Journal of Mathematical Research with Applications
May, 2015, Vol. 35, No. 3, pp. 343–349
DOI:10.3770/j.issn:2095-2651.2015.03.012
Http://jmre.dlut.edu.cn

Measures of Countable Compactness and the Lindelöf Property in L-Fuzzy Topological Spaces

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Abstract In this paper, the concept of countable compactness degree and the concept of Lindelöf property degree are defined in L-fuzzy topological spaces by means of implication operator $\mapsto$. Many properties of them are discussed.

Keywords L-fuzzy topology; countable compactness; Lindelöf property; implication operator; fuzzy compactness degree

MR(2010) Subject Classification 54A40; 54D30; 03E72

1. Introduction

Chang [1] first introduced the concept of compactness to $[0, 1]$-topological spaces by means of open cover. Afterward, many researchers have tried successfully to generalize the compactness theory of general topology to fuzzy setting [2–6].

For the more general case, in an L-fuzzy topology, open sets are not crisp subset, and topology comprising those open sets is a fuzzy set of L^X. There have been many research results about fuzzy compactness in L-fuzzy topological spaces [7–14]. The definitions of countable compactness and the Lindelöf property in L-topological spaces were introduced by Shi [4]. The aim of this paper is to introduce the notion of countable compactness degree and the Lindelöf property degree to L-fuzzy topological spaces, thus some properties of them are researched.

2. Preliminaries

In this paper, $(L, \bigvee, \bigwedge, ')$ is a completely distributive DeMorgan algebra (i.e., completely distributive lattice with order-reversing involution) [2]. The largest element and the smallest element in L are denoted by $\top$ and $\perp$, respectively.

The binary wedge below relation $\prec$ in L is defined as follows: For $a, b \in L$, a is called wedge below b in L(i.e., $a \prec b$) if and only if for every subset $D \subseteq L$, $\bigvee D \geq b$ implies $d \geq a$ for some

Received October 13, 2014; Accepted April 21, 2015
Supported by the National Natural Science Foundation of China (Grant No. 11471297).
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$d \in D$. L is completely distributive lattice if and only if $b = \bigvee\{a \in L : a \prec b\}$ for each $b \in L$. For $b \in L$, $\beta(b) = \{a \in L : a \prec b\}$ is called the greatest minimal family of b.

In a completely distributive DeMorgan algebra L, binary operation $a \mapsto b = \bigvee\{c \in L \mid a \wedge c \leq b\}$ is called implication operator. It is easy to get the following results:

(1) $a \leq b \Leftrightarrow a \mapsto b = \top$;

(2) $c \leq (a \mapsto b) \Leftrightarrow a \wedge c \leq b$;

(3) $(\bigvee_i a_i) \mapsto b = \bigwedge_i (a_i \mapsto b)$;

(4) $a \mapsto (\bigwedge_i b_i) = \bigwedge_i (a \mapsto b_i)$.

$[a \leq b] = a \mapsto b$ can be viewed as the degree of $a \leq b$.

Definition 2.1 ([15]) *An L-fuzzy topology on a set X is a map $\tau : L^X \to L$ such that*

(1) $\tau(\underline{\top}) = \tau(\underline{\perp}) = \top$;

(2) $\forall U, V \in L^X$, $\tau(U \wedge V) \geqslant \tau(U) \wedge \tau(V)$;

(3) $\forall U_j \in L^X$, $j \in J$, $\tau(\bigvee_{j \in J} U_j) \geqslant \bigwedge_{j \in J} \tau(U_j)$.

Generally, $\tau(U)$ can be regarded as degree to which $U \in L^X$ is an open set; while $\tau^*(U) = \tau(U')$ is called the degree of closedness of U. $\forall \mathcal{U} \subseteq L^X, \tau(\mathcal{U}) = \bigwedge_{A \in \mathcal{U}} \tau(A)$ will be called the degree of openness of $\mathcal{U}$. The pair (X, τ) is called an L-fuzzy topological space.

A map $f : (X, \tau) \to (Y, \delta)$ is called continuous with respect to L-fuzzy topologies τ and δ if $\tau(f_L^{\leftarrow}(U)) \geqslant \delta(U)$ holds for all $U \in L^Y$, where $f_L^{\leftarrow}$ is defined by $f_L^{\leftarrow}(U)(x) = U(f(x))$, $x \in X$.

For a subfamily $\Phi \subseteq L^X$, $2^{(\Phi)}$ denotes the set of all finite subfamilies of Φ. And $2^{[\Phi]}$ denotes the set of all countable subfamilies of Φ.

Definition 2.2 ([13]) *A map $\widetilde{\subset} : L^X \times L^X \to L$ is an L-fuzzy inclusion on X, defined as $\widetilde{\subset}(A, B) = \bigwedge_{x \in X} (A'(x) \vee B(x))$, which is denoted by $[A \widetilde{\subset} B]$ for simplicity instead of $\widetilde{\subset}(A, B)$, i.e., $[A \widetilde{\subset} B] = \bigwedge_{x \in X} (A'(x) \vee B(x))$.*

Definition 2.3 ([11]) *If (X, τ) is an L-fuzzy topological space and $G \in L^X$, then*

$$CD_\tau(G) = \bigwedge_{\mathcal{U} \subseteq L^X} \Big(\tau(\mathcal{U}) \mapsto \Big([G \widetilde{\subset} \bigvee \mathcal{U}] \mapsto \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G \widetilde{\subset} \bigvee \mathcal{V}]\Big)\Big)$$

is called the fuzzy compactness degree of G with respect to τ.

Lemma 2.4 ([6]) *Let $f : X \to Y$ be a set map. The fuzzy powerset operators $f_L^{\rightarrow} : L^X \to L^Y$ and $f_L^{\leftarrow} : L^Y \to L^X$ are defined by $f_L^{\rightarrow}(a)(y) = \bigvee\{a(x) : f(x) = y\}$, $f_L^{\leftarrow}(b) = b \circ f$. Then for any $\mathcal{P} \subseteq L^Y$, we have that*

$$\bigwedge_{y \in Y} \Big(f_L^{\rightarrow}(G)'(y) \vee \bigvee_{B \in \mathcal{P}} B(y)\Big) = \bigwedge_{x \in X} \Big(G'(x) \vee \bigvee_{B \in \mathcal{P}} f_L^{\leftarrow}(B)(x)\Big).$$

3. Measures of countable compactness

In the following part, the set $\{\mathcal{U} \mid \mathcal{U}$ is a countable family, $\mathcal{U} \subseteq L^X\}$ is written as L_C^X.

Let $(X, \mathcal{T})$ be an L-topological space and $\mathcal{T}_C = \{\mathcal{U} \mid \mathcal{U}$ is a countable family, $\mathcal{U} \subseteq \mathcal{T}\}$. Then $G \in L^X$ is countably compact [4] if and only if for every $\mathcal{U} \subseteq \mathcal{T}_C$, it follows that $[G \widetilde{\subset} \bigvee \mathcal{U}] \leqslant$

$\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}[G\widetilde{\subset}\bigvee\mathcal{V}]$. This implies that $[[G\widetilde{\subset}\bigvee\mathcal{U}]\leqslant\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}[G\widetilde{\subset}\bigvee\mathcal{V}]]=\top$.

On the other hand, an L-topology $\mathcal{T}$ can be regarded as a map $\chi_{\mathcal{T}}:L^X\to L$ defined by

$$\chi_{\mathcal{T}}(A)=\begin{cases}\top, & A\in\mathcal{T},\\ \bot, & A\notin\mathcal{T}.\end{cases}$$

In this way, $(X,\chi_{\mathcal{T}})$ is a special L-fuzzy topological space and $\chi_{\mathcal{T}}(\mathcal{U})=\top$ for any $\mathcal{U}\subseteq\mathcal{T}$.

From the above analysis, we can obtain that $G\in L^X$ is countably compact if and only if for every family $\mathcal{U}\subseteq\mathcal{T}_C$, it follows that $\chi_{\mathcal{T}}(\mathcal{U})\leqslant[[G\widetilde{\subset}\bigvee\mathcal{U}]\leqslant\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}[G\widetilde{\subset}\bigvee\mathcal{V}]]$.

Naturally, we can introduce a countable compactness degree defined as follows:

Definition 3.1 *Let (X,τ) be an L-fuzzy topological space and $G\in L^X$. Then*

$$\begin{aligned}CCD_\tau(G)&=\bigwedge_{\mathcal{U}\subseteq L_C^X}\Big[\tau(\mathcal{U})\leqslant\Big[[G\widetilde{\subset}\bigvee\mathcal{U}]\leqslant\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}[G\widetilde{\subset}\bigvee\mathcal{V}]\Big]\Big]\\&=\bigwedge_{\mathcal{U}\subseteq L_C^X}\Big(\bigwedge_{A\in\mathcal{U}}\tau(A)\mapsto\Big(\bigwedge_{x\in X}\Big(G'(x)\vee\bigvee_{A\in\mathcal{U}}A(x)\Big)\mapsto\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\bigwedge_{x\in X}\Big(G'(x)\vee\bigvee_{A\in\mathcal{V}}A(x)\Big)\Big)\Big)\end{aligned}$$

is called the countable compactness degree of G with respect to τ.

Obviously, G is countably compact in L-topological space $\mathcal{T}$ if and only if $CCD_{\chi_{\mathcal{T}}}(G)=\top$.

According to the properties of implication operation $\mapsto$, the following lemma can be proved.

Lemma 3.2 *Let (X,τ) be an L-fuzzy topological space and $G\in L^X$. Then $a\leq CCD_\tau(G)$ if and only if for any $\mathcal{U}\subseteq L_C^X$,*

$$\tau(\mathcal{U})\wedge[G\widetilde{\subset}\bigvee\mathcal{U}]\wedge a\leq\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}[G\widetilde{\subset}\bigvee\mathcal{V}].\tag{3.1}$$

Theorem 3.3 *Let (X,τ) be an L-fuzzy topological space and $G\in L^X$. Then*

$$CCD_\tau(G)=\bigvee\Big\{a\in L:\tau(\mathcal{U})\wedge[G\widetilde{\subset}\bigvee\mathcal{U}]\wedge a\leq\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}[G\widetilde{\subset}\bigvee\mathcal{V}]\text{ for any }\mathcal{U}\subseteq L_C^X\Big\}.\tag{3.2}$$

It is easy to get the following theorem according to Definitions 2.3 and 3.1.

Theorem 3.4 *Let (X,τ) be an L-fuzzy topological space and $G\in L^X$. Then $CD_\tau(G)\leq CCD_\tau(G)$.*

By Lemma 3.2, Theorem 3.3 and the properties of implication operation $\mapsto$, we can obtain the following result.

Theorem 3.5 *Let (X,τ) be an L-fuzzy topological space and $G\in L^X$. Then*

$$\begin{aligned}CCD_\tau(G)&=\bigwedge_{\mathcal{U}\subseteq L_C^X}\Big(\bigwedge_{A\in\mathcal{U}}\tau(A)\wedge\bigwedge_{x\in X}\Big(G'(x)\vee\bigvee_{A\in\mathcal{U}}A(x)\Big)\mapsto\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\bigwedge_{x\in X}\Big(G'(x)\vee\bigvee_{A\in\mathcal{V}}A(x)\Big)\Big)\\&=\bigwedge_{\mathcal{U}\subseteq L_C^X}\Big[\tau(\mathcal{U})\wedge[G\widetilde{\subset}\bigvee\mathcal{U}]\leqslant\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}[G\widetilde{\subset}\bigvee\mathcal{V}]\Big].\end{aligned}$$

Theorem 3.6 *Let (X,τ) be an L-fuzzy topological space. Then $\forall G,H\in L^X$, $CCD_\tau(G\wedge H)\geq CCD_\tau(G)\wedge\tau^*(H)$.*

Proof For any $a \in L$ and $a \leq CCD_\tau(G) \wedge \tau^*(H)$, now the proof of $a \leq CCD_\tau(G \wedge H)$ will be conducted. Suppose any $\mathcal{U} \subseteq L_C^X$, since $a \leq CCD_\tau(G) \wedge \tau^*(H)$, we have that $a \leq \tau^*(H)$ and $a \leq CCD_\tau(G)$. Let $\mathcal{W} = \mathcal{U} \bigcup H'$. Then $\mathcal{W} \subseteq L_C^X$. Therefore $\tau(\mathcal{W}) \wedge [G \widetilde{\subset} \bigvee \mathcal{W}] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{W})}} [G \widetilde{\subset} \bigvee \mathcal{V}]$ by Lemma 3.2. And because $\tau(\mathcal{W}) = \tau(\mathcal{U}) \wedge \tau(H') = \tau(\mathcal{U}) \wedge \tau^*(H)$,

$$\begin{aligned}[G \widetilde{\subset} \bigvee \mathcal{W}] &= \bigwedge_{x \in X} \Big(G'(x) \vee \bigvee_{A \in \mathcal{W}} A(x)\Big) = \bigwedge_{x \in X} \Big(G'(x) \vee H'(x) \vee \bigvee_{A \in \mathcal{U}} A(x)\Big) \\ &= \bigwedge_{x \in X} \Big((G(x) \wedge H(x))' \vee \bigvee_{A \in \mathcal{U}} A(x)\Big) = [(G \wedge H) \widetilde{\subset} \bigvee \mathcal{U}]\end{aligned}$$

and

$$\begin{aligned}\bigvee_{\mathcal{V} \in 2^{(\mathcal{W})}} [G \widetilde{\subset} \bigvee \mathcal{V}] &= \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G \widetilde{\subset} \bigvee \mathcal{V}] \vee \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G \widetilde{\subset} \bigvee (\mathcal{V} \vee H')] \\ &= \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G \widetilde{\subset} \bigvee \mathcal{V}] \vee \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [(G \wedge H) \widetilde{\subset} \bigvee \mathcal{V}] \\ &= \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [(G \wedge H) \widetilde{\subset} \bigvee \mathcal{V}],\end{aligned}$$

we have that $(\tau(\mathcal{U}) \wedge \tau^*(H)) \wedge [(G \wedge H) \widetilde{\subset} \bigvee \mathcal{U}] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [(G \wedge H) \widetilde{\subset} \bigvee \mathcal{V}]$. Since $a \leq \tau^*(H)$, this means that $\tau(\mathcal{U}) \wedge [(G \wedge H) \widetilde{\subset} \bigvee \mathcal{U}] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [(G \wedge H) \widetilde{\subset} \bigvee \mathcal{V}]$.

Thus $a \leq CCD_\tau(G \wedge H)$ by Lemma 3.2. The proof is completed. $\Box$

Corollary 3.7 *Let (X, τ) be an L-fuzzy topological spaces. Then $\forall G \in L^X$, $CCD_\tau(G) \geqslant CCD_\tau(\top) \wedge \tau^*(G)$.*

Theorem 3.8 *Let (X, τ) be an L-fuzzy topological space. Then $\forall G, H \in L^X$, $CCD_\tau(G \vee H) \geq CCD_\tau(G) \wedge CCD_\tau(H)$.*

Proof For any $a \in L$ and $a \leq CCD_\tau(G) \wedge CCD_\tau(H)$, we need to prove that $a \leq CCD_\tau(G \vee H)$. Suppose any $\mathcal{U} \subseteq L_C^X$, since $a \leq CCD_\tau(G) \wedge CCD_\tau(H)$, we have that $a \leq CCD_\tau(G)$ and $a \leq CCD_\tau(H)$. According to Lemma 3.2, we can obtain that $\tau(\mathcal{U}) \wedge [G \widetilde{\subset} \bigvee \mathcal{U}] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G \widetilde{\subset} \bigvee \mathcal{V}]$ and $\tau(\mathcal{U}) \wedge [H \widetilde{\subset} \bigvee \mathcal{U}] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [H \widetilde{\subset} \bigvee \mathcal{V}]$.

Moreover, we have that

$$\tau(\mathcal{U}) \wedge [G \widetilde{\subset} \bigvee \mathcal{U}] \wedge [H \widetilde{\subset} \bigvee \mathcal{U}] \wedge a \leq \Big(\bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G \widetilde{\subset} \bigvee \mathcal{V}]\Big) \wedge \Big(\bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [H \widetilde{\subset} \bigvee \mathcal{V}]\Big).$$

By

$$\begin{aligned}[G \widetilde{\subset} \bigvee \mathcal{U}] \wedge [H \widetilde{\subset} \bigvee \mathcal{U}] &= \Big(\bigwedge_{x \in X} \Big(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x)\Big)\Big) \wedge \Big(\bigwedge_{x \in X} \Big(H'(x) \vee \bigvee_{A \in \mathcal{U}} A(x)\Big)\Big) \\ &= \bigwedge_{x \in X} \Big(\Big(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x)\Big) \wedge \Big(H'(x) \vee \bigvee_{A \in \mathcal{U}} A(x)\Big)\Big) \\ &= \bigwedge_{x \in X} \Big((G \vee H)'(x) \vee \bigvee_{A \in \mathcal{U}} A(x)\Big) = [(G \vee H) \widetilde{\subset} \bigvee \mathcal{U}]\end{aligned}$$

and

$$\begin{aligned}\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}[G\widetilde{\subset}\bigvee\mathcal{V}]\wedge\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}[H\widetilde{\subset}\bigvee\mathcal{V}] &= \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\{[G\widetilde{\subset}\bigvee\mathcal{V}]\wedge[H\widetilde{\subset}\bigvee\mathcal{V}]\}\\ &= \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}[(G\vee H)\widetilde{\subset}\bigvee\mathcal{V}],\end{aligned}$$

we have that $\tau(\mathcal{U})\wedge[(G\vee H)\widetilde{\subset}\bigvee\mathcal{U}]\wedge a\leq\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}[(G\vee H)\widetilde{\subset}\bigvee\mathcal{V}]$.

Thus $a\leq CCD_{\tau}(G\vee H)$ by Lemma 3.2. The proof is completed. □

Theorem 3.9 *Let $f:X\to Y$ be a set map, τ_1 be an L-fuzzy topology on X, τ_2 be an L-fuzzy topology on Y, and $f:(X,\tau_1)\to(Y,\tau_2)$ be continuous. Then $CCD_{\tau_2}(f_L^{\rightarrow}(G))\geq CCD_{\tau_1}(G)$.*

Proof $\forall a\in L$ and $a\leq CCD_{\tau_1}(G)$, the next step is to prove that $a\leq CCD_{\tau_2}(f_L^{\rightarrow}(G))$. Suppose any $\mathcal{U}\subseteq L_C^Y$, let $\mathcal{R}=f_L^{\leftarrow}(\mathcal{U})=\{B|B=f_L^{\leftarrow}(A),A\in\mathcal{U}\}$. Then $\mathcal{R}\subseteq L_C^X$. By $a\leq CCD_{\tau_1}(G)$, we have that $\tau_1(\mathcal{R})\wedge[G\widetilde{\subset}\bigvee\mathcal{R}]\wedge a\leq\bigvee_{\mathcal{S}\in 2^{(\mathcal{R})}}[G\widetilde{\subset}\bigvee\mathcal{S}]$. Since f is continuous, $\tau_1(f_L^{\leftarrow}(A))\geqslant\tau_2(A)$ holds for all $A\in\mathcal{U}$, i.e., $\tau_1(\mathcal{R})=\tau_1(f_L^{\leftarrow}(\mathcal{U}))\geqslant\tau_2(\mathcal{U})$.

By Lemma 2.4, we can obtain that

$$[f_L^{\rightarrow}(G)\widetilde{\subset}\bigvee\mathcal{U}]=\bigwedge_{y\in Y}\Big(f_L^{\rightarrow}(G)'(y)\vee\bigvee_{A\in\mathcal{U}}A(y)\Big)=\bigwedge_{x\in X}\Big(G'(x)\vee\bigvee_{B\in\mathcal{R}}B(x)\Big)=[G\widetilde{\subset}\bigvee\mathcal{R}]$$

and

$$\begin{aligned}\bigvee_{\mathcal{S}\in 2^{(\mathcal{R})}}[G\widetilde{\subset}\bigvee\mathcal{S}] &= \bigvee_{\mathcal{S}\in 2^{(\mathcal{R})}}\bigwedge_{x\in X}\Big(G'(x)\vee\bigvee_{C\in\mathcal{S}}C(x)\Big)=\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\bigwedge_{y\in Y}\Big(f_L^{\rightarrow}(G)'(y)\vee\bigvee_{D\in\mathcal{V}}D(y)\Big)\\ &= \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}[f_L^{\rightarrow}(G)\widetilde{\subset}\bigvee\mathcal{V}].\end{aligned}$$

This shows the following inequality is true.

$$\begin{aligned}\tau_2(\mathcal{U})\wedge[f_L^{\rightarrow}(G)\widetilde{\subset}\bigvee\mathcal{U}]\wedge a &\leq \tau_1(\mathcal{R})\wedge[G\widetilde{\subset}\bigvee\mathcal{R}]\wedge a\\ &\leq \bigvee_{\mathcal{S}\in 2^{(\mathcal{R})}}[G\widetilde{\subset}\bigvee\mathcal{S}]\\ &= \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}[f_L^{\rightarrow}(G)\widetilde{\subset}\bigvee\mathcal{V}].\end{aligned}$$

Thus $a\leq CCD_{\tau_2}(f_L^{\rightarrow}(G))$ by Lemma 3.2. The proof is completed. □

4. Measures of Lindelöf property

In an L-topological space $(X,\mathcal{T})$, if $G\in L^X$ has the Lindelöf property [11], then it can be understood that for every $\mathcal{U}\subseteq\mathcal{T}$, it follows that $[G\widetilde{\subset}\bigvee\mathcal{U}]\leqslant\bigvee_{\mathcal{V}\in 2^{[\mathcal{U}]}}[G\widetilde{\subset}\bigvee\mathcal{V}]$, which means that $[[G\widetilde{\subset}\bigvee\mathcal{U}]\leqslant\bigvee_{\mathcal{V}\in 2^{[\mathcal{U}]}}[G\widetilde{\subset}\bigvee\mathcal{V}]]=\top$, for any $\mathcal{U}\subseteq\mathcal{T}$.

For $\mathcal{U}\subseteq\mathcal{T}$ we have $\chi_{\mathcal{T}}(\mathcal{U})=\top$, where $\chi_{\mathcal{T}}(A)=\top$, when $A\in\mathcal{T}$; $\chi_{\mathcal{T}}(A)=\bot$, when $A\notin\mathcal{T}$.

Thus, the following results can be obtained.

$G\in L^X$ has Lindelöf property $\Leftrightarrow\chi_{\mathcal{T}}(\mathcal{U})\leqslant[[G\widetilde{\subset}\bigvee\mathcal{U}]\leqslant\bigvee_{\mathcal{V}\in 2^{[\mathcal{U}]}}[G\widetilde{\subset}\bigvee\mathcal{V}]]$, $\forall\mathcal{U}\subseteq\mathcal{T}$.

In this way, we can very naturally introduce the definition of degree of G with Lindelöf property as follows:

Definition 4.1 *Let (X,τ) be an L-fuzzy topological space and $G \in L^X$. Then*

$$\begin{aligned}LPD_\tau(G) &= \bigwedge_{\mathcal{U}\subseteq L^X}\Big[\tau(\mathcal{U}) \leqslant \Big[[G\widetilde{\subset}\bigvee\mathcal{U}] \leqslant \bigvee_{\mathcal{V}\in 2^{[\mathcal{U}]}}[G\widetilde{\subset}\bigvee\mathcal{V}]\Big]\Big]\\ &= \bigwedge_{\mathcal{U}\subseteq L^X}\Big(\bigwedge_{A\in\mathcal{U}}\tau(A) \mapsto \Big(\bigwedge_{x\in X}\Big(G'(x)\vee\bigvee_{A\in\mathcal{U}}A(x)\Big) \mapsto \bigvee_{\mathcal{V}\in 2^{[\mathcal{U}]}}\bigwedge_{x\in X}\Big(G'(x)\vee\bigvee_{A\in\mathcal{V}}A(x)\Big)\Big)\Big)\end{aligned}$$

is called the degree to which G has the Lindelöf property with respect to τ.

Obviously, G has Lindelöf property in L-topological space $\mathcal{T}$ if and only if $LPD_{\chi_\mathcal{T}}(G) = \top$.

Analogous to countable compactness degree, we have the following results.

Lemma 4.2 *Let (X,τ) be an L-fuzzy topological space and $G \in L^X$. Then $a \le LPD_\tau(G)$ if and only if for any $\mathcal{U} \subseteq L^X$,*

$$\tau(\mathcal{U}) \wedge [G\widetilde{\subset}\bigvee\mathcal{U}] \wedge a \le \bigvee_{\mathcal{V}\in 2^{[\mathcal{U}]}}[G\widetilde{\subset}\bigvee\mathcal{V}]. \tag{4.1}$$

Theorem 4.3 *Let (X,τ) be an L-fuzzy topological space and $G \in L^X$. Then*

$$LPD_\tau(G) = \bigvee\Big\{a\in L : \tau(\mathcal{U}) \wedge [G\widetilde{\subset}\bigvee\mathcal{U}] \wedge a \le \bigvee_{\mathcal{V}\in 2^{[\mathcal{U}]}}[G\widetilde{\subset}\bigvee\mathcal{V}] \text{ for any } \mathcal{U}\subseteq L^X\Big\}. \tag{4.2}$$

Theorem 4.4 *Let (X,τ) be an L-fuzzy topological space and $G \in L^X$. Then*

$$\begin{aligned}LPD_\tau(G) &= \bigwedge_{\mathcal{U}\subseteq L^X}\Big(\bigwedge_{A\in\mathcal{U}}\tau(A) \wedge \bigwedge_{x\in X}\Big(G'(x)\vee\bigvee_{A\in\mathcal{U}}A(x)\Big) \mapsto \bigvee_{\mathcal{V}\in 2^{[\mathcal{U}]}}\bigwedge_{x\in X}\Big(G'(x)\vee\bigvee_{A\in\mathcal{V}}A(x)\Big)\Big)\\ &= \bigwedge_{\mathcal{U}\subseteq L^X}\Big[\tau(\mathcal{U}) \wedge [G\widetilde{\subset}\bigvee\mathcal{U}] \leqslant \bigvee_{\mathcal{V}\in 2^{[\mathcal{U}]}}[G\widetilde{\subset}\bigvee\mathcal{V}]\Big].\end{aligned}$$

Theorem 4.5 *Let (X,τ) be an L-fuzzy topological space. Then $\forall G, H \in L^X$, $LPD_\tau(G\wedge H) \ge LPD_\tau(G) \wedge \tau^*(H)$.*

Corollary 4.6 *Let (X,τ) be an L-fuzzy topological space. Then $\forall G \in L^X$, $LPD_\tau(G) \geqslant LPD_\tau(\underline{\top}) \wedge \tau^*(G)$.*

Theorem 4.7 *Let (X,τ) be an L-fuzzy topological space. Then $\forall G, H \in L^X$, $LPD_\tau(G\vee H) \ge LPD_\tau(G) \wedge LPD_\tau(H)$.*

Theorem 4.8 *Let (X,τ) be an L-fuzzy topological space and $G \in L^X$. Then $LPD_\tau(G) \wedge CCD_\tau(G) \le CD_\tau(G)$.*

Proof For any $a \in L$ and $a \le LPD_\tau(G) \wedge CCD_\tau(G)$, we need to prove that $a \le CD_\tau(G)$. Suppose any $\mathcal{U} \subseteq L^X$, since $a \le LPD_\tau(G) \wedge CCD_\tau(G)$, we have that $a \le LPD_\tau(G)$ and $a \le CCD_\tau(G)$. Thus $\tau(\mathcal{U}) \wedge [G\widetilde{\subset}\bigvee\mathcal{U}] \wedge a \le \bigvee_{\mathcal{V}\in 2^{[\mathcal{U}]}}[G\widetilde{\subset}\bigvee\mathcal{V}]$ by Lemma 4.2. For any $\mathcal{V} \in 2^{[\mathcal{U}]}$, $\mathcal{V} \subseteq L_C^X$. Therefore $\tau(\mathcal{U}) \le \tau(\mathcal{V})$ and $\tau(\mathcal{V}) \wedge [G\widetilde{\subset}\bigvee\mathcal{V}] \wedge a \le \bigvee_{\mathcal{W}\in 2^{(\mathcal{V})}}[G\widetilde{\subset}\bigvee\mathcal{W}]$ by Lemma 3.2.

Further, we have the following results

$$\begin{aligned}\tau(\mathcal{U})\wedge[G\widetilde{\subset}\bigvee\mathcal{U}]\wedge a &\le \tau(\mathcal{U})\wedge\bigvee_{\mathcal{V}\in 2^{[\mathcal{U}]}}[G\widetilde{\subset}\bigvee\mathcal{V}]\wedge a\le\bigvee_{\mathcal{V}\in 2^{[\mathcal{U}]}}(\tau(\mathcal{V})\wedge[G\widetilde{\subset}\bigvee\mathcal{V}]\wedge a)\\ &\le\bigvee_{\mathcal{V}\in 2^{[\mathcal{U}]}}(\bigvee_{\mathcal{W}\in 2^{(\mathcal{V})}}[G\widetilde{\subset}\bigvee\mathcal{W}])\le\bigvee_{\mathcal{V}\in 2^{[\mathcal{U}]}}(\bigvee_{\mathcal{W}\in 2^{(\mathcal{U})}}[G\widetilde{\subset}\bigvee\mathcal{W}])\\ &\le\bigvee_{\mathcal{W}\in 2^{(\mathcal{U})}}[G\widetilde{\subset}\bigvee\mathcal{W}]=\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}[G\widetilde{\subset}\bigvee\mathcal{V}].\end{aligned}$$

This implies $(\tau(\mathcal{U})\wedge a)\le([G\widetilde{\subset}\bigvee\mathcal{U}]\mapsto\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}[G\widetilde{\subset}\bigvee\mathcal{V}])$ for any $\mathcal{U}\subseteq L^X$ by the properties of implication operation $\mapsto$.

Further, we have $a\le\tau(\mathcal{U})\mapsto([G\widetilde{\subset}\bigvee\mathcal{U}]\mapsto\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}[G\widetilde{\subset}\bigvee\mathcal{V}])$ for any $\mathcal{U}\subseteq L^X$.

Thus $a\le CD_\tau(G)$. The proof is completed. $\Box$

Corollary 4.9 *Let (X,τ) be an L-fuzzy topological space and $G\in L^X$. Then $LPD_\tau(G)\wedge CCD_\tau(G)\le LPD_\tau(G)\wedge CD_\tau(G)$.*

The proposition can be regarded as the multi-value generalization of the result "if G has the Lindelöf property, then G is compact if and only if it is countably compact".

Acknowledgements We thank the referees for their time and comments.

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