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G-Cleft Extension of Semilattice Graded Weak Hopf Algebra

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Abstract We give the decomposition of a Clifford monoid algebra into the semilattice direct sum of crossed products, and generalize the crossed-product result to a G-crossed product with an H-G-cleft extension.

Keywords semilattice graded weak Hopf algebra; G-crossed product; H-G-cleft extension

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1. Introduction

Because of the important role of Hopf algebras in the theory of quantum groups and related notions in mathematical physics, along with the deepening of the research, the meanings of some weaker concepts within Hopf-algebra theory have come under closer attention and are becoming better understood. A well-known example is the weak Hopf algebra, which was introduced in [4] in studying the non-invertible solution of the Yang-Baxter Equation based on this class of bialgebras [5]. Because of the importance of the Yang-Baxter Equation in theoretical physics, its solution is a keystone in research. The theory of singular solutions extends largely the scope of research fields. In another aspect, there is a tight relationship between weak Hopf algebras and regular monoids; for example, a semigroup algebra is a weak Hopf algebra if and only if the semigroup is a regular monoid. Obviously, it is necessary to find more non-trivial weak Hopf algebras to study these two aspects deeply.

Since the concept of weak Hopf algebras was introduced in [4], one has only two ways to produce examples, through semigroup algebras of regular monoids (in particular, Clifford monoids) and the weak quantum algebras $\mathrm{wsl}_q(2)$ and $\mathrm{vsl}_q(2)$ (see [7]). The term "weak Hopf algebra" was also used as another generalization of Hopf algebras in [1], [2] and [8] where comultiplication is no longer required to preserve the unit (equivalently, the counit is not required to be an algebra homomorphism). We must point out that these two generalizations are completely distinct as the only common subclass just consists of Hopf algebras [4]. The initial motivation of the latter was its connection with the theory of algebra extensions.

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Semilattice graded weak Hopf algebras were introduced in [6] and a singular solution of the quantum Yang-Baxter equation has been obtained by the quantum G-double. It is well known that an H-extension $A \subset B$ is an H-cleft [3] if and only if $B \cong A\#_\sigma H$. Our focus here is to give a decomposition of a Clifford monoid algebra into the semilattice direct sum of crossed products, and generalize the crossed-product result to the G-crossed product with the H-G-cleft extension.

2. Preliminary

We wish to characterize the G-crossed product $B = \bigoplus_{\alpha\in Y}(A_\alpha\#_\sigma H_\alpha)$ as a special type of extension $A \subset B$ with $A = \bigoplus_{\alpha\in Y} A_\alpha$, termed the semilattice graded algebra. To do this, we first need some definitions.

Definition 2.1 ([6]) *A weak Hopf algebra H with weak antipode T is called a semilattice graded weak Hopf algebra if $H = \bigoplus_{\alpha\in Y} H_\alpha$ is a semilattice grading sum where H_α are Hopf subalgebras of H with antipodes $T|_{H_\alpha}$ for all $\alpha \in Y$ and there are Hopf-algebra homomorphisms $\varphi_{\alpha,\beta}$ from H_α to H_β if $\alpha\beta = \beta$, such that for any $a \in H_\alpha$ and $b \in H_\beta$, the multiplication $a * b$ in H can be given by $a * b = \varphi_{\alpha,\alpha\beta}(a)\varphi_{\beta,\alpha\beta}(b)$.*

Similarly to the discussion on semilattice graded weak Hopf algebras, we obtain the following results for semilattice graded algebras A:

(1) $\{1_{A_\alpha}\}_{\alpha\in Y} \subset C(A)$, the center of A;

(2) A is a semilattice graded algebra if and only if $A = \bigoplus_{\alpha\in Y} A_\alpha$ and $A_\alpha A_\beta \subseteq A_{\alpha\beta}$ for any $\alpha, \beta \in Y$.

Definition 2.2 *Let $H = \bigoplus_{\alpha\in Y} H_\alpha$ be a semilattice graded weak Hopf algebra and $A = \bigoplus_{\alpha\in Y} A_\alpha$ be a semilattice graded algebra with the same semilattice Y. Then $B = \bigoplus_{\alpha\in Y}(A_\alpha\#_\sigma H_\alpha)$ is called a G-crossed product if it satisfies:*

(1) There is a k-linear map $H \otimes A \to A$, given by $h \otimes a \mapsto h \cdot a$, such that for any $h \in H_\alpha$, $a \in A_\beta$, $b \in A_\gamma$, $h\cdot a \in A_{\alpha\beta}$, $h\cdot 1_{A_\beta} = \varepsilon(h)1_{A_{\alpha\beta}}$ and $h\cdot(ab) = \sum_{(h)}(h'\cdot a)(h''\cdot b) \in A_{\alpha\beta}A_{\alpha\gamma} \in A_{\alpha\beta\gamma}$, and a map σ from $H \otimes H$ to A which satisfies $\sigma(h,k) \in A_{\alpha\beta}$ for any $h \in H_\alpha$ and $k \in H_\beta$;

(2) For any $\alpha \in Y$, H_α measures A_α and $\sigma|_{H_\alpha\otimes H_\alpha}$ is an (convolution) invertible map from $H_\alpha \otimes H_\alpha$ to A_α;

(3) For any $\alpha \in Y$, A_α is a twisted H_α-module and $\sigma|_{H_\alpha\otimes H_\alpha}$ is a cocycle.

Definition 2.3 *Let $A = \bigoplus_{\alpha\in Y} A_\alpha \subset B = \bigoplus_{\alpha\in Y} B_\alpha$ be semilattice graded k-algebras and $H = \bigoplus_{\alpha\in Y} H_\alpha$ a semilattice graded weak Hopf algebra with the same semilattice Y.*

(1) $A \subset B$ is a (right) H-G-extension if B is a right H-comodule algebra with $\rho : B \to B\otimes H$ such that B_α is a right H_α-comodule algebra with $\rho|_{B_\alpha} = \rho_\alpha : B_\alpha \to B_\alpha \otimes H_\alpha$ and $B_\alpha^{\mathrm{co}H_\alpha} = A_\alpha$ for any $\alpha \in Y$.

(2) The H-G-extension $A \subset B$ is an H-G-cleft if there exists a right H-comodule graded map $\gamma : H \to B$ which is regular (convolution) invertible with $\gamma^{-1} : H \to B$ satisfies: $\gamma(H_\alpha) \subset B_\alpha$ and $\gamma_\alpha = \gamma|_{H_\alpha} : H_\alpha \to B_\alpha$ is an invertible right H_α-comodule map with inverse $\gamma_\alpha^{-1} = \gamma^{-1}|_{H_\alpha}$.

3. G-cleft extension

The main result of this section is the following:

Theorem 3.1 *An H-G-extension $A=\bigoplus_{\alpha\in Y}A_\alpha\subset B=\bigoplus_{\alpha\in Y}B_\alpha$ with the same semilattice Y is an H-G-cleft $\Leftrightarrow B\cong\bigoplus_{\alpha\in Y}(A_\alpha\#_\sigma H_\alpha)$.*

The theorem follows from the next two propositions.

Proposition 3.2 *Let $A\subset B$ be a right H-G-extension, which is an H-G-cleft via $\gamma:H\to B$ such that $\gamma(1_{H_\alpha})=1_{B_\alpha}$ for any $\alpha\in Y$ and $\gamma(1)=1$. Then there is a crossed product action of H on A and H_α on A_α for any $\alpha\in Y$, which is given by:*

$$h\cdot a=\sum_{(h)}\gamma(h')a\gamma^{-1}(h'')\quad \text{for all}\ \ a\in A,\ h\in H$$

and a convolution regular invertible map $\sigma:H\otimes H\to A$ which is given by:

$$\sigma(h,k)=\sum_{(h),(k)}\gamma(h')\gamma(k')\gamma^{-1}(h''k'')\quad \text{for all}\ \ h,k\in H.$$

Moreover, σ satisfies $\sigma|_{H_\alpha\otimes H_\alpha}=\sigma_\alpha$ is invertible for any $\alpha\in Y$. This action and cocycle σ_α give B_α the structure of an H_α-crossed product over A_α for any $\alpha\in Y$ and give the structure of an H-crossed product over A. Moreover, the semilattice graded algebra isomorphism Φ : $\bigoplus_{\alpha\in Y}(A_\alpha\#_\sigma H_\alpha)\to B=\bigoplus_{\alpha\in Y}B_\alpha$ given by $a\#h\mapsto a\gamma(h)$ is both a left A-module and right H-comodule map, where $\bigoplus_{\alpha\in Y}(A_\alpha\#_\sigma H_\alpha)$ is a left A-module via $a\cdot(b\#h)=(a\#1_{H_\alpha})(b\#h)$ for any $a\in A_\alpha$ and $b\#h\in A_\beta\#_\sigma H_\beta$, $\bigoplus_{\alpha\in Y}(A_\alpha\#H_\alpha)$ is a right H-comodule via $a\#h\mapsto\sum_{(h)}a\#h'\otimes h''$.

We require a technical lemma.

Lemma 3.3 *Assume that $A\subset B$ is a right H-G-extension via $\rho:B\to B\otimes H$ and that $A\subset B$ is an H-G-cleft via γ with $\gamma(1_{H_\alpha})=1_{B_\alpha}$ for any $\alpha\in Y$. Then:*

(1) $\rho\circ\gamma^{-1}=(\gamma^{-1}\otimes T)\circ\tau\circ\Delta$;

(2) For any $b\in B_\alpha$, $\sum_{(b)}b^{(0)}\gamma^{-1}(b')\in A_\alpha=B_\alpha^{\mathrm{co}H_\alpha}$.

Proof (1) First observe that $\rho\circ\gamma^{-1}$ is the regular inverse of $\rho\circ\gamma=(\gamma\otimes\mathrm{id})\circ\Delta$. Because ρ is an algebra map, we can get

$$(\rho\circ\gamma^{-1})*(\rho\circ\gamma)*(\rho\circ\gamma^{-1})=\rho\circ(\gamma^{-1}*\gamma*\gamma^{-1})=\rho\circ\gamma^{-1}$$

$$(\rho\circ\gamma)*(\rho\circ\gamma^{-1})*(\rho\circ\gamma)=\rho\circ(\gamma*\gamma^{-1}*\gamma)=\rho\circ\gamma$$

that is, $\rho\circ\gamma^{-1}$ is a regular inverse of $\rho\circ\gamma$. Moreover, $\rho\circ\gamma^{-1}|_{H_\alpha}=\rho_\alpha\circ\gamma^{-1}$ is the inverse of $\rho\circ\gamma|_{H_\alpha}=\rho_\alpha\circ\gamma_\alpha=(\gamma_\alpha\otimes\mathrm{id})\circ\Delta$. Let $\theta=(\gamma^{-1}\otimes T)\circ\tau\circ\Delta$ and $\theta_\alpha=\theta|_{H_\alpha}=(\gamma_\alpha^{-1}\otimes S|_\alpha)\circ\tau\circ\Delta$ with $S_\alpha=T|_{H_\alpha}$ the antipode of H_α for any $\alpha\in Y$. Because, for any $h\in H_\alpha$,

$$((\rho_\alpha\circ\gamma_\alpha)*\theta_\alpha)(h_\alpha)=\sum_{(h)}(\rho_\alpha\circ\gamma_\alpha)(h')\theta_\alpha(h'')=\sum_{(h)}(\gamma_\alpha\otimes\mathrm{id})\Delta(h')(\gamma_\alpha^{-1}\otimes S_\alpha)\circ\tau\circ\Delta(h'')$$

$$=\sum_{(h)}(\gamma_\alpha(h')\otimes h'')(\gamma_\alpha^{-1}(h^{(4)})\otimes S_\alpha(h'''))=\sum_{(h)}\gamma_\alpha(h')\gamma_\alpha^{-1}(h^{(4)})\otimes h''S_\alpha(h''')$$

$$= \sum_{(h)} \gamma_\alpha(h')\gamma_\alpha^{-1}(h''') \otimes \varepsilon(h'')1_{H_\alpha} = \sum_{(h)} \gamma_\alpha(h')\gamma_\alpha^{-1}(h'') \otimes 1_{H_\alpha}$$
$$= \varepsilon(h)1_{B_\alpha} \otimes 1_{H_\alpha},$$

θ_α is a right inverse of $\rho_\alpha \circ \gamma_\alpha$, and so $\theta_\alpha = \rho_\alpha \circ \gamma_\alpha^{-1}$ by uniqueness of the inverse. Moreover, $(\rho\circ\gamma) * \theta * (\rho\circ\gamma) = \rho\circ\gamma$, $\theta * (\rho\circ\gamma) * \theta = \theta$ since for nay $h \in H_\alpha$ with $\alpha \in Y$,

$$(\rho\circ\gamma) * \theta * (\rho\circ\gamma)(h) = \sum_{(h)} (\rho\circ\gamma)(h')\theta(h'')(\rho\circ\gamma)(h''')$$
$$= \sum_{(h)} ((\gamma\otimes \mathrm{id})\circ\Delta)(h')((\gamma^{-1}\otimes T)\circ\tau\circ\Delta)(h'')((\gamma\otimes\mathrm{id})\circ\Delta)(h''')$$
$$= (\gamma(h')\otimes h'')(\gamma^{-1}(h^{(4)})\otimes T(h'''))(\gamma(h^{(5)})\otimes h^{(6)})$$
$$= \sum_{(h)} \gamma(h')\gamma^{-1}(h^{(4)})\gamma(h^{(5)}) \otimes h''T(h''')h^{(6)}$$
$$= \sum_{(h)} \gamma(h')\varepsilon(h''')1_{B_\alpha} \otimes \varepsilon(h_2)1_{H_\alpha}h^{(4)} = \sum_{(h)} \gamma(h')\otimes h'' = (\rho\circ\gamma)(h).$$

Similarly, $\theta * (\rho\circ\gamma) * \theta = \theta$; that is, θ is a regular inverse of $\rho\circ\gamma$ such that $\theta_\alpha = \rho_\alpha\circ\gamma_\alpha^{-1}$. Hence, $\rho\circ\gamma^{-1} = \theta = (\gamma^{-1}\otimes T)\circ\tau\circ\Delta$.

(2) For any $b \in B_\alpha$,

$$\rho(\sum_{(b)} b^{(0)}\gamma^{-1}(b')) = \sum_{(b)} \rho(b^{(0)})(\rho\circ\gamma^{-1})(b') = \sum_{(b)} \rho(b^{(0)})((\gamma^{-1}\otimes T)\circ\tau\circ\Delta)(b')$$
$$= \sum_{(b)} \rho(b^{(0)})(\gamma^{-1}(b'')\otimes T(b')) = \sum_{(b)} (b^{(0)}\otimes b')(\gamma^{-1}(b''')\otimes T(b''))$$
$$= \sum_{(b)} b^{(0)}\gamma^{-1}(b''')\otimes b'T(b'') = \sum_{(b)} b^{(0)}\gamma^{-1}(b'')\otimes\varepsilon(b')1_{H_\alpha}$$
$$= \sum_{(b)} b^{(0)}\gamma^{-1}(b')\otimes 1_{H_\alpha}.$$

That is, $\sum_{(b)} b^{(0)}\gamma^{-1}(b') \in B_\alpha^{\mathrm{co}H_\alpha} = A_\alpha$ $\square$.

This lemma enables us to define an inverse to Φ. Namely, define

$$\Psi : B \to \bigoplus_{\alpha\in Y}(A_\alpha \#_\sigma H_\alpha) \ \text{ by } \ b \mapsto \sum_{(b)} b^{(0)}\gamma^{-1}(b')\#b''.$$

Proof of Proposition 3.2 (1) $h\cdot a$ is a module action.

For any $h\in H_\alpha$, $a \in A_\beta = B_\beta^{\mathrm{co}H_\beta}$,

$$\rho(h\cdot a) = \rho(\sum_{(h)}\gamma(h')a\gamma^{-1}(h'')) = \sum_{(h)}(\rho\circ\gamma)(h')\rho(a)(\rho\circ\gamma^{-1})(h'')$$
$$= \sum_{(h)}((\gamma\otimes\mathrm{id})\circ\Delta)(h')(a\otimes 1_{H_\beta})((\gamma^{-1}\otimes T)\circ\tau\circ\Delta)(h'')$$
$$= \sum_{(h)}(\gamma(h')\otimes h'')(a\otimes 1_{H_\beta})(\gamma^{-1}(h^{(4)})\otimes T(h'''))$$
$$= \sum_{(h)}\gamma(h')a\gamma^{-1}(h^{(4)})\otimes h''1_{H_\beta}T(h''') = \sum_{(h)}\gamma(h')a\gamma^{-1}(h''')\otimes\varepsilon(h'')1_{H_\alpha}1_{H_\beta}$$

$$= \sum_{(h)} \gamma(h')a\gamma^{-1}(h'') \otimes 1_{H_{\alpha\beta}} = h \cdot a \otimes 1_{H_{\alpha\beta}}.$$

As $h \in H_\alpha$, $\gamma(h'), \gamma^{-1}(h'') \in B_\alpha$ and $a \in A_\beta$, we have $h \cdot a = \sum_{(h)} \gamma(h')a\gamma^{-1}(h'') \in B_\alpha A_\beta B_\alpha \subset B_{\alpha\beta}$, then, $\rho(h \cdot a) = h \cdot a \otimes 1_{H_{\alpha\beta}} \in B_{\alpha\beta} \otimes H_{\alpha\beta}$, that is $h \cdot a \in B_{\alpha\beta}^{\mathrm{co}H_{\alpha\beta}} = A_{\alpha\beta} \subset A$. Obviously, for any $h \in H_\alpha$ and $a \in A_\alpha$, $h \cdot a \in A_\alpha$. Moreover, H_α measures A_α for any $\alpha \in Y$.

(i) For any $a \in B_\alpha$, $b \in B_\beta$ and $h \in H_\gamma$,

$$h \cdot (ab) = \sum_{(h)} \gamma(h')ab\gamma(h'')$$

$$\sum_{(h)} (h' \cdot a)(h'' \cdot b) = \sum_{(h)} \gamma(h')a\gamma^{-1}(h'')\gamma(h''')b\gamma^{-1}(h^{(4)}) = \sum_{(h)} \gamma(h')a\varepsilon(h'')1_{B_\gamma} b\gamma^{-1}(h''')$$
$$= \sum_{(h)} \gamma(h')a1_{B_\gamma} b\gamma^{-1}(h'') = \sum_{(h)} \gamma(h')ab\gamma^{-1}(h'') = h \cdot (ab).$$

(ii) For any $h \in H_\alpha$ with $\alpha \in Y$,

$$h \cdot 1_{A_\beta} = \sum_{(h)} \gamma(h')1_{A_\beta}\gamma^{-1}(h'') = \sum_{(h)} \gamma(h')\gamma^{-1}(h'')1_{A_\beta} = \varepsilon(h)1_{B_\alpha}1_{A_\beta} = \varepsilon(h)1_{A_{\alpha\beta}}.$$

Hence, $h \cdot 1_{A_\alpha} = \varepsilon(h)1_{A_\alpha}$ and $h \cdot 1_A = \varepsilon(h)1_{A_\alpha}$ for any $\alpha \in Y$.

(iii) For any $a \in A_\beta$ with $\beta \in Y$,

$$1_{H_\alpha} \cdot a = \gamma(1_{H_\alpha})a\gamma^{-1}(1_{H_\alpha}) = 1_{B_\alpha}a1_{B_\alpha} = a1_{B_\alpha}.$$

Hence, $1_H \cdot a = \gamma(1_H)a\gamma^{-1}(1_H) = a$ and $1_{H_\beta} \cdot a = a1_{B_\beta} = a$.

Finally, we want to prove that $h \cdot (k \cdot a) = \sum_{(h),(k)} \sigma(h', k')(h''k'' \cdot a)\sigma^{-1}(h''', k''')$ for any $h \in H_\alpha$, $k \in H_\beta$ and $a \in A_\gamma$ if we define $\sigma^{-1}(h, k) = \sum_{(h),(k)} \gamma(h'k')\gamma^{-1}(k'')\gamma^{-1}(h'')$.

We say that σ^{-1} is a regular inverse of σ. Because, for any $h \in H_\alpha$ and $k \in H_\beta$,

$$\begin{aligned}
(\sigma * \sigma^{-1})(h, k) &= \sum_{(h),(k)} \sigma(h', k')\sigma^{-1}(h'', k'') \\
&= \sum_{(h),(k)} \gamma(h')\gamma(k')\gamma^{-1}(h''k'')\gamma(h'''k''')\gamma^{-1}(k^{(4)})\gamma^{-1}(h^{(4)}) \\
&= \sum_{(h),(k)} \gamma(h')\gamma(k')\varepsilon(h'')\varepsilon(k'')1_{B_{\alpha\beta}}\gamma^{-1}(k''')\gamma^{-1}(h''') \\
&= \sum_{(h),(k)} \gamma(h')\gamma(k')\gamma^{-1}(k'')\gamma^{-1}(h'')1_{B_{\alpha\beta}} \\
&= \sum_{(h)} \gamma(h')\varepsilon(k)1_{B_\beta}\gamma^{-1}(h'')1_{B_{\alpha\beta}} = \varepsilon(h)\varepsilon(k)1_{B_\alpha}1_{B_\beta}1_{B_{\alpha\beta}} = \varepsilon(h)\varepsilon(k)1_{B_{\alpha\beta}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
(\sigma * \sigma^{-1} * \sigma)(h, k) &= \sum_{(h),(k)} (\sigma * \sigma^{-1})(h', k')\sigma(h'', k'') = \sum_{(h),(k)} \varepsilon(h')\varepsilon(k')1_{B_{\alpha\beta}}\gamma(h'')\gamma(k'')\gamma^{-1}(h'''k''') \\
&= \sum_{(h),(k)} 1_{B_{\alpha\beta}}\gamma(h')\gamma(k')\gamma^{-1}(h''k'') = \sum_{(h),(k)} \gamma(h')\gamma(k')\gamma^{-1}(h''k'') = \sigma(h, k);
\end{aligned}$$

that is $\sigma * \sigma^{-1} * \sigma = \sigma$. Similarly, we obtain $\sigma^{-1} * \sigma * \sigma^{-1} = \sigma^{-1}$. Moreover, if $\alpha = \beta$, that is $h, k \in H_\alpha$ for some $\alpha \in Y$, then obviously, $\sigma_\alpha^{-1} = \sigma^{-1}|_{H_\alpha \otimes H_\alpha}$ is the inverse of $\sigma_\alpha = \sigma|_{H_\alpha \otimes H_\alpha}$.

Thus we have

$$h \cdot (k \cdot a) = h \cdot (\sum_{(k)} \gamma(k')a\gamma^{-1}(k'')) = \sum_{(h),(k)} \gamma(h')\gamma(k')a\gamma^{-1}(k'')\gamma^{-1}(h'')$$

and

$$\begin{aligned}
&\sum_{(h),(k)} \sigma(h', k')(h''k'' \cdot a)\sigma^{-1}(h''', k''') \\
&= \sum_{(h),(k)} \gamma(h')\gamma(k')\gamma^{-1}(h''k'')\gamma(h'''k''')a\gamma^{-1}(h^{(4)}k^{(4)})\gamma(h^{(5)}, k^{(5)})\gamma^{-1}(k^{(6)})\gamma^{-1}(h^{(6)}) \\
&= \sum_{(h),(k)} \gamma(h')\gamma(k')\epsilon(h''k'')1_{B_{\alpha\beta}}a\varepsilon(h'''k''')1_{B_{\alpha\beta}}\gamma^{-1}(k^{(4)})\gamma^{-1}(h^{(4)}) \\
&= \sum_{(h),(k)} \gamma(h')\gamma(k')a\gamma^{-1}(k'')\gamma^{-1}(h'')1_{B_{\alpha\beta}} \\
&= \sum_{(h),(k)} \gamma(h')\gamma(k')a\gamma^{-1}(k'')\gamma^{-1}(h'')
\end{aligned}$$

as $\gamma(h'), \gamma^{-1}(h'') \in B_\alpha$ and $\gamma(k'), \gamma^{-1}(k'') \in B_\beta$; that is, $\gamma(h')\gamma(k'), \gamma^{-1}(k'')\gamma^{-1}(h'') \in B_{\alpha\beta}$.

(2) $\sigma|_{H_\alpha \otimes H_\alpha} = \sigma_\alpha$ is a cocycle.

For any $h \in H_\alpha$ and $k \in H_\beta$ with $\alpha, \beta \in Y$,

$$\begin{aligned}
\rho(\sigma(h,k)) &= \rho(\sum_{(h),(k)} \gamma(h')\gamma(k')\gamma^{-1}(h''k'')) = \sum_{(h),(k)} (\rho \circ \gamma)(h')(\rho \circ \gamma)(k')(\rho \circ \gamma^{-1})(h''k'') \\
&= \sum_{(h),(k)} (\gamma((h')') \otimes (h')'')(\gamma((k')') \otimes (k')'')(\gamma^{-1}((h'')''(k'')'') \otimes T((h'')'(k'')')) \\
&= \sum_{(h),(k)} \gamma(h')\gamma(k')\gamma^{-1}(h^{(4)}k^{(4)}) \otimes h''k''T(h'''k''') \\
&= \sum_{(h),(k)} \gamma(h')\gamma(k')\gamma^{-1}(h'''k''') \otimes \varepsilon(h''k'')1_{H_{\alpha\beta}} \\
&= \sum_{(h),(k)} \gamma(h')\gamma(k')\gamma^{-1}(h''k'') \otimes 1_{H_{\alpha\beta}} = \sigma(h,k) \otimes 1_{H_{\alpha\beta}}.
\end{aligned}$$

Because $\sigma(h,k) = \sum_{(h),(k)} \gamma(h')\gamma(k')\gamma^{-1}(h''k'')$ and $\gamma(h') \in B_\alpha$, $\gamma(k') \in B_\beta$, $\gamma^{-1}(h''k'') \in B_{\alpha\beta}$, we have $\sigma(h,k) \in B_\alpha B_\beta B_{\alpha\beta} \subset B_{\alpha\beta}$; that is, $\rho(\sigma(h,k)) = \sigma(h,k) \otimes 1_{H_{\alpha\beta}} \in B_{\alpha\beta} \otimes H_{\alpha\beta}$, thus $\sigma(h,k) \in B_{\alpha\beta}^{coH_{\alpha\beta}} = A_{\alpha\beta} \subset A$. Obviously, if $h, k \in H_\alpha$ for some $\alpha \in Y$, $\sigma(h,k) \in A_\alpha$.

Moreover,

$$\sigma(1_{H_\alpha}, k) = \sum_{(k)} \gamma(1_{H_\alpha})\gamma(k')\gamma^{-1}(1_{H_\alpha}k'').$$

Hence, if $k \in H_\alpha$, we have $\sigma(1_{H_\alpha}, k) = \varepsilon(k)1_{B_\alpha}$ and $\sigma(1_H, k) = \varepsilon(k)1_{B_\alpha}$. Similarly, $\sigma(h, 1_{H_\alpha}) = \varepsilon(h)1_{B_\alpha}$ and $\sigma(h, 1_H) = \varepsilon(h)1_{B_\alpha}$ for any $h \in H_\alpha$. Additionally,

$$\sigma(1_{H_\alpha}, 1_{H_\beta}) = \gamma(1_{H_\alpha})\gamma(1_{H_\beta})\gamma^{-1}(1_{H_\alpha}1_{H_\beta}) = 1_{B_{\alpha\beta}}.$$

Next, we need to prove that $\sum_{(h),(k),(m)}[h'\cdot\sigma(k',m')]\sigma(h'',k''m'')=\sum_{(h),(k)}\sigma(h',k')\sigma(h''k'',m)$ for any $h\in H_\alpha$, $k\in H_\beta$ and $m\in H_\gamma$ with $\alpha,\beta,\gamma\in Y$. We develop the left- and right-hand sides separately:

$$\begin{aligned}
\text{Left}&=\sum_{(h),(k),(m)}\gamma(h')(\gamma(k')\gamma(m')\gamma^{-1}(k''m''))\gamma^{-1}(h'')\gamma(h''')\gamma(k'''m''')\gamma^{-1}(h^{(4)}k^{(4)}m^{(4)})\\
&=\sum_{(h),(k),(m)}\gamma(h')\gamma(k')\gamma(m')\gamma^{-1}(k''m'')\varepsilon(h'')1_{B_\alpha}\gamma(k'''m''')\gamma^{-1}(h'''k^{(4)}m^{(4)})\\
&=\sum_{(h),(k),(m)}\gamma(h')\gamma(k')\gamma(m')\gamma^{-1}(k''m'')\gamma(k'''m''')\gamma^{-1}(h''k^{(4)}m^{(4)})1_{B_\alpha}\\
&=\sum_{(h),(k),(m)}\gamma(h')\gamma(k')\gamma(m')\varepsilon(k''m'')1_{B_{\beta\gamma}}\gamma^{-1}(h''k'''m''')1_{B_\alpha}\\
&=\sum_{(h),(k),(m)}\gamma(h')\gamma(k')\gamma(m')\gamma^{-1}(h''k''m'')1_{B_{\beta\gamma}}1_{B_\alpha}\\
&=\sum_{(h),(k),(m)}\gamma(h')\gamma(k')\gamma(m')\gamma^{-1}(h''k''m'')1_{B_{\alpha\beta\gamma}}\\
&=\sum_{(h),(k),(m)}\gamma(h')\gamma(k')\gamma(m')\gamma^{-1}(h''k''m'');\\
\text{Right}&=\sum_{(h),(k),(m)}\gamma(h')\gamma(k')\gamma^{-1}(h''k'')\gamma(h'''k''')\gamma(m')\gamma(h^{(4)}k^{(4)}m'')\\
&=\sum_{(h),(k),(m)}\gamma(h')\gamma(k')\varepsilon(h''k'')1_{B_{\alpha\beta}}\gamma(m')\gamma^{-1}(h'''k'''m'')\\
&=\sum_{(h),(k),(m)}\gamma(h')\gamma(k')\gamma(m')\gamma^{-1}(h''k''m'')1_{B_{\alpha\beta}}\\
&=\sum_{(h),(k),(m)}\gamma(h')\gamma(k')\gamma(m')\gamma^{-1}(h''k''m''),
\end{aligned}$$

as $\gamma(h')\in B_\alpha$, $\gamma(k')\in B_\beta$, $\gamma(m')\in B_\gamma$ and $\gamma^{-1}(h''k''m'')\in B_{\alpha\beta\gamma}$.

We can then state that for any $\alpha\in Y$, σ_α is a cocycle.

From (1) and (2) we obtain $A_\alpha\#_\sigma H_\alpha$ is an associative algebra with identity element $1_{A_\alpha}\#1_{H_\alpha}$ for any $\alpha\in Y$. Moreover, $A\#_\sigma H$ is an associative algebra without identity because for any $a\in A_{\alpha_1}$, $b\in A_{\alpha_2}$ and $c\in A_{\alpha_3}$, $h\in H_{\beta_1}$, $g\in H_{\beta_2}$ and $k\in H_{\beta_3}$,

$$\begin{aligned}
((a\#h)(b\#g))(c\#k)&=(\sum_{(h),(g)}a(h'\cdot b)\sigma(h'',g')\#h'''g'')(c\#k)\\
&=\sum_{(h),(g),(k)}a(h'\cdot b)\sigma(h'',g')((h'''g'')'\cdot c)\sigma((h'''g'')'',k')\#(h'''g'')'''k''\\
&=\sum_{(h),(g),(k)}a(h'\cdot b)\sigma(h'',g')((h'''g'')\cdot c)\sigma(h^{(4)}g''',k')\#h^{(5)}g^{(4)}k''\\
&=\sum_{(h),(g),(k)}a(h'\cdot b)\sigma(h'',g')(h'''\cdot(g''\cdot c))\sigma(h^{(4)}g''',k')\#h^{(5)}g^{(4)}k''
\end{aligned}$$

$$
\begin{aligned}
&(a\#h)((b\#g)(c\#k)) = (a\#h)(\sum_{(g),(k)} b(g'\cdot c)\sigma(g'',k')\#g'''k'')\\
&= \sum_{(h),(g),(k)} a(h'\cdot(b(g'\cdot c)\sigma(g'',k'))\sigma(h'',g'''k'')\#h'''g^{(4)}k'''\\
&= \sum_{(h),(g),(k)} a(h'\cdot b(g'\cdot c))(h''\cdot\sigma(g'',k'))\sigma(h''',g'''k'')\#h^{(4)}g^{(4)}k'''\\
&= \sum_{(h),(g),(k)} a(h'\cdot b(g'\cdot c))\sigma(h'',g'')\sigma(h'''g''',k')\#h^{(4)g^{(4)}}k''\\
&= \sum_{(h),(g),(k)} a(h'\cdot b)h''\cdot(g'\cdot c)\sigma(h''',g'')\sigma(h^{(4)}g''',k')\#h^{(5)}g^{(4)}k''\\
&= \sum_{(h),(g),(k)} a(h'\cdot b)\sigma(h'',g')((h'''g'')\cdot c)\sigma^{-1}(h^{(4)},g''')\sigma(h^{(5)},g^{(4)})\sigma(h^{(6)}g^{(5)},k')\#h^{(7)}g^{(6)}k''\\
&= \sum_{(h),(g),(k)} a(h'\cdot b)\sigma(h'',g')((h'''g'')\cdot c)\varepsilon(h^{(4)})\varepsilon(g''')1_{B_{\alpha\beta}}\sigma(h^{(5)}g^{(4)},k')\#h^{(6)}g^{(5)}k''\\
&= \sum_{(h),(g),(k)} a(h'\cdot b)\sigma(h'',g')((h'''g'')\cdot c)1_{B_{\alpha\beta}}\sigma(h^{(4)}g''',k')\#h^{(5)}g^{(4)}k''\\
&= \sum_{(h),(g),(k)} a(h'\cdot b)\sigma(h'',g')(h'''\cdot(g''\cdot c))\sigma(h^{(4)}g''',k')\#h^{(5)}g^{(4)}k''\\
&= ((a\#h)(b\#g))(c\#k).
\end{aligned}
$$

However, $1\#1$ is not the identity of $A\#_\sigma H$; hence, $A\#_\sigma H$ is an associative algebra without identity and $\bigoplus_{\alpha\in Y}(A_\alpha\#_\sigma H_\alpha)$ is a sub-algebra of $A\#_\sigma H$.

Next we show that

$$\Phi:\bigoplus_{\alpha\in Y}(A_\alpha\#_\sigma H_\alpha)\to B=\bigoplus_{\alpha\in Y}B_\alpha \quad \text{by} \quad a\#h\mapsto a\gamma(h)$$

and

$$\Psi: B=\bigoplus_{\alpha\in Y}B_\alpha\to\bigoplus_{\alpha\in Y}(A_\alpha\#_\sigma H_\alpha) \quad \text{by} \quad b\mapsto\sum_{(b)} b^{(0)}\gamma^{-1}(b')\#b''$$

are mutual inverses. First, if $b\in B_\alpha$ for some $\alpha\in Y$, then $\Phi\Psi(b)=\Phi(\sum_{(b)} b^{(0)}\gamma^{-1}(b')\#b'')=\sum_{(b)} b^{(0)}\gamma^{-1}(b')\gamma(b'')=\sum_{(b)} b^{(0)}\varepsilon(b')1_{B_\alpha}=b1_{B_\alpha}=b$. Next, choose $a\#h\in A_\alpha\#_\sigma H_\alpha$ for any $\alpha\in Y$, then

$$
\begin{aligned}
\Psi\Phi(a\#h) &= \Psi(a\gamma(h)) = \sum_{(a\gamma(h))}(a\gamma(h))^{(0)}\gamma^{-1}((a\gamma(h))')\#(a\gamma(h))''\\
&= \sum_{(a),(h)} a^{(0)}\gamma(h)^{(0)}\gamma^{-1}(a'\gamma(h)')\#a''\gamma(h)''\\
&= \sum_{(h)} a\gamma(h)^{(0)}\gamma^{-1}(1_{H_\alpha}\gamma(h)')\#1_{H_\alpha}\gamma(h)'' \quad (\text{since } \rho(a)=a\otimes 1_{H_\alpha})\\
&= \sum_{(h)} a\gamma(h)^{(0)}\gamma^{-1}(\gamma(h)')\#\gamma(h)'' \quad (\text{since } \gamma(h)\in B_\alpha \text{ and } \gamma(h)',\gamma(h)''\in H_\alpha)\\
&= a(\sum_{(h)}\gamma(h)^{(0)}\gamma^{-1}(\gamma(h)')\#\gamma(h)'') \quad (\text{since } \rho\circ\gamma=(\gamma\otimes\mathrm{id})\circ\Delta)
\end{aligned}
$$

$$= a(\sum_{(h)} \gamma(h')\gamma^{-1}(h'')\#h''') = a(\sum_{(h)} \varepsilon(h')1_{B_\alpha}\#h'') = a1_{B_\alpha}\#h = a\#h.$$

Thus $\Psi = \Phi^{-1}$.

Moreover, Φ is a semilattice graded algebra map. Specifically, for any $a\#h \in A_\alpha\#_\sigma H_\alpha$ and $b\#k \in A_\beta\#_\sigma H_\beta$ with $\alpha, \beta \in Y$, and also $h \in H_\alpha$, $h', h'', h''' \in H_\alpha$ and $h' \cdot b \in A_{\alpha\beta}$, $\sigma(h'', k') \in A_{\alpha\beta}$, we have $(a\#h)(b\#k) = \sum_{(h),(k)} a(h' \cdot b)\sigma(h'', k')\#h'''k'' \in A_{\alpha\beta}\#H_{\alpha\beta}$; that is, $(A_\alpha\#_\sigma H_\alpha)(A_\beta\#_\sigma H_\beta) \subset A_{\alpha\beta}\#_\sigma H_{\alpha\beta}$. Therefore,

$$\begin{aligned}
\Phi((a\#h)(b\#k)) &= \Phi(\sum_{(h),(k)} a(h' \cdot b)\sigma(h'', k')\#h'''k'') = \sum_{(h),(k)} a(h' \cdot b)\sigma(h'', k')\gamma(h'''k'') \\
&= \sum_{(h),(k)} a\gamma(h')b\gamma^{-1}(h'')\gamma(h''')\gamma(k')\gamma^{-1}(h^{(4)}k'')\gamma(h^{(5)}k''') \\
&= \sum_{(h),(k)} a\gamma(h')b\varepsilon(h'')1_{B_\alpha}\gamma(k')\gamma^{-1}(h'''k'')\gamma(h^{(4)}k''') \\
&= \sum_{(h),(k)} a\gamma(h')b1_{B_\alpha}\gamma(k')\varepsilon(h''k'')1_{B_{\alpha\beta}} \\
&= a\gamma(h)b\gamma(k)1_{B_{\alpha\beta}} \quad (\text{since } a\gamma(h)b\gamma(k) \in B_\alpha B_\beta \subset B_{\alpha\beta}) \\
&= a\gamma(h)b\gamma(k) = \Phi(a\#h)\Phi(b\#k).
\end{aligned}$$

Obviously, $\Phi(A_\alpha\#_\sigma H_\alpha) \subset B_\alpha$ for any $\alpha \in Y$. Thus $B \cong \bigoplus_{\alpha\in Y}(A_\alpha\#_\sigma H_\alpha)$ as semilattice graded algebras.

Also, Φ is a left A-module map. For any $a \in A_\alpha$, $b \in A_\beta$, and $c\#h \in A_\gamma\#_\sigma H_\gamma$, then obviously

$$\begin{aligned}
a \cdot (b \cdot (c\#h)) &= (a\#1_{H_\alpha})((b\#1_{H_\beta})(c\#h)) = ((a\#1_{H_\alpha})(b\#1_{H_\beta}))(c\#h) \\
&= (a(1_{H_\alpha} \cdot b)\sigma(1_{H_\alpha}, 1_{H_\beta})\#1_{H_\alpha}1_{H_\beta})(c\#h) \\
&= (ab1_{B_{\alpha\beta}}\#1_{H_{\alpha\beta}})(c\#h) = (ab) \cdot (c\#h)
\end{aligned}$$

and

$$1 \cdot (c\#h) = (1\#1)(c\#h) = c\#h.$$

Hence, $\bigoplus_{\alpha\in Y}(A_\alpha\#_\sigma H_\alpha)$ is an A-module. Moreover, for any $a \in A_\alpha$ and $b\#h \in A_\beta\#H_\beta$ with $\alpha, \beta \in Y$,

$$a \cdot (\Phi(b\#h)) = a \cdot (b\gamma(h)) = ab\gamma(h)$$

and

$$\begin{aligned}
\Phi(a \cdot (b\#h)) &= \Phi((a\#1_{H_\alpha})(b\#h) = \Phi(\sum_{(h)} a(1_{H_\alpha} \cdot b)\sigma(1_{H_\alpha}, h')\#1_{H_\alpha}h'') \\
&= \Phi(\sum_{(h)} ab\gamma(h')\gamma^{-1}(1_{H_\alpha}h'')\#1_{H_\alpha}h''') = \sum_{(h)} ab\gamma(h')\gamma^{-1}(1_{H_\alpha}h'')\gamma(1_{H_\alpha}h''') \\
&= \sum_{(h)} ab\gamma(h')\varepsilon(1_{H_\alpha}h'')1_{B_{\alpha\beta}} = ab\gamma(h)1_{B_{\alpha\beta}} = ab\gamma(h) = a \cdot \Phi(b\#h).
\end{aligned}$$

It is also a right H-comodule map, since for any $a \in A_\alpha$, $h \in H_\alpha$ with $\alpha \in Y$,

$$\begin{aligned}\rho(\Phi(a\#h)) &= \rho(a\gamma(h)) = \rho(a)(\rho\circ\gamma)(h) = (a\otimes 1_{H_\alpha})(\sum_{(h)}\gamma(h')\otimes h'') \\ &= \sum_{(h)} a\gamma(h')\otimes h''1_{H_\alpha} = \sum_{(h)} a\gamma(h')\otimes h'' = \sum_{(h)}\Phi(a\#h')\otimes h'' \\ &= (\Phi\otimes \mathrm{id})\circ(\mathrm{id}\otimes\Delta)(a\#h).\end{aligned}$$

This completes the proof of the Proposition 3.2. □

Proposition 3.4 *Let $\bigoplus_{\alpha\in Y}(A_\alpha\#_\sigma H_\alpha)$ be a G-crossed product, and define the map $\gamma : H \to \bigoplus_{\alpha\in Y}(A_\alpha\#_\sigma H_\alpha)$ by $\gamma(h) = 1_{A_\alpha}\#h$ for any $h \in H_\alpha$ with $\alpha \in Y$. Then γ is convolution regular invertible with regular inverse*

$$\gamma^{-1}(h) = \sum_{(h)}\sigma^{-1}(T(h''), h''')\#T(h')$$

and for any $\alpha \in Y$, $\gamma|_{H_\alpha}$ is invertible in $\mathrm{Hom}(H_\alpha, A_\alpha\#_\sigma H_\alpha)$ with inverse $\gamma^{-1}|_{H_\alpha}$. In particular $A = \bigoplus_{\alpha\in Y} A_\alpha \subset \bigoplus_{\alpha\in Y}(A_\alpha\#_\sigma H_\alpha)$ is an H-G-cleft.

Proof Set $\mu(h) = \sum_{(h)}\sigma^{-1}(T(h''), h''')\#T(h')$. Then it is straightforward to verify that μ is a regular inverse for γ. For any $h \in H_\alpha$,

$$\begin{aligned}(\mu*\gamma)(h) &= \sum_{(h)}\mu(h')\gamma(h'') = \sum_{(h)}(\sigma^{-1}(T(h''), h''')\#T(h'))(1_{A_\alpha}\#h^{(4)}) \\ &= \sum_{(h)}\sigma^{-1}(T(h^{(4)}), h^{(5)})(T(h''')\cdot 1_{A_\alpha})\sigma(T(h''), h^{(6)})\#T(h')h^{(7)} \\ &= \sum_{(h)}\sigma^{-1}(T(h^{(4)}, h^{(5)}))\varepsilon(T(h'''))1_{A_\alpha}\sigma(T(h''), h^{(6)})\#T(h')h^{(7)} \\ &= \sum_{(h)}\sigma^{-1}(T(h'''), h^{(4)})\sigma(T(h''), h^{(5)})\#T(h')h^{(6)} \\ &= \sum_{(h)}\varepsilon(T(h''))\varepsilon(h''')1_{A_\alpha}\#T(h')h^{(4)} = 1_{A_\alpha}\#T(h')h'' \\ &= 1_{A_\alpha}\#\varepsilon(h)1_{H_\alpha} = \varepsilon(h)(1_{A_\alpha}\#1_{H_\alpha})\end{aligned}$$

and

$$\begin{aligned}(\mu*\gamma*\mu)(h) &= \sum_{(h)}(\mu*\gamma)(h')\mu(h'') = \sum_{(h)}\varepsilon(h')(1_{A_\alpha}\#1_{H_\alpha})(\sigma^{-1}(T(h'''), h^{(4)})\#T(h'')) \\ &= \sum_{(h)}\varepsilon(h')1_{A_\alpha}(1_{H_\alpha}\cdot\sigma^{-1}(T(h^{(4)}), h^{(5)}))\sigma(1_{H_\alpha}, T(h'''))\#1_{H_\alpha}T(h'') \\ &= \sum_{(h)}\sigma^{-1}(T(h'''), h^{(4)})\varepsilon(T(h''))1_{A_\alpha}\#T(h') = \sum_{(h)}\sigma^{-1}(T(h''), h''')\#T(h') \\ &= \mu(h);\end{aligned}$$

that is, $\mu*\gamma*\mu = \mu$. Similarly, we can get $\gamma*\mu*\gamma = \gamma$.

Obviously, $(\mu|_{H_\alpha} * \gamma|_{H_\alpha})(h) = \varepsilon(h)1_{A_\alpha}\#1_{H_\alpha}$ and $(\gamma|_{H_\alpha} * \mu|_{H_\alpha})(h) = \varepsilon(h)1_{A_\alpha}\#1_{H_\alpha}$; thus

$\mu|_{H_\alpha}$ is the inverse of $\gamma_\alpha=\gamma|_{H_\alpha}$, and hence μ is uniquely defined. Therefore, $A=\bigoplus_{\alpha\in Y}A_\alpha\subset\bigoplus_{\alpha\in Y}(A_\alpha\#_\sigma H_\alpha)$ is an H-G-cleft. □

Corollary 3.5 *Let $A=\bigoplus_{\alpha\in Y}A_\alpha$ be a semilattice graded algebra, $H=\bigoplus_{\alpha\in Y}H_\alpha$ a semilattice graded weak Hopf algebra and $\bigoplus_{\alpha\in Y}(A_\alpha\#_\sigma H_\alpha)$ a G-crossed product. Then $\bigoplus_{\alpha\in Y}(A_\alpha\#_\sigma H_\alpha)\cong\bigoplus_{\alpha\in Y}(H_\alpha\otimes A_\alpha)$ as right A-modules, provided that the weak antipode T of H is bijective.*

Proof We know that $\gamma:H\to\bigoplus_{\alpha\in Y}(A_\alpha\#_\sigma H_\alpha)$ via $h\mapsto 1_{A_\alpha}\#h$ for any $h\in H_\alpha$ with $\alpha\in Y$ is a regular invertible right H-comodule map with regular inverse γ^{-1} such that $\gamma*\gamma^{-1}(h)=1_{A_\alpha}\varepsilon(h)$ for any $h\in H_\alpha$; that is $\gamma*\gamma^{-1}|_{H_\alpha}=\eta\circ\varepsilon$, where the comodule structure maps for H and $\bigoplus_{\alpha\in Y}(A_\alpha\#_\sigma H_\alpha)$ are given by Δ and by $\rho=\mathrm{id}\otimes\Delta$.

Because

$$\rho(\gamma(h))=(\mathrm{id}\otimes\Delta)(1_{A_\alpha}\#h)=1_{A_\alpha}\#\Delta(h)=\sum_{(h)}1_{A_\alpha}\#h'\otimes h''$$

and

$$(\gamma\otimes\mathrm{id})\circ\Delta(h)=\sum_{(h)}(\gamma\otimes\mathrm{id})(h'\otimes h'')=\sum_{(h)}\gamma(h')\otimes h''=\sum_{(h)}1_{A_\alpha}\#h'\otimes h'',$$

we have $\rho\circ\gamma=(\gamma\otimes\mathrm{id})\circ\Delta$. Then, γ is a right H-comodule map.

Let $\overline{T}$ denote the (composition) inverse of T and set $\mu=\gamma^{-1}\circ\overline{T}$. Note that μ is regular invertible under twist convolution with regular inverse $\hat{\mu}=\gamma\circ\overline{T}$. For any $h\in H_\alpha$, we have

$$\begin{aligned}(\gamma\circ\overline{T})*(\gamma^{-1}\circ\overline{T})(h)&=\sum_{(h)}(\gamma\circ\overline{T})(h')(\gamma^{-1}\circ\overline{T})(h'')=\sum_{(h)}\gamma(\overline{T}(h'))\gamma^{-1}(\overline{T}(h''))\\&=\gamma*\gamma^{-1}(\overline{T}(h))=\varepsilon(\overline{T}(h))1_{A_\alpha}\#1_{H_\alpha}=\varepsilon(h)1_{A_\alpha}\#1_{H_\alpha}\end{aligned}$$

and

$$\begin{aligned}(\gamma^{-1}\circ\overline{T})*(\gamma\circ\overline{T})(h)&=\sum_{(h)}(\gamma^{-1}\circ\overline{T})(h')(\gamma\circ\overline{T})(h'')=\sum_{(h)}\gamma^{-1}(\overline{T}(h'))\gamma(\overline{T}(h''))\\&=(\gamma^{-1}*\gamma)(\overline{T}(h))=\varepsilon(\overline{T}(h))1_{A_\alpha}\#1_{H_\alpha}=\varepsilon(h)1_{A_\alpha}\#1_{H_\alpha}.\end{aligned}$$

Hence,

$$(\gamma\circ\overline{T})*(\gamma^{-1}\circ\overline{T})*(\gamma\circ\overline{T})=\gamma\circ\overline{T}\quad\text{and}\quad(\gamma^{-1}\circ\overline{T})*(\gamma\circ\overline{T})*(\gamma^{-1}\circ\overline{T})=\gamma^{-1}\circ\overline{T}.$$

It follows that if $B=\bigoplus_{\alpha\in Y}B_\alpha=\bigoplus_{\alpha\in Y}(A_\alpha\#_\sigma H_\alpha)\subset A\#_\sigma H$, then B^{op} is a right H^{op}-comodule algebra and B_α^{op} is a right H_α^{op}-comodule algebra which is an H-G-cleft via $\mu:H^{\mathrm{op}}\to B^{\mathrm{op}}$. Thus, $\bigoplus_{\alpha\in Y}(A_\alpha\otimes H_\alpha)$ is a left A^{op}-module via the action: $b\cdot(a\otimes h)=(b\otimes 1_{H_\beta})(a\otimes h)$ for any $b\in A_\beta$ and $a\otimes h\in A_\alpha\otimes H_\alpha$ and $\bigoplus_{\alpha\in Y}(A_\alpha\otimes H_\alpha)\cong B^{\mathrm{op}}=\bigoplus_{\alpha\in Y}B_\alpha^{\mathrm{op}}$ as left A^{op}-modules, where the isomorphism is given by

$$\begin{aligned}f:&\bigoplus_{\alpha\in Y}(A_\alpha\otimes H_\alpha)\to B^{\mathrm{op}}\\&a^{\mathrm{op}}\otimes h^{\mathrm{op}}\mapsto a^{\mathrm{op}}\mu(h)^{\mathrm{op}}=(\gamma^{-1}(\overline{T}(h))a)^{\mathrm{op}}.\end{aligned}$$

Because, for any $b^{\mathrm{op}} \in B_\beta$ and $a^{\mathrm{op}} \otimes h^{\mathrm{op}} \in A^{\mathrm{op}}_\alpha \otimes H^{\mathrm{op}}_\alpha$ with $\alpha, \beta \in Y$,

$$\begin{aligned} f(b^{\mathrm{op}} \cdot (a^{\mathrm{op}} \otimes h^{\mathrm{op}})) &= f((b^{\mathrm{op}} \otimes 1^{\mathrm{op}}_{H_\beta})(a^{\mathrm{op}} \otimes h^{\mathrm{op}})) \\ &= f(\sum_{(h)} b^{\mathrm{op}} a^{\mathrm{op}} \mu(H^{op'})^{op} \mu^{-1}(1^{\mathrm{op}}_{H_\beta} h^{\mathrm{op}''})^{\mathrm{op}} \otimes 1^{\mathrm{op}}_{H_\beta} h^{\mathrm{op}'''}) \\ &= \sum_{(h)} b^{\mathrm{op}} a^{\mathrm{op}} \mu(h^{\mathrm{op}'})^{\mathrm{op}} \mu^{-1}(1^{\mathrm{op}}_{H_\beta} h^{\mathrm{op}''})^{\mathrm{op}} \mu(1^{\mathrm{op}}_{H_\beta} h^{\mathrm{op}'''})^{\mathrm{op}} \\ &= b^{\mathrm{op}} a^{\mathrm{op}} \mu(h^{\mathrm{op}}) 1_{B_{\alpha\beta}} = b^{\mathrm{op}} a^{\mathrm{op}} \mu(h^{\mathrm{op}}) = (\gamma^{-1}(\overline{T}(h))ab)^{\mathrm{op}} \end{aligned}$$

and

$$b^{\mathrm{op}} \cdot f(a^{\mathrm{op}} \otimes h^{\mathrm{op}}) = b^{\mathrm{op}} a^{\mathrm{op}} \mu(h)^{\mathrm{op}}.$$

Hence, f is a left A^{op}-module map; therefore, the map $\alpha : \bigoplus_{\alpha \in Y} (H_\alpha \otimes A_\alpha) \to B = \bigoplus_{\alpha \in Y} B_\alpha$, given by $h \otimes a \mapsto \gamma^{-1}(\overline{T}(h))a$, is an isomorphism of right A-module. □

Next, we consider the G-crossed product in the special case when σ is trivial (that is, we have a smash product). $A\#H = \bigoplus_{\alpha,\beta \in Y} A_\alpha \# H_\beta$ is an associative algebra with identity $1\#1$. For any $a\#h \in A_\alpha \# H_\beta$ with $\alpha, \beta \in Y$, we have

$$(1\#1)(a\#h) = 1(1 \cdot a)\#1h = a\#h$$

$$(a\#h)(1\#1) = \sum_{(h)} a(h' \cdot 1)\#h''1 = \sum_{(h)} a\varepsilon(h')1\#h'' = a\#h.$$

Moreover, we actually have

$$A\#H = \bigoplus_{\alpha,\beta \in Y} A_\alpha \# H_\beta \cong \bigoplus_{\alpha,\beta \in Y} (1\#H_\beta)(A_\alpha\#1) \quad \text{via } a\#h \mapsto (1\#h)(a\#1) = \sum_{(h)} (h' \cdot a)\#h''.$$

Hence, the corollary for a smash product is very easy as then $\gamma^{-1}(h) = 1_{A_\alpha} \# T(h)$, and therefore

$$\alpha : \bigoplus_{\alpha \in Y} (H_\alpha \otimes A_\alpha) \to B = \bigoplus_{\alpha \in Y} B_\alpha \text{ via } h \otimes a \mapsto (1_{A_\alpha}\#h)(a\#1_{H_\alpha}), \quad h \otimes a \in A_\alpha \otimes H_\alpha$$

is an A-module isomorphism with inverse

$$\beta : B \to \bigoplus_{\alpha \in Y} (H_\alpha \otimes A_\alpha) \text{ via } a\#h \mapsto \sum_{(h)} h'' \otimes (\overline{T}(h') \cdot a), \quad a\#h \in A_\alpha \# H_\alpha$$

because

$$\begin{aligned} (\alpha \circ \beta)(a\#h) &= \alpha(\sum_{(h)} h'' \otimes (\overline{T}(h') \cdot a)) = \sum_{(h)} (1_{A_\alpha}\#h'')(\overline{T}(h')a\#1_{H_\alpha}) \\ &= \sum_{(h)} 1_{A_\alpha}(h'' \cdot (\overline{T}(h') \cdot a))\#h'''1_{H_\alpha} = \sum_{(h)} 1_{A_\alpha}(h''\overline{T}(h')) \cdot a\#h''' \\ &= \sum_{(h)} 1_{A_\alpha}(\varepsilon(h')1_{H_\alpha} \cdot a)\#h'' = 1_{A_\alpha} a\#\varepsilon(h')h'' = a\#h \end{aligned}$$

and

$$(\beta \circ \alpha)(h \otimes a) = \beta((1_{A_\alpha}\#h)(a\#1_{H_\alpha})) = \beta(\sum_{(h)} 1_{A_\alpha}(h' \cdot a)\#h''1_{H_\alpha})$$

$$= \beta(\sum_{(h)} (h' \cdot a) \# h'' 1_{H_\alpha}) = \sum_{(h)} h''' \otimes \overline{T}(h'') \cdot (h' \cdot a)$$
$$= \sum_{(h)} h''' \otimes (\overline{T}(h'')h') \cdot a) = \sum_{(h)} h'' \otimes \varepsilon(h') 1_{H_\alpha} \cdot a$$
$$= \sum_{(h)} \varepsilon(h')h'' \otimes a = h \otimes a;$$

that is; $\alpha \circ \beta = \mathrm{id}$ and $\beta \circ \alpha = \mathrm{id}$. □

Obviously, we can get the following result:

Corollary 3.6 *Let $kS = \bigoplus_{\alpha \in Y} kG_\alpha$ be a Clifford monoid, $H = \bigoplus_{\alpha \in Y} H_\alpha$ a semilattice graded weak Hopf algebra and $\bigoplus_{\alpha \in Y}(kG_\alpha \#_\sigma H_\alpha)$ a G-crossed product. Then $\bigoplus_{\alpha \in Y}(kG_\alpha \#_\sigma H_\alpha) \cong \bigoplus_{\alpha \in Y}(H_\alpha \otimes kG_\alpha)$ as right kS-modules, provided that the weak antipode T of H is bijective.*

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