} & 0 & 0 & 0 \\ 0 & \frac{2}{\eta_1-\eta_2} & 0 & \frac{2(\eta_1-\eta_3)}{(\eta_1-\eta_2)(\eta_3-\eta_2)} & 0 & \frac{-2}{(\eta_3-\eta_2)} \\ \frac{-2}{\eta_1-\eta_2} & 0 & \frac{-2(\eta_1-\eta_3)}{(\eta_1-\eta_2)(\eta_3-\eta_2)} & 0 & \frac{2}{(\eta_3-\eta_2)} & \\ 0 & 0 & 0 & \frac{2}{\eta_2-\eta_3} & 0 & \frac{2\eta_2}{\eta_3(\eta_3-\eta_2)} \\ 0 & 0 & \frac{-2}{\eta_2-\eta_3} & 0 & \frac{-2\eta_2}{\eta_3(\eta_3-\eta_2)} & 0 \end{pmatrix}. \tag{5.15}$$

6. Conclusions

Making use of a new Lie algebra, which is the linear combination form of the simple Lie algebra A_1, two kinds of higher-dimensional Lie algebras are introduced, which are much convenient in generating nonlinear integrable and bi-integrable couplings. By employing the new Lie algebra and Tu scheme an integrable Hamiltonian hierarchy is obtained, from which some reduced evolution equations are given. Then, starting from the two kinds of higher-dimensional Lie algebras the nonlinear integrable and bi-integrable couplings of the integrable Hamiltonian hierarchy are worked out, respectively. Moreover their corresponding Hamiltonian structures are generated by the variational identity. Actually, using the method of constructing Lie algebras presented in the paper can generate nonlinear integrable and bi-integralbe couplings of other soliton hierarchies.

Acknowledgements We thank the referees for their time and comments.

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