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Duality for Multiobjective Bilevel Programming Problems with Extremal-Value Function

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Abstract For a multiobjective bilevel programming problem (P) with an extremal-value function, its dual problem is constructed by using the Fenchel-Moreau conjugate of the functions involved. Under some convexity and monotonicity assumptions, the weak and strong duality assertions are obtained.

Keywords multiobjective optimization; bilevel programming problems; conjugate duality; convex programming; composed convex functions

MR(2010) Subject Classification 90C25; 90C29

1. Introduction

Recently, both researchers and practitioners have paid an increasing amount of attention to bilevel programming problems. This class of programming problems offers a very suitable modeling framework for a number of practical problems [1,2]. Many papers have been published in the last two decades studying either single objective or multiobjective bilevel programming problems [3–14]. Bard [3] studied a linear bilevel programming problem and developed first order necessary optimality conditions for it. Dempe [4,5] gave some necessary optimality conditions under the assumption that the solution set of the lower level problem is a singleton. Ye and Zhu [6] derived optimality conditions for the general bilevel programming problems without the assumption that the solution set of the lower level problem is a singleton. Ye [7] considered a nondifferentiable bilevel programming problem and gave some optimality conditions for it by using the value function approach. Suneja and Kohli [10] derived some sufficient optimality conditions for a bilevel programming problem and established various duality results associating the primal problem with two dual problems. Aboussoror et al. [11,12] considered the bilevel programs with extremal value function, and obtained the global optimality conditions and the Fenchel-Lagrange duality results.

The multiobjective bilevel optimization problems have also been studied in the literature. For a nonsmooth multiobjective bilevel programming problem, the necessary optimality conditions were given in [13] by combining the value function and KKT condition of the lower level problem in the constraints. In [14], a numerical method for solving nonlinear multiobjective

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bilevel problems without convexity assumptions was given. There are also many interesting applications of multiobjective bilevel optimization problems, such as medical engineering ([14]), city bus transportation system financed by the public authorities illustrated in [21] and water resources optimal allocation [22], etc.

In this paper we consider the following multiobjective bilevel programming problem with an extremal-value function

$$\begin{aligned}(P)\qquad \mathrm{v}-\min\quad & F(x,v(x)),\\ \text{s.t.}\quad & x\in X,\\ & G(x,v(x))\leq_{R^q_+} 0,\end{aligned}$$

where $v(x)$ is the optimal value of the lower level problem

$$(P_x)\quad \min_{y\in A} f(x,y)$$

and $F=(F_1,\dots,F_p)^T: R^n\times R\to R^p$, $G=(G_1,\dots,G_q)^T: R^n\times R\to R^q$, $f: R^n\times R^m\to R$, $A\subset R^m$ is a nonempty compact convex set, $X\subset R^n$ is nonempty convex set. We always assume that F_i $(i=1,\dots,p)$ and G_j $(j=1,\dots,q)$ are convex functions and R^{n+1}_+-increasing, and that f is a convex function.

Despite some duality results having been obtained in [10,12] for single objective bilevel programming problems, and an increasing amount attention having been paid to the multiobjective bilevel programming problems, there are not many papers studying the duality theory for multiobjective bilevel programming problems. Thias paper aims at developing the duality theory for the above multiobjective bilevel optimization problem by extending the approach in [12]. First, we associate the primal problem (P) to a scalar problem by using a parameter $\lambda\in R^p$. To the scalar problem we derive the duality results and some optimality conditions for it by using the conjugate duality approach [15–19]. Then a dual problem to (P) is constructed, and some duality results are proved.

This paper is organized as follows. In Section 2 we recall some notations, definitions and some well-known results. In Section 3 we construct a dual problem to (P) by using the scalar method, and prove the weak and strong duality theorems between the primal problem and its dual problem. In Section 4, we give an example to illustrate the strong duality assertion and propose a practical model which may be solved by using our results.

2. Preliminary

In this section, we recall some notations and some known facts which can be found in [20].

For any $x,y\in R^n$, we define $x\leq_{R^n_+} y$ (or $y\geq_{R^n_+} x$) if $y-x\in R^n_+$. Let X be a subset of R^n. The relative interior of the set X is denoted by ri$X$. The support function σ_X:$R^n\to\overline{R}=R\cup\{\pm\infty\}$ of X is defined by $\sigma_X(x^*)=\sup_{x\in X}{x^*}^Tx$. The indicator function δ_X: $R^n\to\overline{R}$ of X is defined by $\delta_X(x)=0$ if $x\in X$, and $\delta_X(x)=+\infty$ if $x\notin X$.

For a function $g: R^n\to\overline{R}$, the effective domain of g is given by $\mathrm{dom}(g)=\{x\in R^n: g(x)<$

$+\infty\}$. We say that g is proper if $\mathrm{dom}(g) \neq \emptyset$ and $g(x) > -\infty$ for all $x \in R^n$, and that g is R^n_+-increasing if $y - x \in R^n_+$ implies $g(x) \leq g(y)$. The conjugate function $g^*_X : R^n \to \overline{R}$ of g relative to the set X is defined by $g^*_X(x^*) = \sup\{x^{*T}x - g(x) : x \in X\}$. Note that if $X = R^n$, the conjugate function of g relative to the set X is just the (Fenchel-Moreau) conjugate function of g, denoted by g^*. By definition, the Young-Fenchel inequality holds:

$$g^*(x^*) + g(x) \geq x^{*T}x, \quad \text{for all} \quad x^*, x \in R^n.$$

It is well known that for a nonnegative real number λ,

$$(\lambda g)^*(x^*) = \begin{cases} \lambda g^*(\frac{x^*}{\lambda}), & \text{if } \lambda > 0, \\ \delta_{\{0\}}(x^*), & \text{if } \lambda = 0. \end{cases}$$

In this paper we adopt the conventions that $0 \times (\pm\infty) = 0$ and $a \times (\pm\infty) = \pm\infty$ for all $a > 0$ as in [20].

Lemma 2.1 ([20]) *Let $g_i : R^n \to \overline{R}$ $(i = 1, \ldots, m)$ be proper convex functions. If $\bigcap_{i=1}^m \mathrm{ri}(\mathrm{dom}(g_i))$ is nonempty, then*

(i) $(\sum_{i=1}^m g_i)^*(x^*) = \inf\{\sum_{i=1}^m g_i^*(x_i^*) : x^* = \sum_{i=1}^m x_i^*\}$;

(ii) for all $x^* \in R^n$, the infimum in (i) is attained.

Lemma 2.2 ([16]) *Let $h = (h_1, \ldots, h_n)^T$ with $h_i : R^m \to R$ $(i = 1, \ldots, n)$ be convex functions, and $g : R^n \to \overline{R}$ be proper convex and R^n_+-increasing function. If $h(\cap_{i=i}^n \mathrm{dom} h_i) \cap \mathrm{int}(\mathrm{dom} g) \neq \emptyset$, then*

$$(g \circ h)^*(x^*) = \inf_{r \in R^n_+} \Big\{g^*(r) + (\sum_{i=1}^n r_i h_i)^*(x^*)\Big\},$$

where for any $x^* \in R^m$ the infimum is attained.

For the multiobjective bilevel optimization problem (P), some definitions of solution can be introduced, such as feasible solution, efficient solution, properly efficient solution and weakly efficient solution.

Definition 2.3 *An element $\bar{x} \in X$ is said to be a feasible solution of (P), if $v(\bar{x})$ is the optimal value of the lower level problem $(P_{\bar{x}})$ and $G(\bar{x}, v(\bar{x})) \leq_{R^q_+} 0$.*

We will denote the set of the feasible solution of (P) as Ω, that is $\Omega = \{x \in X | G(x, v(x)) \leq_{R^q_+} 0$ and $v(x)$ is the optimal value of $(P_x)\}$.

Definition 2.4 *An element $\bar{x} \in \Omega$ is said to be an efficient solution of the problem (P), if there is no $x \in \Omega$ such that $F(x, v(x)) \leq_{R^p_+} F(\bar{x}, v(\bar{x}))$ with $F_i(x, v(x)) < F_i(\bar{x}, v(\bar{x}))$ for some $i \in \{1, \ldots, p\}$.*

Definition 2.5 *An element $\bar{x} \in \Omega$ is said to be a properly efficient solution of the problem (P), if there exists $\lambda = (\lambda_1, \ldots, \lambda_p) \in \mathrm{int} R^p_+$ such that $\lambda^T F(\bar{x}, v(\bar{x})) \leq \lambda^T F(x, v(x))$ for all $x \in \Omega$.*

Let us notice that any properly efficient solution turns out to be an efficient solution.

3. Main results

In this section, we first investigate the following scalar optimization problem.

$$
\begin{aligned}
(P^\lambda) \qquad & \inf \quad \lambda^T F(x, v(x)), \\
& \text{s.t.} \quad x \in X, \\
& \qquad\quad G(x, v(x)) \leq_{R^q_+} 0,
\end{aligned}
$$

where $\lambda \in R^p_+ \backslash \{0\}$ is a fixed vector and $v(x)$ is the optimal value of the lower level problem

$$(P_x) \quad \inf_{y \in A} f(x, y).$$

For the case of the above scalar bilevel optimization problem without the constraint function G, the Fenchel-Lagrange duality results have been obtained in [12]. In the following, we present the duality results for (P^λ).

By using the technique in [12], let function $h : R^n \to R^{n+1}$ be defined as

$$h(x) = (h_1(x), \ldots, h_n(x), h_{n+1}(x))^T = (x_1, \ldots, x_n, v(x))^T,$$

where $v(x)$ is the optimal value of lower level problem (P_x). We can rewrite the problem (P^λ) as

$$
\begin{aligned}
(P^\lambda) \qquad & \inf \quad \lambda^T F(h(x)), \\
& \text{s.t.} \quad x \in X, \\
& \qquad\quad G(h(x)) \leq_{R^q_+} 0.
\end{aligned}
$$

Thus the problem (P^λ) can be regarded as a composite programming problem which is studied in many papers such as [16,17]. By using the similar method to that of in [12,16,17], we can construct the following Lagrangian dual problem to the scalar problem (P^λ).

$$(D^\lambda) \quad \sup_{r \in R^q_+} \inf_{x \in R^n} \Big\{ (\lambda^T F + r^T G)(h(x)) + \delta_X(x) \Big\}.$$

Since f is a convex function, we know that f is continuous [21, Corollary 10.1.1], and so the marginal function $v(x)$ is a finite convex function. Hence, h_i is a finite convex function for every $i \in \{1, \ldots, n+1\}$. On the other hand, $\lambda \in R^p_+ \backslash \{0\}$ and $r \in R^q_+$ imply that the function $\lambda^T F + r^T G$ is finite valued convex and R^p_+-increasing. Then, it follows from Lemmas 2.1 and 2.2 that the dual (D^λ) can be rewritten as

$$(D^\lambda) \quad \sup_{(r,u,s,t) \in Y^\lambda} \Big\{ -\sum_{i=1}^p \lambda_i F_i^*(u_i) - (r^T G)^*(s - \sum_{i=1}^p \lambda_i u_i) - (s^T h)^*(t) - \sigma_X(-t) \Big\},$$

where $Y^\lambda = \{(r, s, t, u) : r \in R^q_+, s \in R^{n+1}_+, t \in R^n, u = (u_1, \ldots, u_p)^T, u_i \in R^{n+1}, i = 1, \ldots, p\}$.

The optimal values of the problem (P^λ) and (D^λ) are denoted as $\text{val}(P^\lambda)$ and $\text{val}(D^\lambda)$, respectively. It is obvious that

$$\text{val}(P^\lambda) \geq \text{val}(D^\lambda). \tag{1}$$

In order to obtain the optimility conditions of (P^λ), the following generalized interior point constraint qualification is required:

$$(CQ)\quad \exists \bar{x} \in \mathrm{ri}X \text{ such that } \begin{cases} G_i(h(\bar{x})) \leq 0, & \text{if } i \in L, \\ G_i(h(\bar{x})) < 0, & \text{if } i \in N, \end{cases}$$

where $L = \{i \in \{1, \ldots, q\} : G_i \circ h \text{ is an affine function }\}$ and $N = \{1, \ldots, q\} \setminus L$.

For any $\lambda \in R^p_+ \setminus \{0\}$, let $I_\lambda = \{i \in \{1, \ldots, p\} : \lambda_i > 0\}$. Using the similar method of Theorem 3.3 in [17], we can get the following necessary and sufficient optimality conditions for the problem (P^λ) and its dual (D^λ).

Theorem 3.1 *(a) Let $\lambda \in R^p_+ \backslash \{0\}$ be chosen arbitrarily. If $\bar{x} \in X$ is an optimal solution to (P^λ) and the constraint qualification (CQ) is fulfilled, then there exists an optimal solution $(\bar{r}, \bar{s}, \bar{t}, \bar{u}) \in Y^\lambda$ to the dual problem (D^λ), such that*

(i) $F_i(h(\bar{x})) + F_i^*(\bar{u}_i) = \bar{u}_i^T h(\bar{x}), \forall i \in I_\lambda,$

(ii) $\bar{r}^T G(h(\bar{x})) + (\bar{r}^T G)^*(\bar{s} - \sum_{i=1}^p \lambda_i \bar{u}_i) = (\bar{s} - \sum_{i=1}^p \lambda_i \bar{u}_i)^T h(\bar{x}),$

(iii) $\bar{s}^T h(\bar{x}) + (\bar{s}^T h)^*(\bar{t}) = \bar{x}^T \bar{t},$

(iv) $\sigma_X(-\bar{t}) = -\bar{x}^T \bar{t},$

(v) $\bar{r}^T G(h(\bar{x})) = 0.$

(b) For a given $\lambda \in R^p_+ \backslash \{0\}$, assume that $\bar{x} \in \Omega$ and $(\bar{r}, \bar{s}, \bar{t}, \bar{u}) \in Y^\lambda$ satisfy the condition $(i) - (v)$. Then $\bar{x}$ is an optimal solution to (P^λ), $(\bar{r}, \bar{s}, \bar{t}, \bar{u})$ is an optimal solution to (D^λ) and $\mathrm{val}(P^\lambda) = \mathrm{val}(D^\lambda)$.

From the duality results developed above for the scalar problem, we can introduce the following multiobjective dual problem to (P) and prove some duality results with respect to properly efficient solution of (P).

$$(D)\quad \mathrm{v} - \max_{(r,s,t,u,\lambda,\alpha) \in \Pi} H(r, s, t, u, \lambda, \alpha),$$

where

$$\Pi = \Big\{(r, s, t, u, \lambda, \alpha) : r \in R^q_+, s \in R^{n+1}_+, t \in R^n, u = (u_1, \ldots, u_p)^T, u_i \in R^{n+1}, \\ i = 1, \ldots, p, \lambda \in \mathrm{int} R^p_+, \alpha = (\alpha_1, \ldots, \alpha_p) \in R^p, \sum_{i=1}^p \lambda_i \alpha_i = 0\Big\},$$

and

$$H(r, s, t, u, \lambda, \alpha) = \begin{pmatrix} H_1(r, s, t, u, \lambda, \alpha) \\ \vdots \\ H_p(r, s, t, u, \lambda, \alpha) \end{pmatrix}$$

with $H_i(r, s, t, u, \lambda, \alpha) = -F_i^*(u_i) - \frac{1}{p\lambda_i}((r^T G)^*(s - \sum_{i=1}^p \lambda_i u_i) + (s^T h)^*(t) + \sigma_X(-t)) + \alpha_i.$

The efficient solution for (D) can be defined in a similar manner as for (P).

Definition 3.2 *An element $(\bar{r}, \bar{s}, \bar{t}, \bar{u}, \bar{\lambda}, \bar{\alpha}) \in \Pi$ is said to be an efficient solution of the problem (D), if there is no $(r, s, t, u, \lambda, \alpha) \in \Pi$ such that $H(r, s, t, u, \lambda, \alpha) \geq_{R^p_+} H(\bar{r}, \bar{s}, \bar{t}, \bar{u}, \bar{\lambda}, \bar{\alpha})$ with $H_i(r, s, t, u, \lambda, \alpha) > H_i(\bar{r}, \bar{s}, \bar{t}, \bar{u}, \bar{\lambda}, \bar{\alpha})$ for some $i \in \{1, \ldots, p\}$.*

Now, we present the duality results between the vector problem (P) and (D).

Theorem 3.3 (weak duality) *There are no $x \in \Omega$ and $(r, s, t, u, \lambda, \alpha) \in \Pi$ such that $F(h(x)) \leq_{R^p_+} H(r, s, t, u, \lambda, \alpha)$ and $F_i(h(x)) < H_i(r, s, t, u, \lambda, \alpha)$ for at least one $i \in \{1, \dots, p\}$.*

Proof Let us suppose the contrary. Then there exist $x \in \Omega$ and $(r, s, t, u, \lambda, \alpha) \in \Pi$ satisfying that $F(h(x)) \leq_{R^p_+} H(r, s, t, u, \lambda, \alpha)$ and $F_i(h(x)) < H_i(r, s, t, u, \lambda, \alpha)$ for at least one $i \in \{1, \dots, p\}$. This implies

$$\lambda^T F(h(x)) < \lambda^T H(r, s, t, u, \lambda, \alpha). \tag{2}$$

On the other hand,

$$\begin{aligned}
\lambda^T H(r, s, t, u, \lambda, \alpha) &= \sum_{i=1}^{p} \lambda_i H_i(r, s, t, u, \lambda, \alpha) \\
&= -\sum_{i=1}^{p} \lambda_i F_i^*(u_i) - \sum_{i=1}^{p} \lambda_i \frac{1}{p\lambda_i} \Big((r^T G)^*(s - \sum_{i=1}^{p} \lambda_i u_i) + (s^T h)^*(t) + \sigma_X(-t) \Big) + \sum_{i=1}^{p} \lambda_i \alpha_i \\
&= -\sum_{i=1}^{p} \lambda_i F_i^*(u_i) - (r^T G)^*(s - \sum_{i=1}^{p} \lambda_i u_i) - (s^T h)^*(t) - \sigma_X(-t).
\end{aligned}$$

It follows from (1) that $\lambda^T F(h(x)) \geq \lambda^T H(r, s, t, u, \lambda, \alpha)$, which contradicts (2). $\square$

Theorem 3.4 (strong duality) *Assume that $\bar{x} \in \Omega$ is a properly efficient solution to (P) and that the constraint qualification (CQ) holds. Then there exists an efficient solution $(\bar{r}, \bar{s}, \bar{t}, \bar{u}, \bar{\lambda}, \bar{\alpha}) \in \Pi$ to the dual problem (D) such that $F(h(\bar{x})) = H(\bar{r}, \bar{s}, \bar{t}, \bar{u}, \bar{\lambda}, \bar{\alpha})$.*

Proof Since $\bar{x} \in \Omega$ is a properly efficient solution to the problem (P), there exists a vector $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_p) \in \operatorname{int} R^p_+$ such that $\bar{x}$ is an optimal solution to the scalar problem $(P^{\bar{\lambda}})$. By using Theorem 3.1, there exists an optimal solution $(\bar{r}, \bar{s}, \bar{t}, \bar{u}) \in Y^{\bar{\lambda}}$ to the dual problem $(D^{\bar{\lambda}})$ such that the optimality conditions $(i) - (v)$ are fulfilled. Further let $\bar{\alpha} = (\bar{\alpha}_1, \dots, \bar{\alpha}_p) \in R^p$ as

$$\bar{\alpha}_i = \frac{1}{p\bar{\lambda}_i} \Big((\bar{r}^T G)^*(\bar{s} - \sum_{i=1}^{p} \bar{\lambda}_i \bar{u}_i) + (\bar{s}^T h)^*(\bar{t}) + \sigma_X(-\bar{t}) \Big) + \bar{u}_i^T h(\bar{x}).$$

From the optimal condition in Theorem 3.1, we have

$$\sum_{i=1}^{p} \bar{\lambda}_i \bar{\alpha}_i = (\bar{r}^T G)^*(\bar{s} - \sum_{i=1}^{p} \bar{\lambda}_i \bar{u}_i) + (\bar{s}^T h)^*(\bar{t}) + \sigma_X(-\bar{t}) + \sum_{i=1}^{p} \bar{\lambda}_i \bar{u}_i^T h(\bar{x}) = 0.$$

This shows that $(\bar{r}, \bar{s}, \bar{t}, \bar{u}, \bar{\lambda}, \bar{\alpha}) \in \Pi$.

Using the assertions (i)–(v) of Theorem 3.1, we get for $i = 1, \dots, p$, $H_i(\bar{r}, \bar{s}, \bar{t}, \bar{u}, \bar{\lambda}, \bar{\alpha}) = -F_i^*(\bar{u}_i) - \frac{1}{p\bar{\lambda}_i}((\bar{r}^T G)^*(\bar{s} - \sum_{i=1}^{p} \bar{\lambda}_i \bar{u}_i) + (\bar{s}^T h)^*(\bar{t}) + \sigma_X(-\bar{t})) + \bar{\alpha}_i = -F_i^*(\bar{u}_i) + \bar{u}_i h(\bar{x}) = F_i(h(\bar{x}))$. From Theorem 3.3, it follows that $(\bar{r}, \bar{s}, \bar{t}, \bar{u}, \bar{\lambda}, \bar{\alpha})$ is an efficient solution of (D) and $F(h(\bar{x})) = H(\bar{r}, \bar{s}, \bar{t}, \bar{u}, \bar{\lambda}, \bar{\alpha})$. $\square$

Note that the dual problem (D) has the conjugate function $(s^T h)^*$ which depends on the marginal function of the lower level problem. One can calculate the conjugate functions of the marginal function $v(\cdot)$ as follows [12, Proposition 5.1].

$$v^*(x^*) = f^*_{R^n \times A}(x^*, 0), \quad \forall x^* \in R^n. \tag{3}$$

Therefore, for any $s \in R_+^{n+1}$, the conjugate function $(s^T h)^*$ can be calculated as

$$\begin{aligned}(s^T h)^*(t) &= \sup_{x \in R^n} \{\langle t, x\rangle - \sum_{i=1}^{n+1} s_i h_i(x)\} \\ &= \sup_{x \in R^n} \{\langle t, x\rangle - (\sum_{i=1}^{n} s_i x_i + s_{n+1} v(x))\} \\ &= \begin{cases} s_{s+1} f^*_{R^n \times A}(\frac{t-(s_1,\ldots,s_n)}{s_{n+1}}, 0), & \text{if } s_{n+1} > 0, \\ 0, & \text{if } s_{n+1} = 0, t_i = s_i, i = 1, \ldots, n, \\ +\infty, & \text{otherwise.} \end{cases}\end{aligned} \tag{4}$$

Then the dual problem (D) can be considered as following two problems.

$$(D^1) \quad \text{v} - \max_{(r,s,t,u,\lambda,\alpha) \in \Pi^1} H^1(r, s, t, u, \lambda, \alpha),$$

where

$$\Pi^1 = \Big\{(r, s, t, u, \lambda, \alpha) : r \in R_+^q, s \in R_+^{n+1} \text{ with } s_{n+1} > 0, t \in R^n, u = (u_1, \ldots, u_p)^T, u_i \in R^{n+1}, \\ i = 1, \ldots, p, \lambda \in \text{int} R_+^p, \alpha = (\alpha_1, \ldots, \alpha_p) \in R^p, \sum_{i=1}^{p} \lambda_i \alpha_i = 0\Big\},$$

and

$$H^1(r, s, t, u, \lambda, \alpha) = \begin{pmatrix} H_1^1(r, s, t, u, \lambda, \alpha) \\ \vdots \\ H_p^1(r, s, t, u, \lambda, \alpha) \end{pmatrix}$$

with $H_i^1(r, s, t, u, \lambda, \alpha) = -F_i^*(u_i) - \frac{1}{p\lambda_i}((r^T G)^*(s - \sum_{i=1}^{p} \lambda_i u_i) + s_{n+1} f^*_{R^n \times A}(\frac{t-(s_1,\ldots,s_n)}{s_{n+1}}, 0) + \sigma_X(-t)) + \alpha_i$.

$$(D^2) \quad \text{v} - \max_{(r,s,t,u,\lambda,\alpha) \in \Pi^2} H^2(r, s, t, u, \lambda, \alpha),$$

where

$$\Pi^2 = \Big\{(r, s, t, u, \lambda, \alpha) : r \in R_+^q, s \in R_+^{n+1} \text{ with } s_{n+1} = 0, t \in R^n \text{ with } t_j = s_j, j = 1, \ldots, n, \\ u = (u_1, \ldots, u_p)^T, u_i \in R^{n+1}, i = 1, \ldots, p, \lambda \in \text{int} R_+^p, \alpha = (\alpha_1, \ldots, \alpha_p) \in R^p, \sum_{i=1}^{p} \lambda_i \alpha_i = 0\Big\},$$

and

$$H^2(r, s, t, u, \lambda, \alpha) = \begin{pmatrix} H_1^2(r, s, t, u, \lambda, \alpha) \\ \vdots \\ H_p^2(r, s, t, u, \lambda, \alpha) \end{pmatrix}$$

with $H_i^2(r, s, t, u, \lambda, \alpha) = -F_i^*(u_i) - \frac{1}{p\lambda_i}((r^T G)^*((s_1, \ldots, s_n, 0)^T - \sum_{i=1}^{p} \lambda_i u_i) + \sigma_X(-t)) + \alpha_i$.

From (3) and (4), we can easily obtain the following result.

Theorem 3.5 *An element $(\bar{r}, \bar{s}, \bar{t}, \bar{u}, \bar{\lambda}, \bar{\alpha}) \in \Pi$ is an efficient solution of problem (D) if and only* *if it is an efficient solution of problem* (D^a) *satisfying that there is no* $(r,s,t,u,\lambda,\alpha)\in\Pi^b$ *satisfying that* $H^b(r,s,t,u,\lambda,\alpha)\geq_{R^p_+} H^a(\bar r,\bar s,\bar t,\bar u,\bar\lambda,\bar\alpha)$ *with* $H^b_i(r,s,t,u,\lambda,\alpha)>H^a_i(\bar r,\bar s,\bar t,\bar u,\bar\lambda,\bar\alpha)$ *for some* $i\in\{1,\ldots,p\}$*, where* $a,b\in\{1,2\}$ *and* $a\neq b$.

Proof By using (4) one can see that for any $(r,s,t,u,\lambda,\alpha)\in\Pi$,

$$H_i(r,s,t,u,\lambda,\alpha)=\begin{cases} H^1_i(r,s,t,u,\lambda,\alpha), & \text{if } (r,s,t,u,\lambda,\alpha)\in\Pi^1,\\ H^2_i(r,s,t,u,\lambda,\alpha), & \text{if } (r,s,t,u,\lambda,\alpha)\in\Pi^2,\\ -\infty, & \text{otherwise.}\end{cases}$$

Therefore, the conclusion is fulfilled from the definition of efficient solution. □

Notice that the objective functions of the problems (D^1) and (D^2) are only related to the functions of F, G, f and δ_X. Theorem 3.5 shows that the efficient solutions of the dual problem (D) can be considered equivalently as the efficient solutions of two single-level problems (D^1) and (D^2).

4. Example and application

In this section, we will give an example to illustrate the strong duality assertion and propose a practical model to show the applications of our results.

Let $F=(F_1,F_2)^T: R\times R\to R\times R$, $G: R\times R\to R$, $f: R\times R\to R$ be defined as follows:

$$F_1(x,t)=2x+t,\qquad F_2(x,t)=\tfrac{1}{2}x^2+x+t,$$
$$G(x,t)=x+2t,\qquad f(x,y)=-x+\tfrac{1}{2}y^2.$$

Let $X=[0,+\infty)$, $D=[0,1]$.

For any $x\in R$, we have

$$v(x)=\min_{y\in D} f(x,y)=-x.$$

Then $F_1(x,v(x))=x,\ F_2(x,v(x))=\frac{1}{2}x^2,\ G(x,v(x))=-x,\ h(x)=(x,v(x))^T=(x,-x)^T$.

The primal problem that we consider becomes

$$\begin{aligned}(P)\qquad \mathrm{v-min}\quad & F(x,v(x))=(x,\ \frac{1}{2}x^2)^T\\ \text{s.t.}\quad & x\in[0,+\infty)\\ & G(x,v(x))=-x\leq 0\end{aligned}$$

In order to construct the dual problem, let us calculate following conjugate functions.

$$F^*_1(x^*,t^*)=\begin{cases}0, & x^*=2,\ t^*=1,\\ +\infty, & \text{otherwise,}\end{cases}\qquad F^*_2(x^*,t^*)=\begin{cases}\frac{1}{2}(x^*-1)^2, & t^*=1,\\ +\infty, & \text{otherwise,}\end{cases}$$

$$(rG)^*(x^*,t^*)=\begin{cases}0, & x^*=r,\ t^*=2r,\\ +\infty, & \text{otherwise,}\end{cases}\qquad (s^Th)^*(x^*)=\begin{cases}0, & x^*=s_1-s_2,\\ +\infty, & \text{otherwise,}\end{cases}$$

$$\sigma_X(x^*)=\begin{cases}0, & x^*\leq 0,\\ +\infty, & \text{otherwise.}\end{cases}$$

Then the dual problem (D) looks like

$$(D)\quad \text{v}-\max_{(r,s,t,u,\lambda,\alpha)\in\Pi} H(r,s,t,u,\lambda,\alpha)=\text{v}-\max_{(r,s,t,u,\lambda,\alpha)\in\Pi}\begin{pmatrix} H_1(r,s,t,u,\lambda,\alpha)\\ H_2(r,s,t,u,\lambda,\alpha)\end{pmatrix},$$

where

$$H_i(r,s,t,u,\lambda,\alpha)=-F_i^*(u_i)-\frac{1}{p\lambda_i}\Big((r^TG)^*(s-\sum_{i=1}^{p}\lambda_i u_i)+(s^Th)^*(t)+\sigma_X(-t)\Big)+\alpha_i.$$

One can see that the two objective functions of the dual problem are greater than $-\infty$ only if $u_1=(2,1)^T,\ u_2=(u_2^1,1)^T,\ s-\sum_{i=1}^{2}\lambda_1 u_i=(r,2r)^T,\ t=s_1-s_2\le 0$. Therefore the dual problem (D) becomes

$$(D)\quad \text{v}-\max_{(r,s,t,u,\lambda,\alpha)\in\Pi}\begin{pmatrix} H_1(r,s,t,u,\lambda,\alpha)\\ H_2(r,s,t,u,\lambda,\alpha)\end{pmatrix},$$

where $H_1(r,s,t,u,\lambda,\alpha)=\alpha_1,\ H_2=(r,s,t,u,\lambda,\alpha)=-\frac{1}{2}(u_2^1-1)^2+\alpha_2$ and

$$\Pi=\Big\{(r,s,t,u,\lambda,\alpha): r\ge 0, s=(s_1,s_2)^T\in R_+^2, t=s_1-s_2\le 0, u=(u_1,u_2)^T, u_1=(2,1)^T,$$
$$u_2=(u_2^1,1)^T,\lambda\in \text{int}R_+^2, s-\sum_{i=1}^{2}\lambda_1 u_i=(r,2r)^T,\alpha=(\alpha_1,\alpha_2)^T\in R^2,\sum_{i=1}^{2}\lambda_i\alpha_i=0\Big\}.$$

We see that $\bar{x}=0$ is a properly efficient solution to the primal problem (P), and that the condition (CQ) is fulfilled. By the Theorem 3.4, there exists an efficient solution $(\bar{r},\bar{s},\bar{t},\bar{u},\bar{\lambda},\bar{\alpha})\in\Pi$ to the dual problem (D) such that $F(h(\bar{x}))=H(\bar{r},\bar{s},\bar{t},\bar{u},\bar{\lambda},\bar{\alpha})$. In fact, we can find the efficient solution of dual problem (D) as follows.

$$\bar{r}=1,\bar{s}=(4,4)^T,\bar{t}=0,\bar{u}=(\bar{u}_1,\bar{u}_2)^T=\begin{pmatrix}(2,1)^T\\(1,1)^T\end{pmatrix},\bar{\lambda}=(1,1)^T,\bar{\alpha}=(0,0)^T.$$

The multiobjective bilevel programming problem has been studied in water resources optimal allocation [22], regarding the optimal income of the society and optimal water quality as the upper level decision makers; and the optimal income of using water as the lower level decision maker. If we assume that the water amount of each user is the same and regard the optimal quantity of water as the lower level decision maker, then we deduce the following multiobjective bilevel programming problem

$$\begin{array}{lll}(P) & \text{v}-\min_{r_i,w,t} & (F_1(r_i,w,t,v(r_i,w,t)),F_2(r_i,w,t,v(r_i,w,t)))\\ & \text{s.t.} & \displaystyle\sum_{i=1}^{n} r_i,+w=Q, nv+w\le Q\\ & & r_i\ge r, w\ge\alpha,\beta\le t\le\gamma\end{array}$$

$v(r_i,w,t)$ is the optimal value of the lower level problem

$$(P_v)\quad \min_y g(r_i,w,t,y)$$

where $-F_1(r_i,w,t,v(r_i,w,t))=-h(w)-\sum_{i=1}^{n}[f(v)+s_i(d_1-v)+a(r_i-v)-b\sum_{j=1}^{n}(r_j-v)(r_i-v)]$ is the society income function; $F_2(r_i,w,t,v(r_i,w,t))=\sum_{i=1}^{n}0.01d_ip_iv$ is the function of water quality. $g(r_i, w, t, y)$ is the water withdrawal function of each user, which is always a convex or linear function.

For solving such a problem it is always converted to a single level multiobjective problem by using the Karush-Kuhn-Tucker optimality conditions of the lower level problem [13,14,22]. According to the duality results obtained in this paper, some other methods may be provided to solve the multiobjective bilevel programming problems.

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