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On Intuitionistic Fuzzy *LI*-ideals in Lattice Implication Algebras

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Abstract In the present paper, the intuitionistic fuzzy LI-ideal theory in lattice implication algebras is further studied. Some new properties and equivalent characterizations of intuitionistic fuzzy LI-ideals are given. Representation theorem of intuitionistic fuzzy LI-ideal which is generated by an intuitionistic fuzzy set is established. It is proved that the set consisting of all intuitionistic fuzzy LI-ideals in a lattice implication algebra, under the inclusion order, forms a complete distributive lattice.

Keywords lattice-valued logic; lattice implication algebra; intuitionistic fuzzy LI-ideal

MR(2010) Subject Classification 03B50; 03G25; 03E72

1. Introduction

In the field of many-valued logic, lattice-valued logic [1] plays an important role for two aspects: Firstly, it extends the chain-type truth-value field of some well-known presented logics (such as two-valued logic, three-valued logic, n-valued logic, the Łukasiewicz logic with truth values in the interval $[0, 1]$ and Zadeh's infinite-valued logic, and so on) to some relatively general lattices. Secondly, the incompletely comparable property of truth value characterized by general lattice can more efficiently reflect the uncertainty of people's thinking, judging and decision. Hence, lattice-valued logic is becoming a research field which strongly influences the development of algebraic logic, computer science and artificial intelligence technology. In order to establish a logic system with truth value in a relatively general lattice, in 1990, Xu [2] firstly proposed the concept of lattice implication algebra by combining lattice and implication algebra, and researched many useful properties. It provided the foundation to establish the corresponding logic system from the algebraic viewpoint. Since then, this kind of logical algebra has been extensively investigated by many authors [3–9]. For the general development of lattice implication algebras, the ideal theory plays an important role. Jun introduced the notions of LI-ideals and prime LI-ideals in lattice implication algebras and investigated their properties in [10] and [11]. Liu etc. [12] studied several properties of prime LI-ideals and ILI-ideals in lattice implication algebras. The concept of fuzzy sets was presented firstly by Zadeh [13] in 1965. At present, fuzzy sets has been applied in the field of algebraic structures, the study of fuzzy algebras has achieved great success. Many wonderful and valuable results have been obtained by some

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mathematical researchers, such as Rosenfeld [14], Mordeson and Malik [15], Shum [16] and Zhan [17]. As a generalization of the concept of fuzzy sets, Atanassov [18] introduced the concept of intuitionistic fuzzy sets. Recently, based on the study of intuitionistic fuzzy sets, more and more researchers have devoted themselves to applying some results of intuitionistic fuzzy sets to algebraic structures [19–22]. Among them, Jun etc. [20] introduced the notions of intuitionistic fuzzy LI-ideals and intuitionistic fuzzy lattice ideals in lattice implication algebras. However, more properties of intuitionistic fuzzy LI-ideals, especially, from the point of lattice theory, are less frequent.

In this paper, we will further research the properties of intuitionistic fuzzy LI-ideals in lattice implication algebras. The rest of this article is organized as follows. In Section 2, we review related basic knowledge of lattice implication algebras and intuitionistic fuzzy sets. In Section 3, we discuss several new properties and characterizations of intuitionistic fuzzy LI-ideals. In Section 4, we introduce the concept of intuitionistic fuzzy LI-ideal which is generated by an intuitionistic fuzzy set and establish its representation theorem. In Section 5, we investigate the lattice structural feature of the set containing all of intuitionistic fuzzy LI-ideals in a lattice implication algebra. Finally, we conclude this paper in Section 6.

2. Preliminaries

In this section, we review related basic knowledge of lattice implication algebras [1,2] and intuitionistic fuzzy sets [18].

Definition 2.1 ([2]) *Let $(L,\vee,\wedge,\prime,\to,O,I)$ be a bounded lattice with an order-reversing involution $'$, where I and O are the greatest and the smallest elements of L respectively, $\to$: $L\times L\to L$ is a mapping. Then $(L,\vee,\wedge,\prime,\to,O,I)$ is called a lattice implication algebra if the following conditions hold for all $x,y,z\in L$:*

(I_1) $x\to(y\to z)=y\to(x\to z)$;
(I_2) $x\to x=I$;
(I_3) $x\to y=y'\to x'$;
(I_4) $x\to y=y\to x=I$ implies $x=y$;
(I_5) $(x\to y)\to y=(y\to x)\to x$;
(l_1) $(x\vee y)\to z=(x\to z)\wedge(y\to z)$;
(l_2) $(x\wedge y)\to z=(x\to z)\vee(y\to z)$.

In the sequel, a lattice implication algebra $(L,\vee,\wedge,\prime,\to,O,I)$ will be denoted by L in short.

Lemma 2.2 ([1]) *Let L be a lattice implication algebra. Then for all $x,y,z\in L$,*

(1) $O\to x=I$, $x\to I=I$, $I\to x=x$ and $x'=x\to O$;
(2) $x\leqslant y$ if and only if $x\to y=I$;
(3) $x\to y\leqslant(y\to z)\to(x\to z)$ and $x\vee y=(x\to y)\to y$;
(4) $x\leqslant y$ implies $y\to z\leqslant x\to z$ and $z\to x\leqslant z\to y$;
(5) $x\to(y\vee z)=(x\to y)\vee(x\to z)$ and $x\to(y\wedge z)=(x\to y)\wedge(x\to z)$;

(6) $x \oplus y = y \oplus x$ *and* $(x \oplus y) \oplus z = x \oplus (y \oplus z)$*;*

(7) $O \oplus x = x$, $I \oplus x = I$ *and* $x \oplus x' = I$*;*

(8) $x \vee y \leqslant x \oplus y$ *and* $x \leqslant (x \to y)' \oplus y$*;*

(9) $x \leqslant y$ *implies* $x \oplus z \leqslant y \oplus z$*,*

where, $x \oplus y = x' \to y$ *for all* $x, y \in L$.

In the unit interval $[0,1]$ equipped with the natural order, $\vee = \max$ and $\wedge = \min$. Let $X \neq \emptyset$. A mapping $\alpha : X \to [0,1]$ is called a fuzzy set on X (see [13]). Let α and β be two fuzzy sets on X. We define $\alpha \cap \beta, \alpha \cup \beta, \alpha \subseteq \beta$ and $\alpha = \beta$ as follows:

(1) $(\alpha \cap \beta)(x) = \alpha(x) \wedge \beta(x)$, for all $x \in X$;

(2) $(\alpha \cup \beta)(x) = \alpha(x) \vee \beta(x)$, for all $x \in X$;

(3) $\alpha \subseteq \beta \Longleftrightarrow \alpha(x) \leqslant \beta(x)$, for all $x \in X$;

(4) $\alpha = \beta \Longleftrightarrow (\alpha \subseteq \beta$ and $\beta \subseteq \alpha)$.

Definition 2.3 ([18]) *Let a set* X *be the fixed domain. An intuitionistic fuzzy set* A *in* X *is an object having the form* $A = \{< x, \alpha_A(x), \beta_A(x) > | x \in X\}$*, where* α_A *and* β_A *are fuzzy sets on* X*, denoting the degree of membership and nonmembership respectively, and satisfying* $0 \leqslant \alpha_A(x) + \beta_A(x) \leqslant 1$*, for all* $x \in X$.

For the sake of simplicity, in the sequel, we shall use the symbol $A = (\alpha_A, \beta_A)$ to denote an intuitionistic fuzzy set in X. Let $A = (\alpha_A, \beta_A)$ and $B = (\alpha_B, \beta_B)$ be two intuitionistic fuzzy sets in X. We define $A \Cap B, A \Cup B, A \Subset B$ and $A = B$ as follows:

(1) $A \Cap B = (\alpha_A \cap \alpha_B, \beta_A \cup \beta_B)$;

(2) $A \Cup B = (\alpha_A \cup \alpha_B, \beta_A \cap \beta_B)$;

(3) $A \Subset B \Longleftrightarrow (\alpha_A \subseteq \alpha_B$ and $\beta_B \subseteq \beta_A)$;

(4) $A = B \Longleftrightarrow (A \Subset B$ and $B \Subset A)$.

3. Intuitionistic fuzzy *LI*-ideals

Definition 3.1 ([20]) *Let* L *be a lattice implication algebra. An intuitionistic fuzzy set* $A = (\alpha_A, \beta_A)$ *in* L *is called an intuitionistic fuzzy LI-ideal of* L *if it satisfies the following conditions for all* $x, y \in L$*,*

(IFI1) $\alpha_A(O) \geqslant \alpha_A(x)$ *and* $\beta_A(O) \leqslant \beta_A(x)$*;*

(IFI2) $\alpha_A(x) \geqslant \alpha_A((x \to y)') \wedge \alpha_A(y)$ *and* $\beta_A(x) \leqslant \beta_A((x \to y)') \vee \beta_A(y)$.

The set of all intuitionistic fuzzy LI-ideals of L *is denoted by* $\mathbf{IFLI}(L)$.

Theorem 3.2 ([20]) *Let* L *be a lattice implication algebra and* $A = (\alpha_A, \beta_A) \in \mathbf{IFLI}(L)$*. Then* A *is intuitionistic non-increasing, i.e., it satisfies the following condition for all* $x, y \in L$*,*

(IFI3) $x \leqslant y \Longrightarrow (\alpha_A(x) \geqslant \alpha_A(y)$ *and* $\beta_A(x) \leqslant \beta_A(y))$.

Theorem 3.3 *Let* L *be a lattice implication algebra and* $A = (\alpha_A, \beta_A)$ *an intuitionistic fuzzy set in* L*. Consider the following conditions for all* $x, y \in L$*,*

(IFI4) $z \leqslant x \oplus y \Longrightarrow (\alpha_A(z) \geqslant \alpha_A(x) \wedge \alpha_A(y)$ *and* $\beta_A(z) \leqslant \beta_A(x) \vee \beta_A(y))$*;*

(IFI5) $\alpha_A(x \oplus y) \geqslant \alpha_A(x) \wedge \alpha_A(y)$ *and* $\beta_A(x \oplus y) \leqslant \beta_A(x) \vee \beta_A(y)$*;*

(IFI6) $\alpha_A(x\oplus z)\geqslant\alpha_A((x\to y)')\wedge\alpha_A(y\oplus z)$ *and* $\beta_A(x\oplus z)\leqslant\beta_A((x\to y)')\vee\beta_A(y\oplus z)$*;*

(IFI7) $\alpha_A((x\to z)')\geqslant\alpha_A((x\to y)')\wedge\alpha_A((y\to z)')$ *and* $\beta_A((x\to z)')\leqslant\beta_A((x\to y)')\vee\beta_A((y\to z)')$.

Then $A\in\mathbf{IFLI}(L)\Longleftrightarrow$*(IFI4)*$\Longleftrightarrow$*(IFI3)+(IFI5)*$\Longleftrightarrow$*(IFI1)+(IFI6)*$\Longleftrightarrow$*(IFI1)+(IFI7).*

Proof (1) $A\in\mathbf{IFLI}(L)\Longleftrightarrow$(IFI4): Assume $A\in\mathbf{IFLI}(L)$ and $x,y,z\in L$. If $z\leqslant x\oplus y$, then $I=z\to(x\oplus y)=z\to(x'\to y)=x'\to(z\to y)=(z\to y)'\to x$, and so $((z\to y)'\to x)'=O$. It follows that $\alpha_A(z)\geqslant\alpha_A((z\to y)')\wedge\alpha_A(y)\geqslant\alpha_A(((z\to y)'\to x)')\wedge\alpha_A(x)\wedge\alpha_A(y)=\alpha_A(O)\wedge\alpha_A(x)\wedge\alpha_A(y)=\alpha_A(x)\wedge\alpha_A(y)$ and $\beta_A(z)\leqslant\beta_A((z\to y)')\vee\beta_A(y)\leqslant\beta_A(((z\to y)'\to x)')\vee\beta_A(x)\vee\beta_A(y)=\beta_A(O)\vee\beta_A(x)\vee\beta_A(y)=\beta_A(x)\vee\beta_A(y)$, i.e., (IFI4) holds. Conversely, assume (IFI4) holds. On the one hand, since $O\leqslant x\oplus x$ for any $x\in L$, we have that $\alpha_A(O)\geqslant\alpha_A(x)\wedge\alpha_A(x)=\alpha_A(x)$ and $\beta_A(O)\leqslant\beta_A(x)\vee\beta_A(x)=\beta_A(x)$ by using (IFI4), i.e., (IFI1) holds. On the other hand, from $x\leqslant(x\to y)'\oplus y$ for all $x,y\in L$, it follows that $\alpha_A(x)\geqslant\alpha_A((x\to y)')\wedge\alpha_A(y)$ and $\beta_A(x)\leqslant\beta_A((x\to y)')\vee\beta_A(y)$, i.e., (IFI2) also holds. By Definition 3.1, we get that $A\in\mathbf{IFLI}(L)$.

(2) (IFI4)$\Longrightarrow$(IFI3)+(IFI5): Assume (IFI4) holds. If $x\leqslant y$, then $x\leqslant y\leqslant y\oplus y$ by Lemma 2.2 (8). From (IFI4), it follows that $\alpha_A(x)\geqslant\alpha_A(y)\wedge\alpha_A(y)=\alpha_A(y)$ and $\beta_A(x)\leqslant\beta_A(y)\vee\beta_A(y)=\beta_A(y)$, i.e., (IFI3) holds. For all $x,y\in L$, since $x\oplus y\leqslant x\oplus y$, by (IFI4) we have that $\alpha_A(x\oplus y)\geqslant\alpha_A(x)\wedge\alpha_A(y)$ and $\beta_A(x\oplus y)\leqslant\beta_A(x)\vee\beta_A(y)$, i.e., (IFI5) holds.

(3) (IFI3)+(IFI5)$\Longrightarrow$(IFI1)+(IFI6): Assume (IFI3) and (IFI5) hold. Obviously, $\alpha_A(O)\geqslant\alpha_A(x)$ and $\beta_A(O)\leqslant\beta_A(x)$ for all $x\in L$ by $O\leqslant x$ and (IFI3), i.e., (IFI1) holds. Let $x,y\in L$. By Lemma 2.2 (3) we have that $x\leqslant(x\to y)'\oplus y$, hence $x\oplus z\leqslant(x\to y)'\oplus y\oplus z=(x\to y)'\oplus(y\oplus z)$ by using Lemma 2.2 (6) and (9), and so we can obtain that $\alpha_A(x\oplus z)\geqslant\alpha_A((x\to y)'\oplus(y\oplus z))\geqslant\alpha_A((x\to y)')\wedge\alpha_A(y\oplus z)$ and $\beta_A(x\oplus z)\leqslant\beta_A((x\to y)'\oplus(y\oplus z))\leqslant\beta_A((x\to y)')\vee\beta_A(y\oplus z)$ by (IFI3) and (IFI5). This means that (IFI6) holds.

(4) (IFI1)+(IFI6)$\Longrightarrow$(IFI4): Assume (IFI1) and (IFI6) hold. By letting $z=O$ and $x\leqslant y$, we have by (IFI1) that $\alpha_A(x)=\alpha_A(x\oplus O)\geqslant\alpha_A((x\to y)')\wedge\alpha_A(y\oplus O)=\alpha_A(O)\wedge\alpha_A(y)=\alpha_A(y)$ and $\beta_A(x)=\beta_A(x\oplus O)\leqslant\beta_A((x\to y)')\vee\beta_A(y\oplus O)=\beta_A(O)\vee\beta_A(y)=\beta_A(y)$. Then for all $x,y\in L$ and $z\leqslant x\oplus y$, we have that $\alpha_A(z)\geqslant\alpha_A(x\oplus y)\geqslant\alpha_A((x\to O)')\wedge\alpha_A(y\oplus O)=\alpha_A(x)\wedge\alpha_A(y)$ and $\beta_A(z)\leqslant\beta_A(x\oplus y)\leqslant\beta_A((x\to O)')\vee\beta_A(y\oplus O)=\beta_A(x)\vee\beta_A(y)$. Hence (IFI4) holds.

(5) $A\in\mathbf{IFLI}(L)\Longleftrightarrow$(IFI1)+(IFI7): Assume $A\in\mathbf{IFLI}(L)$. Then (IFI1) holds by Definition 3.1. Since $((x\to z)'\to(y\to z)')'\to(x\to y)'=(x\to y)\to((y\to z)\to(x\to z))=I$ by (I_3) and Lemma 2.2, we have $((x\to z)'\to(y\to z)')'\leqslant(x\to y)'$. Hence we can obtain that $\alpha_A((x\to z)')\geqslant\alpha_A(((x\to z)'\to(y\to z)')')\wedge\alpha_A((y\to z)')\geqslant\alpha_A((x\to y)')\wedge\alpha_A((y\to z)')$ and $\beta_A((x\to z)')\leqslant\beta_A(((x\to z)'\to(y\to z)')')\vee\beta_A((y\to z)')\leqslant\beta_A((x\to y)')\vee\beta_A((y\to z)')$ by (IFI2) and (IFI3). Hence (IFI7) holds. Conversely, assume (IFI1) and (IFI7) hold. In order to show that $A\in\mathbf{IFLI}(L)$, it suffices to show that A satisfies (IFI2) according to Definition 3.1. In fact, since $(x\to O)'=x$ for any $x\in L$, by (IFI7) we have that $\alpha_A(x)=\alpha_A((x\to O)')\geqslant\alpha_A((x\to y)')\wedge\alpha_A((y\to O)')=\alpha_A((x\to y)')\wedge\alpha_A(y)$ and $\beta_A(x)=\beta_A((x\to O)')\leqslant\beta_A((x\to y)')\vee\beta_A((y\to O)')=\beta_A((x\to y)')\vee\beta_A(y)$. Hence (IFI2) holds.

Definition 3.4 *Let L be a lattice implication algebra and $A = (\alpha_A, \beta_A)$ an intuitionistic fuzzy set in L. An intuitionistic fuzzy set $A_{st} = (\alpha_A^s, \beta_A^t)$ in L is defined as follows:*

$$\alpha_A^s(x) = \begin{cases} \alpha_A(x), & x \neq O; \\ \alpha_A(O) \vee s, & x = O \end{cases} \quad \text{and} \quad \beta_A^t(x) = \begin{cases} \beta_A(x), & x \neq O; \\ \beta_A(O) \wedge t, & x = O \end{cases} \tag{1}$$

for all $x \in L$, where $s, t \in [0, 1]$ and $0 \leqslant s + t \leqslant 1$.

Remark 3.5 Let $A = (\alpha_A, \beta_A)$ be an intuitionistic fuzzy set in L. Then $0 \leqslant \alpha_A(O)+\beta_A(O) \leqslant 1$. Hence, for all $s, t \in [0, 1]$ with $0 \leqslant s+t \leqslant 1$, if $\alpha_A(O) \leqslant s$, we have that $0 \leqslant \alpha_A(O)\vee s+\beta_A(O)\wedge t \leqslant s + t \leqslant 1$; If $\alpha_A(O) > s$, we have that $0 \leqslant \alpha_A(O) \vee s + \beta_A(O) \wedge t \leqslant \alpha_A(O) + \beta_A(O) \leqslant 1$. This shows that the intuitionistic fuzzy set $A_{st} = (\alpha_A^s, \beta_A^t)$ in Definition 3.4 is well defined.

Theorem 3.6 *Let L be a lattice implication algebra and $A = (\alpha_A, \beta_A) \in$ **IFLI**(L). Then for all $s, t \in [0, 1]$ with $0 \leqslant s + t \leqslant 1$, $A_{st} \in$ **IFLI**(L).*

Proof Firstly, for all $x, y \in L$, let $x \leqslant y$. We consider the following two cases:

(i) Assume that $x = O$. If $y = O$, we have that $\alpha_A^s(x) = \alpha_A(O) \vee s = \alpha_A^s(y)$ and $\beta_A^t(x) = \beta_A(O) \wedge t = \beta_A^t(y)$. If $y \neq O$, we have that $\alpha_A^s(x) = \alpha_A(O) \vee s \geqslant \alpha_A(O) \geqslant \alpha_A(y) = \alpha_A^s(y)$ and $\beta_A^t(x) = \beta_A(O) \wedge t \leqslant \beta_A(O) \leqslant \beta_A(y) = \beta_A^t(y)$.

(ii) Assume that $x \neq O$, then $y \neq O$. It follows that $\alpha_A^s(x) = \alpha_A(x) \geqslant \alpha_A(y) = \alpha_A^s(y)$ and $\beta_A^t(x) = \beta_A(x) \leqslant \beta_A(y) = \beta_A^t(y)$ from $A \in$ **IFLI**(L) and (IFI3).

Summarizing these two cases, we conclude that $x \leqslant y$ implies $\alpha_A^s(x) \geqslant \alpha_A^s(y)$ and $\beta_A^t(x) \leqslant \beta_A^t(y)$, for all $x, y \in L$. i.e., A_{st} satisfies (IFI3).

Secondly, for all $x, y \in L$, we consider the following two cases:

(i) Assume that $x \oplus y = O$. If $x = y = O$, it is obvious that $\alpha_A^s(x \oplus y) = \alpha_A^s(x) \wedge \alpha_A^s(y)$ and $\beta_A^t(x \oplus y) = \beta_A^t(x) \vee \beta_A^t(y)$. If $x = O, y \neq O$ or $x \neq O, y = O$, then $x \oplus y \neq O$, it is a contradiction. If $x \neq O$ and $y \neq O$, it follows that $\alpha_A^s(x)\wedge\alpha_A^s(y) = \alpha_A(x)\wedge\alpha_A(y) \leqslant \alpha_A(x\oplus y) \leqslant \alpha_A(O) \leqslant \alpha_A(O) \vee s = \alpha_A^s(x \oplus y)$ and $\beta_A^t(x) \vee \beta_A^t(y) = \beta_A(x) \vee \beta_A(y) \geqslant \beta_A(x \oplus y) \geqslant \beta_A(O) \geqslant \beta_A(O) \wedge t = \beta_A^t(x \oplus y)$ from $A \in$ **IFLI**(L), (IFI5) and (IFI1).

(ii) Assume that $x \oplus y \neq O$. If $x = y = O$, it is obviously a contradiction. If $x = O, y \neq O$ or $x \neq O, y = O$, we assume $x = O, y \neq O$, then $x \oplus y = O' \to y = y$, and so $\alpha_A^s(x) \wedge \alpha_A^s(y) \leqslant \alpha_A(y) = \alpha_A(x \oplus y) = \alpha_A^s(x \oplus y)$ and $\beta_A^t(x) \vee \beta_A^t(y) \geqslant \beta_A(y) = \beta_A(x \oplus y) = \beta_A^t(x \oplus y)$. If $x \neq O$ and $y \neq O$, it follows that $\alpha_A^s(x \oplus y) = \alpha_A(x \oplus y) \geqslant \alpha_A(x) \wedge \alpha_A(y) = \alpha_A^s(x) \wedge \alpha_A^s(y)$ and $\beta_A^t(x \oplus y) = \beta_A(x \oplus y) \leqslant \beta_A(x) \vee \beta_A(y) = \beta_A^t(x) \vee \beta_A^t(y)$ from $A \in$ **IFLI**(L) and (IFI5).

Summarizing these two cases, we conclude that $\alpha_A^s(x\oplus y) \geqslant \alpha_A^s(x)\wedge\alpha_A^s(y)$ and $\beta_A^t(x\oplus y) \leqslant \beta_A^t(x) \vee \beta_A^t(y)$, for all $x, y \in L$. i.e., A_{st} satisfies (IFI5).

Thus, it follows that $A_{st} \in$ **IFLI**(L) from Theorem 3.3.

Definition 3.7 *Let L be a lattice implication algebra and $A = (\alpha_A, \beta_A), B = (\alpha_B, \beta_B)$ intuitionistic fuzzy sets in L. Intuitionistic fuzzy sets $A^B = (\alpha_{A^B}, \beta_{A^B})$ and $B^A = (\alpha_{B^A}, \beta_{B^A})$*

in L are defined as follows: for all $x \in L$,

$$\alpha_{A^B}(x) = \begin{cases} \alpha_A(x), & x \neq O; \\ \alpha_A(O) \vee \alpha_B(O), & x = O \end{cases} \quad \text{and} \quad \beta_{A^B}(x) = \begin{cases} \beta_A(x), & x \neq O; \\ \beta_A(O) \wedge \beta_B(O), & x = O \end{cases} \tag{2}$$

and

$$\alpha_{B^A}(x) = \begin{cases} \alpha_B(x), & x \neq O; \\ \alpha_B(O) \vee \alpha_A(O), & x = O \end{cases} \quad \text{and} \quad \beta_{B^A}(x) = \begin{cases} \beta_B(x), & x \neq O; \\ \beta_B(O) \wedge \beta_A(O), & x = O. \end{cases} \tag{3}$$

Corollary 3.8 *Let L be a lattice implication algebra and $A = (\alpha_A, \beta_A), B = (\alpha_B, \beta_B) \in$ $\mathbf{IFLI}(L)$. Then $A^B = (\alpha_{A^B}, \beta_{A^B}), B^A = (\alpha_{B^A}, \beta_{B^A}) \in \mathbf{IFLI}(L)$.*

Definition 3.9 *Let L be a lattice implication algebra and $A = (\alpha_A, \beta_A), B = (\alpha_B, \beta_B)$ intuitionistic fuzzy sets in L. An intuitionistic fuzzy set $A \uplus B = (\alpha_{A\uplus B}, \beta_{A\uplus B})$ in L is defined as follows: for all $x, a, b \in L$,*

$$\alpha_{A\uplus B}(x) = \bigvee_{x \leqslant a\oplus b} [\alpha_A(a) \wedge \alpha_B(b)] \quad \text{and} \quad \beta_{A\uplus B}(x) = \bigwedge_{x \leqslant a\oplus b} [\beta_A(a) \vee \beta_B(b)]. \tag{4}$$

Theorem 3.10 *Let L be a lattice implication algebra and $A = (\alpha_A, \beta_A), B = (\alpha_B, \beta_B) \in$ $\mathbf{IFLI}(L)$. Then $A^B \uplus B^A \in \mathbf{IFLI}(L)$.*

Proof Firstly, for all $x, y \in L$, let $x \leqslant y$. Then $\{a \oplus b | x \leqslant a \oplus b\} \supseteq \{a \oplus b | y \leqslant a \oplus b\}$, and so $\alpha_{A^B\uplus B^A}(x) = \bigvee_{x\leqslant a\oplus b} [\alpha_{A^B}(a) \wedge \alpha_{B^A}(b)] \geqslant \bigvee_{y\leqslant a\oplus b} [\alpha_{A^B}(a) \wedge \alpha_{B^A}(b)] = \alpha_{A^B\uplus B^A}(y)$ and $\beta_{A^B\uplus B^A}(x) = \bigwedge_{x\leqslant a\oplus b} [\beta_{A^B}(a) \vee \beta_{B^A}(b)] \leqslant \bigwedge_{y\leqslant a\oplus b} [\beta_{A^B}(a) \vee \beta_{B^A}(b)] = \beta_{A^B\uplus B^A}(y)$. Hence $A^B \uplus B^A$ is intuitionistic non-increasing, i.e., it satisfies (IFI3). Secondly, for all $x, y \in L$, we have that

$$\begin{aligned}
\alpha_{A^B\uplus B^A}(x \oplus y) &= \bigvee_{x\oplus y\leqslant a\oplus b} [\alpha_{A^B}(a) \wedge \alpha_{B^A}(b)] \\
&\geqslant \bigvee_{x\leqslant a_1\oplus a_2 \text{ and } y\leqslant b_1\oplus b_2} [\alpha_{A^B}(a_1 \oplus b_1) \wedge \alpha_{B^A}(a_2 \oplus b_2)] \\
&\geqslant \bigvee_{x\leqslant a_1\oplus a_2 \text{ and } y\leqslant b_1\oplus b_2} [\alpha_{A^B}(a_1) \wedge \alpha_{A^B}(b_1) \wedge \alpha_{B^A}(a_2) \wedge \alpha_{B^A}(b_2)] \\
&= \bigvee_{x\leqslant a_1\oplus a_2} [\alpha_{A^B}(a_1) \wedge \alpha_{B^A}(a_2)] \wedge \bigvee_{y\leqslant b_1\oplus b_2} [\alpha_{A^B}(b_1) \wedge \alpha_{B^A}(b_2)] \\
&= \alpha_{A^B\uplus B^A}(x) \wedge \alpha_{A^B\uplus B^A}(y),
\end{aligned}$$

$$\begin{aligned}
\beta_{A^B\uplus B^A}(x \oplus y) &= \bigwedge_{x\oplus y\leqslant a\oplus b} [\beta_{A^B}(a) \vee \beta_{B^A}(b)] \\
&\leqslant \bigwedge_{x\leqslant a_1\oplus a_2 \text{ and } y\leqslant b_1\oplus b_2} [\beta_{A^B}(a_1 \oplus b_1) \vee \beta_{B^A}(a_2 \oplus b_2)] \\
&\leqslant \bigwedge_{x\leqslant a_1\oplus a_2 \text{ and } y\leqslant b_1\oplus b_2} [\beta_{A^B}(a_1) \vee \beta_{A^B}(b_1) \vee \beta_{B^A}(a_2) \vee \beta_{B^A}(b_2)] \\
&= \bigwedge_{x\leqslant a_1\oplus a_2} [\beta_{A^B}(a_1) \vee \beta_{B^A}(a_2)] \vee \bigwedge_{y\leqslant b_1\oplus b_2} [\beta_{A^B}(b_1) \vee \beta_{B^A}(b_2)]
\end{aligned}$$

$$=\beta_{A^B\uplus B^A}(x)\vee\beta_{A^B\uplus B^A}(y),$$

and so $A^B\uplus B^A$ also satisfies (IFI5). Hence $A^B\uplus B^A\in\mathbf{IFLI}(L)$ by Theorem 3.3.

4. Generated intuitionistic fuzzy *LI*-ideals

Definition 4.1 *Let L be a lattice implication algebra, $A=(\alpha_A,\beta_A)$ an intuitionistic fuzzy set in L. An intuitionistic fuzzy LI-ideal $B=(\alpha_B,\beta_B)$ of L is called the generated intuitionistic fuzzy LI-ideal by A, denoted $\langle A\rangle$, if $A\Subset B$ and for any $C\in\mathbf{IFLI}(L)$, $A\Subset C$ implies $B\Subset C$.*

Theorem 4.2 *Let L be a lattice implication algebra and $A=(\alpha_A,\beta_A)$ an intuitionistic fuzzy set in L. An intuitionistic fuzzy set $B=(\alpha_B,\beta_B)$ in L is defined as follows:*

$$\alpha_B(x)=\bigvee\{\alpha_A(a_1)\wedge\cdots\wedge\alpha_A(a_n)|a_1,a_2,\ldots,a_n\in L\text{ and }x\leqslant a_1\oplus a_2\oplus\cdots\oplus a_n\},\tag{5}$$

$$\beta_B(x)=\bigwedge\{\beta_A(a_1)\vee\cdots\vee\beta_A(a_n)|a_1,a_2,\ldots,a_n\in L\text{ and }x\leqslant a_1\oplus a_2\oplus\cdots\oplus a_n\}\tag{6}$$

for all $x\in L$. Then $B=\langle A\rangle$.

Proof Firstly, we prove that $B\in\mathbf{IFLI}(L)$. For all $x,y,z\in L$, let $x\leqslant y\oplus z$. Given an arbitrarily small $\varepsilon>0$, there are $a_1,a_2,\ldots,a_n\in L$ and $b_1,b_2,\ldots,b_m\in L$ such that

$$y\leqslant a_1\oplus a_2\oplus\cdots\oplus a_n\text{ and }z\leqslant b_1\oplus b_2\oplus\cdots\oplus b_m,\tag{7}$$

$$\alpha_B(y)-\varepsilon<\alpha_A(a_1)\wedge\cdots\wedge\alpha_A(a_n)\text{ and }\alpha_B(z)-\varepsilon<\alpha_A(b_1)\wedge\cdots\wedge\alpha_A(b_m),\tag{8}$$

$$\beta_A(a_1)\vee\cdots\vee\beta_A(a_n)<\beta_B(y)+\varepsilon\text{ and }\beta_A(b_1)\vee\cdots\vee\beta_A(b_m)<\beta_B(z)+\varepsilon.\tag{9}$$

By (7) and $x\leqslant y\oplus z$ we have that $x\leqslant y\oplus z\leqslant(a_1\oplus a_2\oplus\cdots\oplus a_n)\oplus(b_1\oplus b_2\oplus\cdots\oplus b_m)=a_1\oplus a_2\oplus\cdots\oplus a_n\oplus b_1\oplus b_2\oplus\cdots\oplus b_m$. So, on the one hand, by (5) and (8) we have that $\alpha_B(x)\geqslant\alpha_A(a_1)\wedge\cdots\wedge\alpha_A(a_n)\wedge\alpha_A(b_1)\wedge\cdots\wedge\alpha_A(b_m)=[\alpha_A(a_1)\wedge\cdots\wedge\alpha_A(a_n)]\wedge[\alpha_A(b_1)\wedge\cdots\wedge\alpha_A(b_m)]>[\alpha_B(y)-\varepsilon]\wedge[\alpha_B(z)-\varepsilon]=[\alpha_B(y)\wedge\alpha_B(z)]-\varepsilon$. And on the other hand, by (6) and (9) we have that $\beta_B(x)\leqslant\beta_A(a_1)\vee\cdots\vee\beta_A(a_n)\vee\beta_A(b_1)\vee\cdots\vee\beta_A(b_m)=[\beta_A(a_1)\vee\cdots\vee\beta_A(a_n)]\vee[\beta_A(b_1)\vee\cdots\vee\beta_A(b_m)]<[\beta_B(y)+\varepsilon]\vee[\beta_B(z)+\varepsilon]=[\beta_B(y)\vee\beta_B(z)]+\varepsilon$. By the arbitrary smallness of ε, we have that $\alpha_B(x)\geqslant\alpha_B(y)\wedge\alpha_B(z)$ and $\beta_B(x)\leqslant\beta_B(y)\vee\beta_B(z)$. It follows from Theorem 3.3 that $B\in\mathbf{IFLI}(L)$.

Secondly, for any $x\in L$, it follows from $x\leqslant x$ and the definition of B that $\alpha_A(x)\leqslant\alpha_B(x)$ and $\beta_A(x)\geqslant\beta_B(x)$. This means that $A\Subset B$.

Finally, assume that $C\in\mathbf{IFLI}(L)$ with $A\Subset C$. Then for any $x\in L$, we have

$$\begin{aligned}\alpha_B(x)&=\bigvee\{\alpha_A(a_1)\wedge\cdots\wedge\alpha_A(a_n)|a_1,a_2,\ldots,a_n\in L\text{ and }x\leqslant a_1\oplus a_2\oplus\cdots\oplus a_n\}\\&\leqslant\bigvee\{\alpha_C(a_1)\wedge\cdots\wedge\alpha_C(a_n)|a_1,a_2,\ldots,a_n\in L\text{ and }x\leqslant a_1\oplus a_2\oplus\cdots\oplus a_n\}\\&\leqslant\bigvee\{\alpha_C(a_1\oplus\cdots\oplus a_n)|a_1,a_2,\ldots,a_n\in L\text{ and }x\leqslant a_1\oplus a_2\oplus\cdots\oplus a_n\}\\&\leqslant\bigvee\{\alpha_C(x)\}\\&=\alpha_C(x).\end{aligned}$$

$$\begin{aligned}\beta_B(x) &= \bigwedge\{\beta_A(a_1)\vee\cdots\vee\beta_A(a_n)|a_1,a_2,\ldots,a_n\in L \text{ and } x\leqslant a_1\oplus a_2\oplus\cdots\oplus a_n\}\\ &\geqslant \bigwedge\{\beta_C(a_1)\vee\cdots\vee\beta_C(a_n)|a_1,a_2,\ldots,a_n\in L \text{ and } x\leqslant a_1\oplus a_2\oplus\cdots\oplus a_n\}\\ &\geqslant \bigwedge\{\beta_C(a_1\oplus\cdots\oplus a_n)|a_1,a_2,\ldots,a_n\in L \text{ and } x\leqslant a_1\oplus a_2\oplus\cdots\oplus a_n\}\\ &\geqslant \bigwedge\{\beta_C(x)\}=\beta_C(x).\end{aligned}$$

Hence $B\Subset C$ holds. To sum up, we have that $B=\langle A\rangle$.

Example 4.3 Let $L=\{O,a,b,c,d,I\}$, $O'=I,a'=c,b'=d,c'=a,d'=b,I'=O$, the Hasse diagram of L be defined as Fig. 1, and the operator $\to$ of L be defined as Table 1.

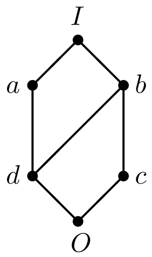

Figure 1 Hasse diagram of L

$\to$	O	a	b	c	d	I
O	I	I	I	I	I	I
a	c	I	b	c	b	I
b	d	a	I	b	a	I
c	a	a	I	I	a	I
d	b	I	I	b	I	I
I	O	a	b	c	d	I

Table 1 Operator $\to$ of L

Then $(L,\vee,\wedge,',\to,O,I)$ is a lattice implication algebra. Define an intuitionistic fuzzy set $A=(\alpha_A,\beta_A)$ in L by $\alpha_A(O)=\alpha_A(a)=1,\alpha_A(b)=\alpha_A(c)=\alpha_A(d)=\alpha_A(I)=s$ and $\beta_A(O)=\beta_A(a)=0,\beta_A(b)=\beta_A(c)=\beta_A(d)=\beta_A(I)=t$, where $s,t\in[0,1]$ and $0\leqslant s+t\leqslant 1$. Since $d\leqslant a$ but $\alpha_A(d)=s\ngeqslant 1=\alpha_A(a)$, we know that $A\notin \mathbf{IFLI}(L)$. It is easy to verify that $\langle A\rangle\in\mathbf{IFLI}(L)$ from Theorem 4.2, where $\alpha_{\langle A\rangle}(O)=\alpha_{\langle A\rangle}(a)=\alpha_{\langle A\rangle}(d)=1,\alpha_{\langle A\rangle}(b)=\alpha_{\langle A\rangle}(c)=\alpha_{\langle A\rangle}(I)=s$ and $\beta_{\langle A\rangle}(O)=\beta_{\langle A\rangle}(a)=\beta_{\langle A\rangle}(d)=0,\beta_{\langle A\rangle}(b)=\beta_{\langle A\rangle}(c)=\beta_{\langle A\rangle}(I)=t$.

5. The lattice of intuitionistic fuzzy *LI*-ideals

In this section, we investigate the lattice structural feature of the set $\mathbf{IFLI}(L)$.

Theorem 5.1 *Let L be a lattice implication algebra. Then $(\mathbf{IFLI}(L),\Subset)$ is a complete lattice.*

Proof For any $\{A_\lambda\}_{\lambda\in\Lambda}\subseteq\mathbf{IFLI}(L)$, where Λ is indexed set. It is easy to verify that $\Cap_{\lambda\in\Lambda}A_\lambda$ is infimum of $\{A_\lambda\}_{\lambda\in\Lambda}$, where $(\Cap_{\lambda\in\Lambda}A_\lambda)(x)=\bigwedge\limits_{\lambda\in\Lambda}A_\lambda(x)$ for all $x\in L$. i.e., $\bigwedge\limits_{\lambda\in\Lambda}A_\lambda=\Cap_{\lambda\in\Lambda}A_\lambda$. Define $\Cup_{\lambda\in\Lambda}A_\lambda$ as follows: $(\Cup_{\lambda\in\Lambda}A_\lambda)(x)=\bigvee\limits_{\lambda\in\Lambda}A_\lambda(x)$ for all $x\in L$. Then $\langle\Cup_{\lambda\in\Lambda}A_\lambda\rangle$ is supermun of $\{A_\lambda\}_{\lambda\in\Lambda}$, where $\langle\Cup_{\lambda\in\Lambda}A_\lambda\rangle$ is the the generated intuitionistic fuzzy LI-ideal by $\Cup_{\lambda\in\Lambda}A_\lambda$ of L. i.e., $\bigvee\limits_{\lambda\in\Lambda}A_\lambda=\langle\Cup_{\lambda\in\Lambda}A_\lambda\rangle$. Therefore $(\mathbf{IFLI}(L),\Subset)$ is a complete lattice. The proof is completed.

Remark 5.2 Let L be a lattice implication algebra. For all $A,B\in\mathbf{IFLI}(L)$, by Theorem 5.1 we know that $A\wedge B=A\Cap B$ and $A\vee B=\langle A\Cup B\rangle$. The following example shows that $A\vee B\neq A\Cup B$ in general.

Example 5.3 Let $L=\{O,a,b,I\}$, $O'=I,a'=b,b'=a,I'=O$, the Hasse diagram of L be defined as Figure 2, and the operator $\to$ of L be defined as Table 2.

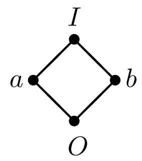

Figure 2 Hasse diagram of L

$\to$	O	a	b	I
O	I	I	I	I
a	b	I	b	I
b	a	a	I	I
I	O	a	b	I

Table 2 Operator $\to$ of L

Then $(L,\vee,\wedge,',\to,O,I)$ is a lattice implication algebra. Define intuitionistic fuzzy sets $A = (\alpha_A,\beta_A)$ and $B=(\alpha_B,\beta_B)$ in L by $\alpha_A(O)=\alpha_A(b)=1, \alpha_A(a)=\alpha_A(I)=0$, $\beta_A(O)=\beta_A(b)=0, \beta_A(a)=\beta_A(I)=1$ and $\alpha_B(O)=\alpha_B(a)=1, \alpha_B(b)=\alpha_B(I)=0$, $\beta_B(O)=\beta_B(a)=0, \beta_B(b)=\beta_B(I)=1$, then $A,B\in \mathbf{IFLI}(L)$. It is easily to verify that $C=A\Cup B\notin \mathbf{IFLI}(L)$, where $\alpha_C(O)=\alpha_C(a)=\alpha_C(b)=1, \alpha_C(I)=0$ and $\beta_C(O)=\beta_C(a)=\beta_C(b)=0, \beta_C(I)=1$. In fact, $\alpha_C(I)=0\not\geqslant 1=\alpha_C((I\to a)')\wedge\alpha_C(a)$.

Theorem 5.4 *Let L be a lattice implication algebra. Then for all $A,B\in\mathbf{IFLI}(L)$, $A\vee B=\langle A\Cup B\rangle = A^B\uplus B^A$ in the complete lattice $(\mathbf{IFLI}(L),\Subset)$.*

Proof For all $A,B\in\mathbf{IFLI}(L)$, it is obvious that $A\Subset A^B\uplus B^A$ and $B\Subset A^B\uplus B^A$, thus $A\Cup B\Subset A^B\uplus B^A$, and thus $\langle A\Cup B\rangle\Subset A^B\uplus B^A$. Let $C\in\mathbf{IFLI}(L)$ such that $A\Cup B\Subset C$, for all $x\in L$, we consider the following two cases:

(i) If $x=O$, then $\alpha_{A^B\uplus B^A}(x)=\bigvee\limits_{O\leqslant a\oplus b}[\alpha_{A^B}(a)\wedge\alpha_{B^A}(b)]=\alpha_{A^B}(O)\wedge\alpha_{B^A}(O)=\alpha_A(O)\vee\alpha_B(O)=\alpha_{A\Cup B}(O)\leqslant\alpha_C(O)=\alpha_C(x)$ and $\beta_{A^B\uplus B^A}(x)=\bigwedge\limits_{O\leqslant a\oplus b}[\beta_{A^B}(a)\vee\beta_{B^A}(b)]=\beta_{A^B}(O)\vee\beta_{B^A}(O)=\beta_A(O)\wedge\beta_B(O)=\beta_{A\Cup B}(O)\geqslant\beta_C(O)=\beta_C(x)$, thus $A^B\uplus B^A\Subset C$ for this case.

(ii) If $x>O$, then we have

$$\begin{aligned}
\alpha_{A^B\uplus B^A}(x) &= \bigvee_{x\leqslant a\oplus b}[\alpha_{A^B}(a)\wedge\alpha_{B^A}(b)]\\
&= \bigvee_{x\leqslant a\oplus b, a\neq O, b\neq O}[\alpha_{A^B}(a)\wedge\alpha_{B^A}(b)]\vee\bigvee_{x\leqslant a}[\alpha_A(a)\wedge(\alpha_A(O)\vee\alpha_B(O))]\\
&\quad\vee\bigvee_{x\leqslant b}[(\alpha_A(O)\vee\alpha_B(O))\wedge\alpha_B(b)]\\
&= \bigvee_{x\leqslant a\oplus b, a\neq O, b\neq O}[\alpha_A(a)\wedge\alpha_B(b)]\vee\bigvee_{x\leqslant a}\alpha_A(a)\vee\bigvee_{x\leqslant b}\alpha_B(b)\\
&\leqslant \bigvee_{x\leqslant a\oplus b, a\neq O, b\neq O}[\alpha_C(a)\wedge\alpha_C(b)]\vee\bigvee_{x\leqslant a}\alpha_C(a)\vee\bigvee_{x\leqslant b}\alpha_C(b)\\
&= \bigvee_{x\leqslant a\oplus b}[\alpha_C(a)\wedge\alpha_C(b)]\leqslant\alpha_C(x),
\end{aligned}$$

$$\begin{aligned}
\beta_{A^B\uplus B^A}(x) &= \bigwedge_{x\leqslant a\oplus b}[\beta_{A^B}(a)\vee\beta_{B^A}(b)]\\
&= \bigwedge_{x\leqslant a\oplus b, a\neq O, b\neq O}[\beta_{A^B}(a)\vee\beta_{B^A}(b)]\wedge\bigwedge_{x\leqslant a}[\beta_A(a)\vee(\beta_B(O)\wedge\beta_A(O))]
\end{aligned}$$

$$\wedge \bigwedge_{x\leqslant b}[(\beta_A(O)\wedge\beta_B(O))\vee\beta_B(b)]$$
$$= \bigwedge_{x\leqslant a\oplus b, a\neq O, b\neq O}[\beta_A(a)\vee\beta_B(b)]\wedge\bigwedge_{x\leqslant a}\beta_A(a)\wedge\bigwedge_{x\leqslant b}\beta_B(b)$$
$$\geqslant \bigwedge_{x\leqslant a\oplus b, a\neq O, b\neq O}[\beta_C(a)\vee\beta_C(b)]\wedge\bigwedge_{x\leqslant a}\beta_C(a)\wedge\bigwedge_{x\leqslant b}\beta_C(b)$$
$$= \bigwedge_{x\leqslant a\oplus b}[\beta_C(a)\vee\beta_C(b)]\geqslant\beta_C(x).$$

Thus $A^B \uplus B^A \Subset C$ for this case too.

By Theorem 3.2, Definition 4.1 and Theorem 5.1 we have that $A\vee B=\langle A\Cup B\rangle = A^B\uplus B^A$.

Finally, we investigate the distributivity of lattice $(\mathbf{IFLI}(L),\Subset)$.

Theorem 5.5 *Let L be a lattice implication algebra. Then $(\mathbf{IFLI}(L),\Subset)$ is a distributive lattice, where, $A\wedge B = A\Cap B$ and $A\vee B=\langle A\Cup B\rangle$, for all $A,B\in\mathbf{IFLI}(L)$.*

Proof To finish the proof, it suffices to show that $C\wedge(A\vee B)=(C\wedge A)\vee(C\wedge B)$, for all $A,B,C\in\mathbf{IFLI}(L)$. Since the inequality $(C\wedge A)\vee(C\wedge B)\Subset C\wedge(A\vee B)$ holds automatically in a lattice, we need only to show the inequality $C\wedge(A\vee B)\Subset(C\wedge A)\vee(C\wedge B)$. i.e., we need only to show that $(\alpha_C\cap\alpha_{A^B\uplus B^A})(x)\leqslant\alpha_{C\Cap A^{C\Cap B}\uplus C\Cap B^{C\Cap A}}(x)$ and $(\beta_C\cup\beta_{A^B\uplus B^A})(x)\geqslant\beta_{C\Cap A^{C\Cap B}\uplus C\Cap B^{C\Cap A}}(x)$, for all $x\in L$. For these, we consider the following two cases:

(i) If $x=O$, we have

$$\begin{aligned}(\alpha_C\cap\alpha_{A^B\uplus B^A})(O) =&\alpha_C(O)\wedge\alpha_{A^B\uplus B^A}(O)=\alpha_C(O)\wedge\bigvee_{O\leqslant a\oplus b}[\alpha_{A^B}(a)\wedge\alpha_{B^A}(b)]\\ =&\alpha_C(O)\wedge[\alpha_{A^B}(O)\wedge\alpha_{B^A}(O)]=\alpha_C(O)\wedge[\alpha_A(O)\vee\alpha_B(O)]\\ =&[\alpha_C(O)\wedge\alpha_A(O)]\vee[\alpha_C(O)\wedge\alpha_B(O)]=\alpha_{C\Cap A}(O)\vee\alpha_{C\Cap B}(O)\\ =&\alpha_{C\Cap A^{C\Cap B}}(O)\wedge\alpha_{C\Cap B^{C\Cap A}}(O)=\bigvee_{O\leqslant a\oplus b}[\alpha_{C\Cap A^{C\Cap B}}(a)\wedge\alpha_{C\Cap B^{C\Cap A}}(b)]\\ =&\alpha_{C\Cap A^{C\Cap B}\uplus C\Cap B^{C\Cap A}}(O),\\ (\beta_C\cup\beta_{A^B\uplus B^A})(O) =&\beta_C(O)\vee\beta_{A^B\uplus B^A}(O)=\beta_C(O)\vee\bigwedge_{O\leqslant a\oplus b}[\beta_{A^B}(a)\vee\beta_{B^A}(b)]\\ =&\beta_C(O)\vee[\beta_{A^B}(O)\vee\beta_{B^A}(O)]=\beta_C(O)\vee[\beta_A(O)\wedge\beta_B(O)]\\ =&[\beta_C(O)\vee\beta_A(O)]\wedge[\beta_C(O)\vee\beta_B(O)]=\beta_{C\Cap A}(O)\wedge\beta_{C\Cap B}(O)\\ =&\beta_{C\Cap A^{C\Cap B}}(O)\vee\beta_{C\Cap B^{C\Cap A}}(O)=\bigwedge_{O\leqslant a\oplus b}[\beta_{C\Cap A^{C\Cap B}}(a)\vee\beta_{C\Cap B^{C\Cap A}}(b)]\\ =&\beta_{C\Cap A^{C\Cap B}\uplus C\Cap B^{C\Cap A}}(O).\end{aligned}$$

(ii) If $x>O$, we have

$$\begin{aligned}(\alpha_C\cap\alpha_{A^B\uplus B^A})(x) =&\alpha_C(x)\wedge\alpha_{A^B\uplus B^A}(x)=\alpha_C(x)\wedge\bigvee_{x\leqslant a\oplus b}[\alpha_{A^B}(a)\wedge\alpha_{B^A}(b)]\\ =&\bigvee_{x\leqslant a\oplus b}[\alpha_C(x)\wedge\alpha_{A^B}(a)\wedge\alpha_{B^A}(b)]\end{aligned}$$

$$
\begin{aligned}
&= \bigvee_{x\leqslant a\oplus b, a\neq O, b\neq O} [\alpha_C(x)\wedge \alpha_{A^B}(a)\wedge \alpha_{B^A}(b)] \vee \\
&\quad \bigvee_{x\leqslant b} [\alpha_C(x)\wedge \alpha_{A^B}(O)\wedge \alpha_B(b)] \vee \bigvee_{x\leqslant a} [\alpha_C(x)\wedge \alpha_A(a)\wedge \alpha_{B^A}(O)] \\
&= \bigvee_{x\leqslant a\oplus b, a\neq O, b\neq O} [\alpha_C(x)\wedge \alpha_A(a)\wedge \alpha_B(b)] \vee \\
&\quad [\alpha_C(x)\wedge \alpha_{A^B}(O)\wedge \alpha_B(x)] \vee [\alpha_C(x)\wedge \alpha_A(x)\wedge \alpha_{B^A}(O)] \\
&= \bigvee_{x\leqslant a\oplus b, a\neq O, b\neq O} [(\alpha_C(x)\wedge \alpha_A(a))\wedge (\alpha_C(x)\wedge \alpha_B(b))] \vee \\
&\quad [\alpha_C(x)\wedge \alpha_C(O)\wedge \alpha_{A^B}(O)\wedge \alpha_B(x)] \vee [\alpha_C(x)\wedge \alpha_C(O)\wedge \alpha_A(x)\wedge \alpha_{B^A}(O)] \\
&= \bigvee_{x\leqslant a\oplus b, a\neq O, b\neq O} [(\alpha_C(x)\wedge \alpha_A(a))\wedge (\alpha_C(x)\wedge \alpha_B(b))] \\
&\quad \vee \{[\alpha_C(O)\wedge (\alpha_A(O)\vee \alpha_B(O)]\wedge [(\alpha_C(x)\wedge \alpha_B(x))\vee (\alpha_C(x)\wedge \alpha_A(x))]\} \\
&= \bigvee_{x\leqslant a\oplus b, a\neq O, b\neq O} [(\alpha_C(x)\wedge \alpha_A(a))\wedge (\alpha_C(x)\wedge \alpha_B(b))] \\
&\quad \vee \{(\alpha_{C\Cap A}(O)\vee \alpha_{C\Cap B}(O))\wedge [(\alpha_C(x)\wedge \alpha_B(x))\vee (\alpha_C(x)\wedge \alpha_A(x))]\} \\
&\leqslant \bigvee_{x\leqslant a\oplus b, a\neq O, b\neq O} [(\alpha_C(a\wedge x)\wedge \alpha_A(a\wedge x))\wedge (\alpha_C(b\wedge x)\wedge \alpha_B(b\wedge x))] \\
&\quad \vee \{\alpha_{C\Cap A^{C\Cap B}}(O)\wedge [(\alpha_C(a\wedge x)\wedge \alpha_A(a\wedge x))\vee (\alpha_C(b\wedge x)\wedge \alpha_B(b\wedge x))]\} \\
&= \bigvee_{x\leqslant a\oplus b, a\neq O, b\neq O} [\alpha_{C\Cap A^{C\Cap B}}(a\wedge x)\wedge \alpha_{C\Cap B^{C\Cap A}}(b\wedge x)] \\
&\quad \vee [\alpha_{C\Cap A^{C\Cap B}}(O)\wedge (\alpha_C\cap \alpha_A)(a\wedge x)] \vee [\alpha_{C\Cap A^{C\Cap B}}(O)\wedge (\alpha_C\cap \alpha_B)(b\wedge x)] \\
&= \bigvee_{x\leqslant a\oplus b, a\neq O, b\neq O} [\alpha_{C\Cap A^{C\Cap B}}(a\wedge x)\wedge \alpha_{C\Cap B^{C\Cap A}}(b\wedge x)] \\
&\quad \vee [\alpha_{C\Cap A^{C\Cap B}}(O)\wedge \alpha_{C\Cap A}(a\wedge x)] \vee [\alpha_{C\Cap A^{C\Cap B}}(O)\wedge \alpha_{C\Cap B}(b\wedge x)] \\
&= \bigvee_{x\leqslant a\oplus b} [\alpha_{C\Cap A^{C\Cap B}}(a\wedge x)\wedge \alpha_{C\Cap B^{C\Cap A}}(b\wedge x)],
\end{aligned}
$$

and

$$
\begin{aligned}
(\beta_C\cup \beta_{A^B\uplus B^A})(x) &= \beta_C(x)\vee \beta_{A^B\uplus B^A}(x) = \beta_C(x)\vee \bigwedge_{x\leqslant a\oplus b} [\beta_{A^B}(a)\vee \beta_{B^A}(b)] \\
&= \bigwedge_{x\leqslant a\oplus b} [\beta_C(x)\vee \beta_{A^B}(a)\vee \beta_{B^A}(b)] \\
&= \bigwedge_{x\leqslant a\oplus b, a\neq O, b\neq O} [\beta_C(x)\vee \beta_{A^B}(a)\vee \beta_{B^A}(b)] \wedge \\
&\quad \bigwedge_{x\leqslant b} [\beta_C(x)\vee \beta_{A^B}(O)\vee \beta_B(b)] \wedge \bigwedge_{x\leqslant a} [\beta_C(x)\vee \beta_A(a)\vee \beta_{B^A}(O)] \\
&= \bigwedge_{x\leqslant a\oplus b, a\neq O, b\neq O} [\beta_C(x)\vee \beta_A(a)\vee \beta_B(b)] \wedge \\
&\quad [\beta_C(x)\vee \beta_{A^B}(O)\vee \beta_B(x)] \wedge [\beta_C(x)\vee \beta_A(x)\vee \beta_{B^A}(O)] \\
&= \bigwedge_{x\leqslant a\oplus b, a\neq O, b\neq O} [(\beta_C(x)\vee \beta_A(a))\vee (\beta_C(x)\vee \beta_B(b))] \wedge
\end{aligned}
$$

$$
\begin{aligned}
&[\beta_C(x)\vee\beta_C(O)\vee\beta_{A^B}(O)\vee\beta_B(x)]\wedge[\beta_C(x)\vee\beta_C(O)\vee\beta_A(x)\vee\beta_{B^A}(O)]\\
=&\bigwedge_{x\leqslant a\oplus b,a\neq O,b\neq O}[(\beta_C(x)\vee\beta_A(a))\vee(\beta_C(x)\vee\beta_B(b))]\\
&\wedge\{[\beta_C(O)\vee(\beta_A(O)\wedge\beta_B(O)]\vee[(\beta_C(x)\vee\beta_B(x))\wedge(\beta_C(x)\vee\beta_A(x))]\}\\
=&\bigwedge_{x\leqslant a\oplus b,a\neq O,b\neq O}[(\beta_C(x)\vee\beta_A(a))\vee(\beta_C(x)\vee\beta_B(b))]\\
&\wedge\{(\beta_{C\Cap A}(O)\vee\beta_{C\Cap B}(O))\vee[(\beta_C(x)\vee\beta_B(x))\wedge(\beta_C(x)\vee\beta_A(x))]\\
\geqslant&\bigwedge_{x\leqslant a\oplus b,a\neq O,b\neq O}[(\beta_C(a\wedge x)\vee\beta_A(a\wedge x))\vee(\beta_C(b\wedge x)\vee\beta_B(b\wedge x))]\\
&\wedge\{\beta_{C\Cap A^{C\Cap B}}(O)\vee[(\beta_C(a\wedge x)\vee\beta_A(a\wedge x))\wedge(\beta_C(b\wedge x)\vee\beta_B(b\wedge x))]\}\\
=&\bigwedge_{x\leqslant a\oplus b,a\neq O,b\neq O}[\beta_{C\Cap A^{C\Cap B}}(a\wedge x)\vee\beta_{C\Cap B^{C\Cap A}}(b\wedge x)]\\
&\wedge[\beta_{C\Cap A^{C\Cap B}}(O)\vee(\beta_C\cup\beta_A)(a\wedge x)]\wedge[\beta_{C\Cap A^{C\Cap B}}(O)\vee(\beta_C\cup\beta_B)(b\wedge x)]\\
=&\bigwedge_{x\leqslant a\oplus b,a\neq O,b\neq O}[\beta_{C\Cap A^{C\Cap B}}(a\wedge x)\vee\beta_{C\Cap B^{C\Cap A}}(b\wedge x)]\\
&\wedge[\beta_{C\Cap A^{C\Cap B}}(O)\vee\beta_{C\Cap A}(a\wedge x)]\wedge[\beta_{C\Cap A^{C\Cap B}}(O)\vee\beta_{C\Cap B}(b\wedge x)]\\
=&\bigwedge_{x\leqslant a\oplus b}[\beta_{C\Cap A^{C\Cap B}}(a\wedge x)\vee\beta_{C\Cap B^{C\Cap A}}(b\wedge x)].
\end{aligned}
$$

Let $a\wedge x=c$ and $b\wedge x=d$. Since $x\leqslant a\oplus b$, using Lemma 2.2, we get that $c\oplus d=(a\wedge x)\oplus(b\wedge x)=((a\wedge x)\oplus b)\wedge((a\wedge x)\oplus x)=(a\oplus b)\wedge(x\oplus b)\wedge(a\oplus x)\oplus(x\oplus x)\geqslant(a\oplus b)\wedge x\wedge x\wedge x=(a\oplus b)\wedge x\geqslant x\wedge x=x$. Hence we can conclude that

$$
\begin{aligned}
(\alpha_C\cap\alpha_{A^B\uplus B^A})(x)&\leqslant\bigvee_{x\leqslant a\oplus b}[\alpha_{C\Cap A^{C\Cap B}}(a\wedge x)\wedge\alpha_{C\Cap B^{C\Cap A}}(b\wedge x)]\\
&\leqslant\bigvee_{x\leqslant c\oplus d}[\alpha_{C\Cap A^{C\Cap B}}(c)\wedge\alpha_{C\Cap B^{C\Cap A}}(d)]\\
&=\alpha_{C\Cap A^{C\Cap B}\uplus C\Cap B^{C\Cap A}}(x),
\end{aligned}
$$

$$
\begin{aligned}
(\beta_C\cup\beta_{A^B\uplus B^A})(x)&\geqslant\bigwedge_{x\leqslant a\oplus b}[\beta_{C\Cap A^{C\Cap B}}(a\wedge x)\vee\beta_{C\Cap B^{C\Cap A}}(b\wedge x)]\\
&\geqslant\bigwedge_{x\leqslant c\oplus d}[\beta_{C\Cap A^{C\Cap B}}(c)\vee\beta_{C\Cap B^{C\Cap A}}(d)]\\
&=\beta_{C\Cap A^{C\Cap B}\uplus C\Cap B^{C\Cap A}}(x).
\end{aligned}
$$

To sum up, we have that $(\alpha_C\cap\alpha_{A^B\uplus B^A})(x)\leqslant\alpha_{C\Cap A^{C\Cap B}\uplus C\Cap B^{C\Cap A}}(x)$ and $(\beta_C\cup\beta_{A^B\uplus B^A})(x)\geqslant\beta_{C\Cap A^{C\Cap B}\uplus C\Cap B^{C\Cap A}}(x)$, for all $x\in L$. The proof is completed.

6. Concluding remarks

As well known, LI-ideals is an important concept for studying the structural features of lattice implication algebras. In this paper, the intuitionistic fuzzy LI-ideal theory in lattice implication algebras is further studied. Some new properties and equivalent characterizations of intuitionistic fuzzy LI-ideals are given. Representation theorem of intuitionistic fuzzy LI-ideal which is generated by an intuitionistic fuzzy set is established. It is proved that the set consisting of all intuitionistic fuzzy LI-ideals in a lattice implication algebra, under the inclusion order, forms a complete distributive lattice. Results obtained in this paper not only enrich the content of intuitionistic fuzzy LI-ideal theory in lattice implication algebras, but also show interactions of algebraic technique and intuitionistic fuzzifying method in the studying logic problems. We hope that more links of intuitionistic fuzzy sets and logics emerge by the stipulating of this work.

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