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Some Equalities and Inequalities for the Hermitian Moore-Penrose Inverse of Triple Matrix Product with Applications

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Abstract We investigate relationships between the Moore-Penrose inverse $(ABA^*)^\dagger$ and the product $[(AB)^{(1,2,3)}]^*B(AB)^{(1,2,3)}$ through some rank and inertia formulas for the difference of $(ABA^*)^\dagger - [(AB)^{(1,2,3)}]^*B(AB)^{(1,2,3)}$, where B is Hermitian matrix and $(AB)^{(1,2,3)}$ is a $\{1, 2, 3\}$-inverse of AB. We show that there always exists an $(AB)^{(1,2,3)}$ such that $(ABA^*)^\dagger = [(AB)^{(1,2,3)}]^*B(AB)^{(1,2,3)}$ holds. In addition, we also establish necessary and sufficient conditions for the two inequalities $(ABA^*)^\dagger \succ [(AB)^{(1,2,3)}]^*B(AB)^{(1,2,3)}$ and $(ABA^*)^\dagger \prec [(AB)^{(1,2,3)}]^*B(AB)^{(1,2,3)}$ to hold in the Löwner partial ordering. Some variations of the equalities and inequalities are also presented. In particular, some equalities and inequalities for the Moore-Penrose inverse of the sum $A + B$ of two Hermitian matrices A and B are established.

Keywords Moore-Penrose inverse; reverse-order law; rank; inertia; Löwner partial ordering

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1. Introduction

It is well known that a fundamental work in matrix theory is to establish equalities and inequalities between matrices. Assume that A, B and C are three matrices such that the product ABC is defined. If the triple matrices are nonsingular, then AB and ABC are nonsingular as well, and the standard inverses of AB and ABC satisfy the equalities $(AB)^{-1} = B^{-1}A^{-1}$ and $(ABC)^{-1} = (BC)^{-1}B(AB)^{-1} = C^{-1}B^{-1}A^{-1}$. These equalities are called the reverse-order laws for inverse operations of matrix products in matrix theory. If A, B and C are singular, we can use generalized inverses of the matrices instead of standard inverses of the matrices. In this case, various equalities for generalized inverses of matrix products, called reverse-order laws for generalized inverses, may hold as well. Motivated by $(ABC)^{-1} = (BC)^{-1}B(AB)^{-1}$, it was shown in [6] that there always exist two generalized inverses $(AB)^{(1,2,3)}$ and $(BC)^{(1,2,4)}$ such that the Moore-Penrose inverse of ABC can be expressed as

$$(ABC)^\dagger = (BC)^{(1,2,4)}B(AB)^{(1,2,3)}. \tag{1.1}$$

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This equality in fact establishes a decomposition for the Moore-Penrose inverse of ABC, and can be applied to analyze properties of $(ABC)^{\dagger}$. In this paper, we consider the triple product ABA^*, with $B = B^*$, and show that the following reverse-order law

$$(ABA^*)^{\dagger} = [(AB)^{(1,2,3)}]^* B (AB)^{(1,2,3)} \tag{1.2}$$

always holds. We also give identifying conditions for the following matrix inequalities

$$(ABA^*)^{\dagger} \succ [(AB)^{(1,2,3)}]^* B (AB)^{(1,2,3)} \quad \text{and} \quad (ABA^*)^{\dagger} \prec [(AB)^{(1,2,3)}]^* B (AB)^{(1,2,3)} \tag{1.3}$$

to hold in the Löwner partial ordering.

As formulated in Lemma 2.1 below, we can use ranks and inertias of matrices to prove possible equalities and inequalities. In this paper, we shall first establish a group of formulas for calculating the maximal and minimal values of

$$\operatorname{rank}\{(ABA^*)^{\dagger} - [(AB)^{(1,2,3)}]^* B (AB)^{(1,2,3)}\}, \tag{1.4}$$

$$\operatorname{inertia}\{(ABA^*)^{\dagger} - [(AB)^{(1,2,3)}]^* B (AB)^{(1,2,3)}\}. \tag{1.5}$$

We then utilize the formulas to characterize (1.2) and (1.3). Some variations of (1.2) and (1.3) are also presented. In particular, a group of equalities and inequalities for the Moore-Penrose inverse of the sum $A + B$ of two Hermitian matrices are established.

Before proceeding, we introduce the notation used in this paper. We use $\mathbb{C}^{m\times n}$ and $\mathbb{C}_{\mathrm{H}}^{m}$ to denote the sets of all complex $m \times n$ matrices and all complex Hermitian $m \times m$ matrices, respectively. The symbols A^*, $r(A)$, and $\mathscr{R}(A)$ stand for the conjugate transpose, the rank, and the range (column space) of a matrix $A \in \mathbb{C}^{m\times n}$, respectively. The Moore-Penrose inverse of $A \in \mathbb{C}^{m\times n}$ is defined to be the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the four equations

$$\text{(i)}\ \ AXA = A, \quad \text{(ii)}\ \ XAX = X, \quad \text{(iii)}\ \ (AX)^* = AX, \quad \text{(iv)}\ \ (XA)^* = XA,$$

and is denoted by $A^{\dagger}$. Further let E_A and F_A stand for the two orthogonal projectors $E_A = I_m - AA^{\dagger}$, and $F_A = I_n - A^{\dagger}A$ induced by A, and both $E_{A^*} = F_A$ and $F_{A^*} = E_A$ always hold. A matrix X is called a g-inverse of A, denoted by A^-, if it satisfies (i); called an $\{i, \dots, j\}$-inverse of A, denoted by $A^{(i,\dots,j)}$, if it satisfies the ith,$\dots$, jth equations. For $A \in \mathbb{C}_{\mathrm{H}}^{m}$, the symbols $i_+(A)$ and $i_-(A)$ stand for the number of the positive and negative eigenvalues of A counted with multiplicities, respectively, where $r(A) = i_+(A) + i_-(A)$. The notation $A \succcurlyeq 0$ ($A \succ 0$) means that A is Hermitian positive semi-definite (positive definite). Two A, $B \in \mathbb{C}_{\mathrm{H}}^{m}$ are said to satisfy the inequality $A \succcurlyeq B$ ($A \succ B$) in the Löwner partial ordering if $A - B$ is Hermitian positive semi-definite (positive definite).

We use the following rank and inertia expansion formulas for calculating (1.4) and (1.5).

Lemma 1.1 ([3]) *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, and $C \in \mathbb{C}^{l\times n}$. Then*

$$r[\,A,\ B\,] = r(A) + r(E_A B) = r(B) + r(E_B A), \tag{1.6}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C), \tag{1.7}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_B A F_C). \tag{1.8}$$

The results on ranks and inertias of matrices in the following lemma are obvious or well known (see also [4] for their references).

Lemma 1.2 *Let $A \in \mathbb{C}_{\mathrm{H}}^{m}$, $B \in \mathbb{C}_{\mathrm{H}}^{n}$, $B \in \mathbb{C}_{\mathrm{H}}^{n}$, and $Q \in \mathbb{C}^{m\times n}$, and assume that $P \in \mathbb{C}^{m\times m}$ is nonsingular. Then*

$$i_{\pm}(PAP^{*}) = i_{\pm}(A), \tag{1.9}$$

$$i_{\pm}(A^{\dagger}) = i_{\pm}(A), \quad i_{\pm}(-A) = i_{\mp}(A), \tag{1.10}$$

$$i_{\pm}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = i_{\pm}(A) + i_{\pm}(B), \tag{1.11}$$

$$i_{+}\begin{bmatrix} 0 & Q \\ Q^{*} & 0 \end{bmatrix} = i_{-}\begin{bmatrix} 0 & Q \\ Q^{*} & 0 \end{bmatrix} = r(Q). \tag{1.12}$$

Lemma 1.3 ([4]) *Let $A \in \mathbb{C}_{\mathrm{H}}^{m}$, $B \in \mathbb{C}^{m\times n}$, and $D \in \mathbb{C}_{\mathrm{H}}^{n}$. Then*

$$i_{\pm}\begin{bmatrix} A & B \\ B^{*} & 0 \end{bmatrix} = r(B) + i_{\pm}(E_{B}AE_{B}), \tag{1.13}$$

$$i_{\pm}\begin{bmatrix} A & B \\ B^{*} & D \end{bmatrix} = i_{\pm}(A) + i_{\pm}\begin{bmatrix} 0 & E_{A}B \\ B^{*}E_{A} & D - B^{*}A^{\dagger}B \end{bmatrix}. \tag{1.14}$$

Theorem 1.4 ([5]) *Let*

$$f(X) = \phi(X;\, A,\, B,\, C,\, D,\, M) = (\, AXB + C\,)M(\, AXB + C)^{*} + D, \tag{1.15}$$

where $A \in \mathbb{C}^{n\times p}$, $B \in \mathbb{C}^{m\times q}$, $C \in \mathbb{C}^{n\times q}$, $D \in \mathbb{C}_{\mathrm{H}}^{n}$, and $M \in \mathbb{C}_{\mathrm{H}}^{q}$ are given, and $X \in \mathbb{C}^{p\times m}$ is a variable matrix. Also, define

$$N_1 = \begin{bmatrix} D + CMC^{*} & A \\ A^{*} & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} D + CMC^{*} & CMB^{*} & A \\ A^{*} & 0 & 0 \end{bmatrix},$$

$$N_3 = \begin{bmatrix} D + CMC^{*} & CMB^{*} \\ BMC^{*} & BMB^{*} \end{bmatrix}, \quad N_4 = \begin{bmatrix} D + CMC^{*} & CMB^{*} & A \\ BMC^{*} & BMB^{*} & 0 \end{bmatrix}.$$

Then, the global maximal and minimal ranks and inertias of $f(X)$ are given by

$$\max_{X\in\mathbb{C}^{p\times m}} r[\, f(X)\,] = \min\{\, r[\, D + CMC^{*},\, CMB^{*},\, A\,], \quad r(N_1), \quad r(N_3)\,\}\,, \tag{1.16}$$

$$\min_{X\in\mathbb{C}^{p\times m}} r[\, f(X)\,] = 2r[\, D + CMC^{*},\, CMB^{*},\, A\,] + \max\{\, s_1, \quad s_2, \quad s_3, \quad s_4\,\}, \tag{1.17}$$

$$\max_{X\in\mathbb{C}^{p\times m}} i_{\pm}[\, f(X)\,] = \min\{\, i_{\pm}(N_1), \quad i_{\pm}(N_3)\,\}\,, \tag{1.18}$$

$$\min_{X\in\mathbb{C}^{p\times m}} i_{\pm}[\, f(X)\,] = r[\, D{+}CMC^{*},\, CMB^{*},\, A\,]{+}\max\{i_{\pm}(N_1) - r(N_2),\, i_{\pm}(N_3) - r(N_4)\}\,, \tag{1.19}$$

where $s_1 = r(N_1) - 2r(N_2)$, $s_2 = r(N_3) - 2r(N_4)$, $s_3 = i_{+}(N_1) + i_{-}(N_3) - r(N_2) - r(N_4)$, and $s_4 = i_{-}(N_1) + i_{+}(N_3) - r(N_2) - r(N_4)$.

2. Main results

The assertions in the following lemma arise directly from the definitions of rank and inertia of matrix.

Lemma 2.1 *Let $\mathcal{S}$ be a set of $\mathbb{C}_{\mathrm{H}}^{m}$. Then, the following results hold.*

(a) $0 \in \mathcal{S}$ *if and only if* $\min_{X\in\mathcal{S}} r(X) = 0$.

(b) $\mathcal{S}$ *has a nonsingular matrix if and only if* $\max_{X\in\mathcal{S}} r(X) = m$.

(c) $\mathcal{S}$ *has a matrix* $X \succ 0$ $(X \prec 0)$ *if and only if* $\max_{X\in\mathcal{S}} i_{+}(X) = m$ $(\max_{X\in\mathcal{S}} i_{-}(X) = m)$.

(d) *All* $X \in \mathcal{S}$ *satisfy* $X \succ 0$ $(X \prec 0)$ *if and only if* $\min_{X\in\mathcal{S}} i_{+}(X) = m$ $(\min_{X\in\mathcal{S}} i_{-}(X) = m)$.

(e) $\mathcal{S}$ *has a matrix* $X \succcurlyeq 0$ $(X \preccurlyeq 0)$ *if and only if* $\min_{X\in\mathcal{S}} i_{-}(X) = 0$ $(\min_{X\in\mathcal{S}} i_{+}(X) = 0)$.

(f) *All* $X \in \mathcal{S}$ *satisfy* $X \succcurlyeq 0$ $(X \preccurlyeq 0)$ *if and only if* $\max_{X\in\mathcal{S}} i_{-}(X) = 0$ $(\max_{X\in\mathcal{S}} i_{+}(X) = 0)$.

This lemma shows that if certain expansion formulas for calculating ranks and inertias of differences of matrices are established, they can be used to characterize the corresponding matrix equalities and inequalities.

The main result of this paper is given below.

Theorem 2.2 *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}_{\mathrm{H}}^{n}$. *Then, the following equalities hold*

$$\max_{(AB)^{(1,2,3)}} r\Big\{ (ABA^{*})^{\dagger} - [(AB)^{(1,2,3)}]^{*}B(AB)^{(1,2,3)} \Big\} = \min\{\, r(AB), \quad r(B) - r(ABA^{*}) \,\}, \tag{2.1}$$

$$\min_{(AB)^{(1,2,3)}} r\Big\{ (ABA^{*})^{\dagger} - [(AB)^{(1,2,3)}]^{*}B(AB)^{(1,2,3)} \Big\} = 0, \tag{2.2}$$

$$\max_{(AB)^{(1,2,3)}} i_{\pm}\Big\{ (ABA^{*})^{\dagger} - [(AB)^{(1,2,3)}]^{*}B(AB)^{(1,2,3)} \Big\} = \min\{\, r(AB),\ i_{\mp}(B) - i_{\mp}(ABA^{*}) \}, \tag{2.3}$$

$$\min_{(AB)^{(1,2,3)}} i_{\pm}\Big\{ (ABA^{*})^{\dagger} - [(AB)^{(1,2,3)}]^{*}B(AB)^{(1,2,3)} \Big\} = 0. \tag{2.4}$$

Proof It is well known that the general expression of $A^{(1,2,3)}$ can be written as

$$A^{(1,2,3)} = A^{\dagger} + F_{A}VAA^{\dagger}, \tag{2.5}$$

where V is arbitrary; see, e.g., [1,2]. So that $(ABA^{*})^{\dagger} - [(AB)^{(1,2,3)}]^{*}B(AB)^{(1,2,3)}$ can be written as

$$\begin{aligned} f(V^{*}) &= (ABA^{*})^{\dagger} - [(AB)^{(1,2,3)}]^{*}B(AB)^{(1,2,3)} \\ &= (ABA^{*})^{\dagger} - [\,(AB)^{\dagger} + F_{AB}VAB(AB)^{\dagger}\,]^{*}B[\,(AB)^{\dagger} + F_{AB}VAB(AB)^{\dagger}\,] \\ &= (ABA^{*})^{\dagger} + \{\,[(AB)^{\dagger}]^{*} + AB(AB)^{\dagger}V^{*}F_{AB}\,\}(-B)\{\,[(AB)^{\dagger}]^{*} + AB(AB)^{\dagger}V^{*}F_{AB}\,\}^{*}, \end{aligned} \tag{2.6}$$

which is a special case of (1.15). Substituting the given matrices in (2.6) into (1.16)–(1.19) and simplifying by

$$\mathscr{R}[(ABA^{*})^{\dagger}] = \mathscr{R}(ABA^{*}) \subseteq \mathscr{R}(AB) = \mathscr{R}[AB(AB)^{\dagger}] \quad \text{and} \quad \mathscr{R}([(AB)^{\dagger}]^{*}) = \mathscr{R}(AB),$$

and Lemmas 1.1–1.3, we first obtain

$$\begin{aligned} i_{\pm}(N_1) &= i_{\pm}\begin{bmatrix} (ABA^{*})^{\dagger} - [(AB)^{\dagger}]^{*}B(AB)^{\dagger} & AB(AB)^{\dagger} \\ AB(AB)^{\dagger} & 0 \end{bmatrix} \\ &= i_{\pm}\begin{bmatrix} 0 & AB(AB)^{\dagger} \\ AB(AB)^{\dagger} & 0 \end{bmatrix} = r(AB), \end{aligned} \tag{2.7}$$

$$i_{\pm}(N_3) = i_{\pm}\begin{bmatrix} (ABA^{*})^{\dagger} - [(AB)^{\dagger}]^{*}B(AB)^{\dagger} & [(AB)^{\dagger}]^{*}BF_{AB} \\ F_{AB}B(AB)^{\dagger} & -F_{AB}BF_{AB} \end{bmatrix}$$

$$= i_{\pm}\begin{bmatrix} (ABA^*)^\dagger - [(AB)^\dagger]^* B(AB)^\dagger & [(AB)^\dagger]^* B & 0 \\ B(AB)^\dagger & -B & (AB)^* \\ 0 & AB & 0 \end{bmatrix} - r(AB)$$

$$= i_{\pm}\begin{bmatrix} (ABA^*)^\dagger - [(AB)^\dagger]^* B(AB)^\dagger & [(AB)^\dagger]^* B & AB(AB)^\dagger \\ B(AB)^\dagger & -B & 0 \\ AB(AB)^\dagger & 0 & ABA^* \end{bmatrix} - r(AB)$$

$$= i_{\pm}\begin{bmatrix} (ABA^*)^\dagger & 0 & AB(AB)^\dagger \\ 0 & -B & 0 \\ AB(AB)^\dagger & 0 & ABA^* \end{bmatrix} - r(AB)$$

$$= i_{\pm}\begin{bmatrix} (ABA^*)^\dagger & AB(AB)^\dagger \\ AB(AB)^\dagger & ABA^* \end{bmatrix} + i_{\pm}(-B) - r(AB)$$

$$= i_{\pm}\begin{bmatrix} (ABA^*)^\dagger - AB(AB)^\dagger(ABA^*)^\dagger AB(AB)^\dagger & AB(AB)^\dagger E_{ABA^*} \\ E_{ABA^*}AB(AB)^\dagger & 0 \end{bmatrix}$$

$$+ \, i_{\pm}(ABA^*) + i_{\pm}(-B) - r(AB)$$

$$= i_{\pm}\begin{bmatrix} 0 & AB(AB)^\dagger E_{ABA^*} \\ E_{ABA^*}AB(AB)^\dagger & 0 \end{bmatrix} + \, i_{\pm}(ABA^*) + i_{\mp}(B) - r(AB)$$

$$= r(E_{ABA^*}AB) + i_{\pm}(ABA^*) + i_{\mp}(B) - r(AB)$$

$$= r[\, AB, \, ABA^* \,] - i_{\mp}(ABA^*) + i_{\mp}(B) - r(AB)$$

$$= i_{\mp}(B) - i_{\mp}(ABA^*), \tag{2.8}$$

$$r(N_2) = r\begin{bmatrix} (ABA^*)^\dagger - [(AB)^\dagger]^* B(AB)^\dagger & [(AB)^\dagger]^* BF_{AB} & AB(AB)^\dagger \\ AB(AB)^\dagger & 0 & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} 0 & 0 & AB \\ (AB)^* & 0 & 0 \end{bmatrix} = 2r(AB), \tag{2.9}$$

$$r(N_4) = r\begin{bmatrix} (ABA^*)^\dagger - [(AB)^\dagger]^* B(AB)^\dagger & [(AB)^\dagger]^* BF_{AB} & AB(AB)^\dagger \\ F_{AB}B(AB)^\dagger & -F_{AB}BF_{AB} & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} 0 & 0 & AB \\ F_{AB}B(AB)^\dagger & -F_{AB}BF_{AB} & 0 \end{bmatrix}$$

$$= r(AB) + r[\, F_{AB}B(AB)^\dagger, \, F_{AB}BF_{AB} \,] = r\begin{bmatrix} (AB)^* & B(AB)^\dagger & B \\ 0 & 0 & AB \end{bmatrix} - r(AB)$$

$$= r\begin{bmatrix} 0 & 0 & B \\ ABA^* & AB(AB)^\dagger & 0 \end{bmatrix} - r(AB) = r\begin{bmatrix} 0 & 0 & B \\ 0 & AB & 0 \end{bmatrix} - r(AB) = r(B), \tag{2.10}$$

and

$$s_1 = r(N_1) - 2r(N_2) = -2r(AB), \tag{2.11}$$

$$s_2 = r(N_3) - 2r(N_4) = -r(B) - r(ABA^*), \tag{2.12}$$

$$s_3 = i_+(N_1) + i_-(N_3) - r(N_2) - r(N_4) = -r(AB) - i_-(B) - i_+(ABA^*), \tag{2.13}$$

$$s_4 = i_-(N_1) + i_+(N_3) - r(N_2) - r(N_4) = -r(AB) - i_+(B) - i_-(ABA^*). \tag{2.14}$$

Applying (1.18) and (1.19) to (2.6), and simplifying by (2.7)–(2.14) and the following basic inertia inequalities (see [4])

$$i_{+}(ABA^{*}) \geqslant r(AB) - i_{-}(B), \quad i_{-}(ABA^{*}) \geqslant r(AB) - i_{+}(B),$$

we obtain

$$\begin{aligned}
&\max_{(AB)^{(1,2,3)}} r\left\{ (ABA^{*})^{\dagger} - [(AB)^{(1,2,3)}]^{*}B(AB)^{(1,2,3)}\right\} \\
&= \max_{V} r\left\{ (ABA^{*})^{\dagger} - [\,(AB)^{\dagger} + F_{AB}VAB(AB)^{\dagger}\,]^{*}B[\,(AB)^{\dagger} + F_{AB}VAB(AB)^{\dagger}\,]\right\} \\
&= \min\left\{ r\big[\,(ABA^{*})^{\dagger} - [(AB)^{\dagger}]^{*}B(AB)^{\dagger},\ [(AB)^{\dagger}]^{*}BF_{AB},\ AB(AB)^{\dagger}\,\big],\ r(N_1),\ r(N_3)\right\} \\
&= \min\{\, r(AB), \quad r(B) - r(ABA^{*})\,\},
\end{aligned}$$

and

$$\begin{aligned}
&\min_{(AB)^{(1,2,3)}} r\left\{ (ABA^{*})^{\dagger} - [(AB)^{(1,2,3)}]^{*}B(AB)^{(1,2,3)}\right\} \\
&= \min_{V} r\left\{ (ABA^{*})^{\dagger} - [\,(AB)^{\dagger} + F_{AB}VAB(AB)^{\dagger}\,]^{*}B[\,(AB)^{\dagger} + F_{AB}VAB(AB)^{\dagger}\,]\right\} \\
&= 2r\big[(ABA^{*})^{\dagger} - [(AB)^{\dagger}]^{*}B(AB)^{\dagger},\ -[(AB)^{\dagger}]^{*}BF_{AB}\,,\ AB(AB)^{\dagger}\big] + \max\{s_1,\ s_2,\ s_3,\ s_4\} \\
&= 2r(AB) + \max\{s_1,\ s_2,\ s_3,\ s_4\} \\
&= \max\{0,\ 2r(AB) - r(B) - r(ABA^{*}),\ r(AB) - i_{-}(B) - i_{+}(ABA^{*}), \\
&\qquad r(AB) - i_{+}(B) - i_{-}(ABA^{*})\} = 0,
\end{aligned}$$

as required for (2.1) and (2.2). Applying (1.20) and (1.21) to (2.6) also gives

$$\begin{aligned}
&\max_{(AB)^{(1,2,3)}} i_{\pm}\left\{ (ABA^{*})^{\dagger} - [(AB)^{(1,2,3)}]^{*}B(AB)^{(1,2,3)}\right\} = \min\{\, i_{\pm}(N_1), \quad i_{\pm}(N_3)\} \\
&= \min\{\, r(AB), \quad i_{\mp}(B) - i_{\mp}(ABA^{*})\,\},
\end{aligned}$$

and

$$\begin{aligned}
&\min_{(AB)^{(1,2,3)}} i_{\pm}\left\{ (ABA^{*})^{\dagger} - [(AB)^{(1,2,3)}]^{*}B(AB)^{(1,2,3)}\right\} \\
&= r\big[(ABA^{*})^{\dagger} - [(AB)^{\dagger}]^{*}B(AB)^{\dagger}\,,\ -[(AB)^{\dagger}]^{*}BF_{AB}\,,\ AB(AB)^{\dagger}\big] \\
&\quad + \max\{i_{\pm}(N_1) - r(N_2), \quad i_{\pm}(N_3) - r(N_4)\} \\
&= \max\{\,0,\ r(AB) - i_{\pm}(B) - i_{\mp}(ABA^{*})\,\} = 0,
\end{aligned}$$

as required for (2.3) and (2.4). $\square$

Note from $BB^{\dagger}B = B$ that ABA^{*} can be rewritten as $ABA^{*} = (AB)B^{\dagger}(AB)^{*}$. Applying Theorem 2.2 to the product also yields the following result.

Theorem 2.3 *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n}_{\mathrm{H}}$. Then, the following equalities hold*

$$\begin{aligned}
&\max_{(ABB^{\dagger})^{(1,2,3)}} r\left\{ (ABA^{*})^{\dagger} - [(ABB^{\dagger})^{(1,2,3)}]^{*}B^{\dagger}(ABB^{\dagger})^{(1,2,3)}\right\} \\
&= \min\{\, r(AB),\ r(B) - r(ABA^{*})\,\}, \qquad (2.15) \\
&\max_{(ABB^{\dagger})^{(1,2,3)}} i_{\pm}\left\{ (ABA^{*})^{\dagger} - [(ABB^{\dagger})^{(1,2,3)}]^{*}B^{\dagger}(ABB^{\dagger})^{(1,2,3)}\right\} \\
&= \min\{\, r(AB),\ i_{\mp}(B) - i_{\mp}(ABA^{*})\,\}, \qquad (2.16)
\end{aligned}$$

and

$$\min_{(ABB^{\dagger})^{(1,2,3)}} r\left\{ (ABA^*)^{\dagger} - [(ABB^{\dagger})^{(1,2,3)}]^* B^{\dagger}(ABB^{\dagger})^{(1,2,3)} \right\} = 0, \tag{2.17}$$

$$\min_{(ABB^{\dagger})^{(1,2,3)}} i_{\pm}\left\{ (ABA^*)^{\dagger} - [(ABB^{\dagger})^{(1,2,3)}]^* B^{\dagger}(ABB^{\dagger})^{(1,2,3)} \right\} = 0. \tag{2.18}$$

Proof Eqs. (2.15)–(2.18) follow directly from (2.1)–(2.4). □

Applying Lemma 2.1 to Theorems 2.2 and 2.3, we obtain the following consequences.

Corollary 2.4 *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n}_{\mathrm{H}}$. Then there always exist $(AB)^{(1,2,3)}$ and $(ABB^{\dagger})^{(1,2,3)}$ such that the following equalities hold*

$$(ABA^*)^{\dagger} = \left[(AB)^{(1,2,3)}\right]^* B(AB)^{(1,2,3)},$$

$$(ABA^*)^{\dagger} = \left[(ABB^{\dagger})^{(1,2,3)}\right]^* B^{\dagger}(ABB^{\dagger})^{(1,2,3)}.$$

Proof It follows directly from (2.2) and (2.17). □

Corollary 2.5 *Let $A \in \mathbb{C}^{m\times n}$ and $B = B^* \in \mathbb{C}^{n\times n}$ be given. Then, the following results hold.*

(a) *The following statements are equivalent:*
  - (i) *There exists an $(AB)^{(1,2,3)}$ such that $(ABA^*)^{\dagger} - [(AB)^{(1,2,3)}]^* B(AB)^{(1,2,3)}$ is nonsingular.*
  - (ii) *There exists an $(AB)^{(1,2,3)}$ such that $(ABA^*)^{\dagger} - [(ABB^{\dagger})^{(1,2,3)}]^* B^{\dagger}(ABB^{\dagger})^{(1,2,3)}$ is nonsingular.*
  - (iii) *$r(AB) = m$ and $r(B) \geqslant r(ABA^*) + m$.*

(b) *The following statements are equivalent:*
  - (i) *There exists an $(AB)^{(1,2,3)}$ such that $(ABA^*)^{\dagger} \succ [(AB)^{(1,2,3)}]^* B(AB)^{(1,2,3)}$.*
  - (ii) *There exists an $(ABB^{\dagger})^{(1,2,3)}$ such that $(ABA^*)^{\dagger} \succ [(ABB^{\dagger})^{(1,2,3)}]^* B^{\dagger}(ABB^{\dagger})^{(1,2,3)}$.*
  - (iii) *$r(AB) = m$ and $i_-(B) \geqslant i_-(ABA^*) + m$.*

(c) *The following statements are equivalent:*
  - (i) *There exists an $(AB)^{(1,2,3)}$ such that $(ABA^*)^{\dagger} \prec [(AB)^{(1,2,3)}]^* B(AB)^{(1,2,3)}$.*
  - (ii) *There exists an $(ABB^{\dagger})^{(1,2,3)}$ such that $(ABA^*)^{\dagger} \prec [(ABB^{\dagger})^{(1,2,3)}]^* B^{\dagger}(ABB^{\dagger})^{(1,2,3)}$.*
  - (iii) *$r(AB) = m$ and $i_+(B) \geqslant i_+(ABA^*) + m$.*

3. Some applications

Assume that $A, B \in \mathbb{C}^{m}_{\mathrm{H}}$. Then their sum can be written as

$$A + B = [\, I_m,\ I_m\,] \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I_m \\ I_m \end{bmatrix} = PJP^*,$$

where $P = [\, I_m,\ I_m\,]$ and $J = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Applying the previous results to this PJP^*, we obtain the following result.

Corollary 3.1 *Let $A, B \in \mathbb{C}^{m}_{\mathrm{H}}$. Then, the following results hold.*

(a) *There always exist $[\, A,\ B\,]^{(1,2,3)}$ and $[\, AA^{\dagger},\ BB^{\dagger}\,]^{(1,2,3)}$ such that*

$$(\, A + B\,)^{\dagger} = \left([\, A,\ B\,]^{(1,2,3)}\right)^* J[\, A,\ B\,]^{(1,2,3)},$$

$$(A+B)^{\dagger} = \left([AA^{\dagger},\ BB^{\dagger}]^{(1,2,3)}\right)^{*} J^{\dagger}[AA^{\dagger},\ BB^{\dagger}]^{(1,2,3)}.$$

(b) *The following statements are equivalent:*

(i) *There exists a* $[A,\ B]^{(1,2,3)}$ *such that*

$$(A+B)^{\dagger} - \left([A,\ B]^{(1,2,3)}\right)^{*} J[A,\ B]^{(1,2,3)}$$

is nonsingular.

(ii) *There exists a* $[AA^{\dagger},\ BB^{\dagger}]^{(1,2,3)}$ *such that*

$$(A+B)^{\dagger} - \left([AA^{\dagger},\ BB^{\dagger}]^{(1,2,3)}\right)^{*} J^{\dagger}[AA^{\dagger},\ BB^{\dagger}]^{(1,2,3)}$$

is nonsingular.

(iii) $r[A,\ B] = m$ *and* $r(A) + r(B) \geqslant r(A+B) + m$.

(c) *The following statements are equivalent:*

(i) *There exists a* $[A,\ B]^{(1,2,3)}$ *such that*

$$(A+B)^{\dagger} \succ \left([A,\ B]^{(1,2,3)}\right)^{*} J[A,\ B]^{(1,2,3)}.$$

(ii) *There exists a* $[AA^{\dagger},\ BB^{\dagger}]^{(1,2,3)}$ *such that*

$$(A+B)^{\dagger} \succ \left([AA^{\dagger},\ BB^{\dagger}]^{(1,2,3)}\right)^{*} J^{\dagger}[AA^{\dagger},\ BB^{\dagger}]^{(1,2,3)}.$$

(iii) $r[A,\ B] = m$ *and* $i_{-}(A) + i_{-}(B) \geqslant i_{-}(A+B) + m$.

(d) *The following statements are equivalent:*

(i) *There exists a* $[A,\ B]^{(1,2,3)}$ *such that*

$$(A+B)^{\dagger} \prec \left([A,\ B]^{(1,2,3)}\right)^{*} J[A,\ B]^{(1,2,3)}.$$

(ii) *There exists a* $[AA^{\dagger},\ BB^{\dagger}]^{(1,2,3)}$ *such that*

$$(A+B)^{\dagger} \prec \left([AA^{\dagger},\ BB^{\dagger}]^{(1,2,3)}\right)^{*} J^{\dagger}[AA^{\dagger},\ BB^{\dagger}]^{(1,2,3)}.$$

(iii) $r[A,\ B] = m$ *and* $i_{+}(A) + i_{+}(B) \geqslant i_{+}(A+B) + m$.

For two given positive definite matrices $M \in \mathbb{C}_{\mathrm{H}}^{m}$ and $N \in \mathbb{C}_{\mathrm{H}}^{n}$, the weighted Moore-Penrose inverse of $A \in \mathbb{C}^{m\times n}$ is defined to be the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the four equations

$$\text{(i)}\ \ AXA = A, \quad \text{(ii)}\ \ XAX = X, \quad \text{(iii)}\ \ (MAX)^{*} = MAX, \quad \text{(iv)}\ \ (NXA)^{*} = NXA,$$

and is denoted by $A_{M,N}^{\dagger}$. A matrix X is called a $\{1,\ 2,\ 3M\}$-inverse of A, if it satisfies (i), (ii), and (iii), and is denoted by $A^{(1,2,3M)}$. In the partial case $M = I_m$ and $N = I_n$, the matrix $A_{M,N}^{\dagger}$ becomes the Moore-Penrose inverse of A.

It is well known (see, e.g., [1]) that the weighted Moore-Penrose inverse $A_{M,N}^{\dagger}$ of A can be rewritten as

$$A_{M,N}^{\dagger} = N^{-\frac{1}{2}}(M^{\frac{1}{2}}AN^{-\frac{1}{2}})^{\dagger}M^{\frac{1}{2}}, \tag{3.1}$$

where $M^{\frac{1}{2}}$ and $N^{\frac{1}{2}}$ are the positive definite square roots of M and N, respectively. Assume that $M \in \mathbb{C}_{\mathrm{H}}^{m}$ is positive definite. It turns out from (3.1) that

$$(ABA^{*})_{M,M^{-1}}^{\dagger} = M^{\frac{1}{2}}(M^{\frac{1}{2}}ABA^{*}M^{\frac{1}{2}})^{\dagger}M^{\frac{1}{2}},$$

which is also Hermitian matrix. From Corollary 3.1, there exists an $(M^{\frac{1}{2}}AB)^{(1,2,3)}$ such that

$$(M^{\frac{1}{2}}ABA^*M^{\frac{1}{2}})^{\dagger} = [(M^{\frac{1}{2}}AB)^{(1,2,3)}]^*B(M^{\frac{1}{2}}AB)^{(1,2,3)},$$
$$(ABA^*)^{\dagger}_{M,M^{-1}} = M^{\frac{1}{2}}[(M^{\frac{1}{2}}AB)^{(1,2,3)}]^*B(M^{\frac{1}{2}}AB)^{(1,2,3)}M^{\frac{1}{2}}$$

hold. Hence, we have the following consequence.

Corollary 3.2 *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^n_{\mathrm{H}}$, and assume that $M \in \mathbb{C}^m_{\mathrm{H}}$ is positive definite. Then, there always exist $(AB)^{(1,2,3M)}$ and $(ABB^{\dagger})^{(1,2,3M)}$ such that the following equalities hold*

$$(ABA^*)^{\dagger}_{M,M^{-1}} = [(AB)^{(1,2,3M)}]^*B(AB)^{(1,2,3M)},$$
$$(ABA^*)^{\dagger}_{M,M^{-1}} = [(ABB^{\dagger})^{(1,2,3M)}]^*B^{\dagger}(ABB^{\dagger})^{(1,2,3M)}.$$

In addition to (1.2) and (1.3), rank and inertia formulas for the difference

$$[(AB)^{(1,2,3)}]^*B(AB)^{(1,2,3)} - C$$

can be established, where C is a general Hermitian matrix, and identifying conditions can also be obtained for the following matrix equality and inequalities

$$[(AB)^{(1,2,3)}]^*B(AB)^{(1,2,3)} = C\ (\succcurlyeq C,\ \preccurlyeq C, \succ C,\ \prec C)$$

to hold. Further, both (1.1) and (1.2) could be regarded as special cases of the following mixed-type reverse-order laws

$$(ABC)^{\dagger} = (BC)^{(i,\ldots,j)}B(AB)^{(i,\ldots,j)}, \quad (ABA^*)^{\dagger} = [(AB)^{(i,\ldots,j)}]^*B(AB)^{(i,\ldots,j)},$$

where the right-hand sides of these two equalities are quadratic matrix-valued functions with one or more arbitrary matrices for different choices of $(AB)^{(i,\ldots,j)}$ and $(BC)^{(i,\ldots,j)}$. It is, however, a challenging task to establish necessary and sufficient conditions for these two equalities to hold in general cases.

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