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Characterizations of Semi-Metric Spaces

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Abstract We use the notions of weak base g-functions and pair networks to give some new characterizations of semi-metric spaces.

Keywords semi-metric spaces; weak base g-functions; pair networks; co-cushioned families

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1. Introduction

The notion of semi-metric spaces was introduced by Wilson [1] which plays an important role in the theory of generalized metric spaces. Properties of semi-metric spaces were investigated by Heath [2,3], Brown [4], McAuley [5,6] and Jones [7], etc and several important characterizations were obtained. In this paper, we shall give some new characterizations of semi-metric spaces in terms of weak base g-functions and pair networks.

Throughout, the set of all positive integers is denoted by $\mathbb{N}$ while $\langle x_n \rangle$ denotes a sequence. Let X be a topological space. Then τ is the topology on X and τ^c is the family of all closed subsets of X. For a subset A of a space X, the closure and the interior of A will be denoted by $\overline{A}$ and $\operatorname{int} A$, respectively.

A collection $\mathcal{P}$ of pairs of subsets of a space X is called a pair network [8] if $P_1 \subset P_2$ for each $(P_1, P_2) \in \mathcal{P}$ and if $x \in U \in \tau$, then there exists $(P_1, P_2) \in \mathcal{P}$ such that $x \in P_1 \subset P_2 \subset U$.

Definition 1.1 ([1]) *A space X is called a semi-metric space if there is a function $d : X \times X \to [0, \infty)$ satisfying the following conditions:*

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) a subset U of X is open if and only if for each $x \in U$, there exists $r > 0$ such that $B(x, r) \subset U$, where $B(x, r) = \{y \in X : d(x, y) < r\}$;

(4) for each $x \in X$ and $r > 0$, $x \in \operatorname{int}(B(x, r))$.

Definition 1.2 *Let $\mathcal{P}$ be a collection of pairs of subsets of a space X. Then:*

(1) $\mathcal{P}$ is called co-cushioned [9] if for each $\mathcal{P}' \subset \mathcal{P}$, $\cap\{P_1 : (P_1, P_2) \in \mathcal{P}'\} \subset \operatorname{int}(\cap\{P_2 : (P_1, P_2) \in \mathcal{P}'\})$;

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(2) $\mathcal{P}$ is called cushioned [10] if for each $\mathcal{P}' \subset \mathcal{P}$, $\overline{\cup\{P_1 : (P_1, P_2) \in \mathcal{P}'\}} \subset \cup\{P_2 : (P_1, P_2) \in \mathcal{P}'\}$.

Definition 1.3 ([11]) *A weak base for a space X is a collection $\mathcal{B} = \cup\{\mathcal{B}_x : x \in X\}$ of subsets of X such that*

(1) For each $x \in X$, $x \in \cap\mathcal{B}_x$ and $\mathcal{B}_x$ is closed under finite intersections;

(2) A subset U of X is open if and only if for each $x \in U$ there exists $B \in \mathcal{B}_x$ such that $x \in B \subset U$.

Definition 1.4 ([12]) *A weak base g-function for a space X is a map $g : \mathbb{N} \times X \to 2^X$ satisfying:*

(1) For every $x \in X$ and $n \in \mathbb{N}$, $x \in g(n+1, x) \subset g(n, x)$;

(2) $\{g(n, x) : n \in \mathbb{N}, x \in X\}$ is a weak base for X, i.e., a subset U of X is open if and only if for each $x \in U$ there exists $n \in \mathbb{N}$ such that $g(n, x) \subset U$.

A g-function [13] for a space X is a mapping $g : \mathbb{N} \times X \to \tau$ such that for every $x \in X$ and $n \in \mathbb{N}$, $x \in g(n+1, x) \subset g(n, x)$.

Let g be a g-function or a weak base g-function for a space X. For a set $A \subset X$, denote $g(n, A) = \cup\{g(n, x) : x \in A\}$.

All spaces are assumed to be T_2 unless otherwise stated.

2. Main results

In this section, we shall give some new characterizations of semi-metric spaces.

The following theorem is a reformulation of a theorem in [3].

Theorem 2.1 *A space X is a semi-metric space if and only if for each $x \in X$ there is a decreasing (open) neighborhood base $\{U_n(x) : n \in \mathbb{N}\}$ for x such that for each $A \subset X$ and $n \in \mathbb{N}$, $\overline{A} \subset \cup\{U_n(x) : x \in A\}$.*

Proof Let X be a semi-metric space. For each $x \in X$ and $n \in \mathbb{N}$, let $V_n(x) = B(x, 1/n)$. For each $n \in \mathbb{N}$, if $V_n(x) \subset \mathrm{int}(V_m(x))$ for some $m \in \mathbb{N}$, then let $k_n = \max\{i \leq n : V_n(x) \subset \mathrm{int}(V_i(x))\}$ and $U_n(x) = \mathrm{int}(V_{k_n}(x))$; otherwise, let $U_n(x) = X$. Then $\{U_n(x) : n \in \mathbb{N}\}$ is a decreasing open neighborhood base for x. Indeed, if $x \in U \in \tau$, then there exist $m, n \in \mathbb{N}$ such that $V_n(x) \subset \mathrm{int}(V_m(x)) \subset U$, from which it follows that $U_n(x) = \mathrm{int}(V_{k_n}(x))$ with $m \leq k_n \leq n$. Consequently, $U_n(x) \subset \mathrm{int}(V_m(x)) \subset U$ and so $\{U_n(x) : n \in \mathbb{N}\}$ is a neighborhood base for x. Now, let $A \subset X$ and $n \in \mathbb{N}$. If $x \in \overline{A}$, then $\mathrm{int}(V_n(x)) \cap A \neq \emptyset$. Choose $y \in \mathrm{int}(V_n(x)) \cap A$, then $x \in V_n(y)$. If $V_n(y) \nsubseteq \mathrm{int}(V_m(y))$ for all $m \in \mathbb{N}$, then $U_n(y) = X$ and so $x \in U_n(y)$. If $V_n(y) \subset \mathrm{int}(V_m(y))$ for some $m \in \mathbb{N}$, then $U_n(y) = \mathrm{int}(V_{k_n}(y))$ with $V_n(y) \subset \mathrm{int}(V_{k_n}(y))$ which also follows that $x \in U_n(y)$. As a consequence, $\overline{A} \subset \cup\{U_n(x) : x \in A\}$.

Conversely, for each $x \in X$, let $\{U_n(x) : n \in \mathbb{N}\}$ be a decreasing neighborhood base for x that satisfies the condition of the theorem. For each $x \in X$ and $n \in \mathbb{N}$, put $o(n, x) = \{y \in U_n(x) : x \in U_n(y)\}$. Then $x \in \mathrm{int}(o(n, x))$. Indeed, since $\overline{A} \subset \cup\{U_n(x) : x \in A\}$ for each $A \subset X$, $x \in X \setminus \overline{\{y \in X : x \notin U_n(y)\}}$. So if we let $V_n(x) = U_n(x) \setminus \overline{\{y \in X : x \notin U_n(y)\}}$, then

$V_n(x)$ is a neighborhood of x and $V_n(x) \subset o(n,x)$ which implies that $x \in \mathrm{int}(o(n,x))$. Now, for every pair of distinct $x, y \in X$, let $m(x,y) = \min\{n \in \mathbb{N} : y \notin o(n,x)\}$. Define a function $d : X \times X \to [0,\infty)$ as follows: $d(x,x) = 0$ and $d(x,y) = 1/m(x,y)$ whenever $x \neq y$. It is easy to verify that $B(x,1/n) = o(n,x)$ for each $x \in X$ and $n \in \mathbb{N}$, from which it follows that $x \in \mathrm{int}(B(x,r))$ for each $r > 0$. ($o(n,x)$ is decreasing with respect to n, so if $x \neq y$ then $y \in o(n,x)$ if and only if $n < m(x,y)$ if and only if $y \in B(x,1/n)$). Now it is clear that U is open if and only if for each $x \in U$ there exists $r > 0$ such that $B(x,r) \subset U$. Consequently, X is a semi-metric space. $\square$

Corollary 2.2 *A space X is a semi-metric space if and only if for each $x \in X$ there exist two sequences $\{U_n(x) : n \in \mathbb{N}\}$ and $\{V_n(x) : n \in \mathbb{N}\}$ of neighborhoods of x satisfying the following conditions:*

(i) $\{U_n(x) : n \in \mathbb{N}\}$ *is a neighborhood base of x;*

(ii) For each $n \in \mathbb{N}$, if $x \in V_n(y)$, then $y \in U_n(x)$.

Proof Let X be a semi-metric space. For each $x \in X$ and $n \in \mathbb{N}$, let $U_n(x) = B(x,1/n)$ and $V_n(x) = \mathrm{int}(B(x,1/n))$, then $U_n(x)$ and $V_n(x)$ are the desired neighborhoods of x.

For the converse, let $\{U_n(x) : n \in \mathbb{N}\}$ and $\{V_n(x) : n \in \mathbb{N}\}$ satisfy the conditions. For each $n \in \mathbb{N}$ and $A \subset X$, if $x \in \overline{A}$, then $V_n(x) \cap A \neq \emptyset$. Choose $y \in V_n(x) \cap A$, then by (ii), $x \in U_n(y)$. Hence, $\overline{A} \subset \cup\{U_n(x) : x \in A\}$. By Theorem 2.1, X is a semi-metric space. $\square$

Lemma 2.3 ([8]) *Let $\cup\{\mathcal{B}_x : x \in X\}$ be a weak base for a space X. For each $x \in X$ and $B \in \mathcal{B}_x$, if $x_n \to x$, then $\{x_n : n \geq m\} \subset B$ for some $m \in \mathbb{N}$.*

With the above lemma, we can give a characterization of semi-metric spaces in terms of weak base g-functions as follows.

Theorem 2.4 *A space X is a semi-metric space if and only if there exists a weak base g-function g for X satisfying the following conditions:*

(1) If $x \in g(n,x_n)$ for all $n \in \mathbb{N}$, then $x_n \to x$;

(2) For each $A \subset X$ and $n \in \mathbb{N}$, $\overline{A} \subset g(n,A)$.

Proof Let X be a semi-metric space. For each $x \in X$ and $n \in \mathbb{N}$, let $g(n,x) = B(x,1/n)$. Then it is easy to verify that g is a weak base g-function for X which satisfies conditions (1) and (2).

Conversely, let g be a weak base g-function for X satisfying the conditions. For each $x \in X$ and $n \in \mathbb{N}$, let $U_n(x) = X \setminus \overline{\{y \in X : x \notin U_n(y))\}}$. Then $U_n(x)$ is an open neighborhood of x by (2). We shall show that $g(n,x)$ is a neighborhood of x for each $n \in \mathbb{N}$. Suppose not, then $x \in \overline{X \setminus g(m,x)}$ for some $m \in \mathbb{N}$. So $U_n(x) \setminus g(m,x) \neq \emptyset$ for each $n \in \mathbb{N}$. Choose $x_n \in U_n(x) \setminus g(m,x)$, then $x \in g(n,x_n)$ for all $n \in \mathbb{N}$. It follows from (1) that $x_n \to x$, a contradiction with Lemma 2.3. Now, for each $x \in X$ and $n \in \mathbb{N}$, put $V_n(x) = g(n,x) \cap U_n(x)$ and $o(n,x) = \{y \in g(n,x) : x \in g(n,y)\}$. Then $V_n(x)$ is a neighborhood of x and $V_n(x) \subset o(n,x)$ which implies that $o(n,x)$ is also a neighborhood of x. Next, define a function d as that in the proof of Theorem 2.1. It is easy to check that $B(x,1/n) = o(n,x)$ for each $x \in X$ and $n \in \mathbb{N}$, from which it follows that $x \in \mathrm{int}(B(x,r))$ for each $r > 0$. If for each $x \in U$ there exists $r > 0$ such that $B(x,r) \subset U$, then it is clear that U is open. Suppose now that U is open and $x \in U$. If $B(x, 1/n) \setminus U \neq \emptyset$ for all $n \in \mathbb{N}$, choose $x_n \in B(x, 1/n) \setminus U$, then $x \in g(n, x_n)$. By (1), $x_n \to x$, a contradiction. Consequently, X is a semi-metric space. □

Corollary 2.5 *A space X is a semi-metric space if and only if there exists a weak base g-function g for X such that for each $A \subset X$, $\overline{A} = \cap_{n\in\mathbb{N}} g(n, A)$.*

Proof Let X be a semi-metric space. By Theorem 2.4, there exists a weak base g-function g for X satisfying conditions (1) and (2). Then for each $A \subset X$, $\overline{A} \subset \cap_{n\in\mathbb{N}} g(n, A)$. Now, let $x \in \cap_{n\in\mathbb{N}} g(n, A)$. Then for each $n \in \mathbb{N}$, there is $x_n \in A$ such that $x \in g(n, x_n)$. By (1), $x_n \to x$ which implies that $x \in \overline{A}$. Thus $\cap_{n\in\mathbb{N}} g(n, A) \subset \overline{A}$.

For the converse, let g be a weak base g-function for X satisfying the condition. Then g satisfies condition (2) of Theorem 2.4. Now suppose that $x \in g(n, x_n)$ for all $n \in \mathbb{N}$. If $\langle x_n \rangle$ does not converge to x, then there exists a subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ such that $x \notin \overline{\{x_{n_k} : k \in \mathbb{N}\}}$. Put $A = \{x_{n_k} : k \in \mathbb{N}\}$, then there is $m \in \mathbb{N}$ such that $x \notin g(m, A)$. Choose $k \in \mathbb{N}$ such that $n_k > m$, then $x \notin g(n_k, x_{n_k})$, a contradiction. Thus g also satisfies condition (1) of Theorem 2.4. Consequently, X is a semi-metric space. □

Corollary 2.6 ([3]) *A space X is a semi-metric space if and only if there exists a g-function g for X satisfying: if for each $n \in \mathbb{N}$, $x_n \in g(n, x)$ or $x \in g(n, x_n)$, then $x_n \to x$.*

Proof We only prove the sufficiency.

For each $x \in X$ and $n \in \mathbb{N}$, let $h(n, x) = \{y \in X : y \in g(n, x) \text{ or } x \in g(n, y)\}$. Then $g(n, x) \subset h(n, x)$. Thus, if for each $x \in U$ there exists $n \in \mathbb{N}$ such that $h(n, x) \subset U$, then $g(n, x) \subset U$, therefore U is open. Now, suppose that U is open and $x \in U$. Assume that $h(n, x) \setminus U \neq \emptyset$ for all $n \in \mathbb{N}$. Choose $x_n \in h(n, x) \setminus U$, then $x_n \in g(n, x)$ or $x \in g(n, x_n)$ for each $n \in \mathbb{N}$, therefore $x_n \to x$, a contradiction. Hence there is $m \in \mathbb{N}$ such that $h(m, x) \subset U$. From the above argument, we see that h is a weak base g-function for X.

It is easy to verify that h satisfies conditions (1) and (2) of Theorem 2.4. ($g(n, x) \subset h(n, x)$. If $x \in \overline{A}$, then $g(n, x) \cap A \neq \emptyset$, from which it follows that $h(n, x) \cap A \neq \emptyset$. Choose $y \in h(n, x) \cap A$, then $x \in h(n, y)$ and $y \in A$ which implies that $x \in h(n, A)$). Therefore, X is a semi-metric space. □

Corollary 2.7 *A space X is a semi-metric space if and only if there exists a g-function g for X satisfying: for each $x \in X$ and $F \in \tau^c$, if $x \notin F$, then there exists $m \in \mathbb{N}$ such that $x \notin g(m, F)$ and $F \cap g(m, x) = \emptyset$.*

Proof Let X be a semi-metric space. Then there exists a g-function g for X satisfying the condition of Corollary 2.6. Let $F \in \tau^c$ and $x \notin F$. Assume that for each $n \in \mathbb{N}$, $x \in g(n, F)$ or $F \cap g(n, x) \neq \emptyset$, then there is $x_n \in F$ such that $x_n \in g(n, x)$ or $x \in g(n, x_n)$. By Corollary 2.6, $x_n \to x$ which implies that $x \in F$, a contradiction.

For the converse, let g be a g-function for X satisfying the condition. Suppose that for each $n \in \mathbb{N}$, $x_n \in g(n,x)$ or $x \in g(n,x_n)$. If $\langle x_n \rangle$ does not converge to x, then there exists a subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ such that $x \notin \overline{\{x_{n_k} : k \in \mathbb{N}\}}$. Put $F = \overline{\{x_{n_k} : k \in \mathbb{N}\}}$, then there exists $m \in \mathbb{N}$ such that $x \notin g(m,F)$ and $F \cap g(m,x) = \emptyset$. Choose $k \in \mathbb{N}$ such that $n_k > m$, then $x \notin g(n_k, x_{n_k})$ and $x_{n_k} \notin g(n_k, x)$, a contradiction. Thus $x_n \to x$. By Corollary 2.6, X is a semi-metric space. $\Box$

Theorem 2.8 *A space X is a semi-metric space if and only if it has a pair network $\mathcal{P} = \cup_{n\in\mathbb{N}}\mathcal{P}_n$ such that for each $n \in \mathbb{N}$, $\mathcal{P}_n$ is both cushioned and co-cushioned.*

Proof Suppose that X is a semi-metric space and let g be the weak base g-function for X in Theorem 2.4. For each $n \in \mathbb{N}$, let $\mathcal{P}_n = \{(x, g(n,x)) : x \in X\}$ and $\mathcal{P} = \cup_{n\in\mathbb{N}}\mathcal{P}_n$. Then $\mathcal{P}$ is a pair network for X. Clearly, each $\mathcal{P}_n$ is co-cushioned. By (2), $\mathcal{P}_n$ is also cushioned.

Conversely, let $\mathcal{P} = \cup_{n\in\mathbb{N}}\mathcal{P}_n$ be a pair network for X where for each $n \in \mathbb{N}$, $\mathcal{P}_n$ is both cushioned and co-cushioned. Without loss of generality, we may assume that for each $n \in \mathbb{N}$, $\mathcal{P}_n \subset \mathcal{P}_{n+1}$. For each $x \in X$ and $n \in \mathbb{N}$, put $g(n,x) = \mathrm{int}(\cap\{P_2 : x \in P_1, (P_1,P_2) \in \mathcal{P}_n\}) \setminus \overline{\cup\{P_1 : x \notin P_2, (P_1,P_2) \in \mathcal{P}_n\}}$. Since $\mathcal{P}_n$ is both co-cushioned and cushioned, g is a g-function for X.

Suppose that $F \in \tau^c$ and $x \notin F$. Then there is $m \in \mathbb{N}$ and $(P_1,P_2) \in \mathcal{P}_m$ such that $x \in P_1 \subset P_2 \subset X \setminus F$. Since $x \in P_1$, we have that $g(m,x) \subset P_2$, therefore $F \cap g(m,x) = \emptyset$. Now, for each $y \in F$, $y \notin P_2$. It follows from the definition of g that $g(m,y) \cap P_1 = \emptyset$. Thus $x \notin g(m,y)$ which implies that $x \notin g(m,F)$. By Corollary 2.7, X is a semi-metric space. $\Box$

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References

[1] W. A. WILSON. *On Semi-Metric Spaces.* Amer. J. Math., 1931, **53**(2): 361–373.
[2] R. W. HEATH. *A regular semi-metric space for which there is no semi-metric under which all spheres are open.* Proc. Amer. Math. Soc., 1961, **12**: 810–811.
[3] R. W. HEATH. *Arc-wise connectedness in semi-metric spaces.* Pacific J. Math., 1962, **12**: 1301–1319.
[4] M. BROWN. *Semi-metric spaces Summer.* Institute on Set Theoretic Topology, Madison, Wisconsin, Amer. Math. Soc., 1955, 62–64.
[5] L. F. MCAULEY. *On semi-metric spaces.* Summer Institute on Set Theoretic Topology, Madison, Wisconsin, Amer. Math. Soc., 1955, 58–62.
[6] L. F. MCAULEY. *A relation between perfect separability, completeness, and normality in semi-metric spaces.* Pacific J. Math., 1956, **6**: 315–326.
[7] F. B. JONES. *Introductory remarks on semi-metric spaces.* Summer Institute on Set Theoretic Topology, Madison, Wisconsin, Amer. Math. Soc., 1955, 58.
[8] Shou LIN. *Weak bases and metrization theorems.* Sichuan Daxue Xuebao, 1993, **30**(2): 164–166. (in Chinese)
[9] A. ANDRIKOPOULOS. *The quasimetrization problem in the (Bi)topological spaces.* Int. J. Math. Math. Sci., 2007, 1–19.
[10] J. G. CEDER. *Some generalizations of metric spaces.* Pacific J. Math., 1961, **11**: 105–125.
[11] A. V. ARHANGEL'SKII. *Mappings and spaces.* Russian Math. Surveys, 1966, **21**: 115–162.
[12] Zhimin GAO. *Metrizability of spaces and weak base g-functions.* Topology Appl., 2005, **146/147**: 279–288.
[13] R. E. HODEL. *Spaces defined by sequence of open covers which guarantee that certain sequences have cluster points.* Duke Math. J., 1972, **39**: 253–263.