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On Global Dimensions of A_n-Type Finite Dimensional Algebras

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Abstract A technique is provided to explicitly describe global dimensions of all A_n-type finite dimensional k-algebras for k an algebraic closed field. All possible global dimensions of all A_n-type finite dimensional algebras are explicitly presented. In particular, it is pointed out that the maximum is $n-1$, and the minimum is 1 for $n>1$.

Keywords global dimension; finite dimensional k-algebra; quiver; admissible ideal

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1. Introduction

Let k be an algebraically closed field, A a finite dimension k-algebra. All modules are finite dimensional left A-modules. The supremum of the projective dimension of all A-modules, or equivalently, of all simple A-modules, is called the global dimension of algebra A, and denoted by gl.dim.A (see [1]).

It is known that global dimensions of semi-simple algebras are zero, of hereditary algebras are one, of tilting algebras as well as quasi-tilted algebras are at most two, while those of self-injective algebras except semi-simple ones are infinite [2–5]. Some attractive issues of representation theory of algebras such as representation dimensions of finite dimensional algebras, the finitistic global dimension conjecture for Artin algebras are all connected with global dimensions of corresponding algebras [6–9]. So to find global dimensions of algebras of the particular type is worthwhile and interesting.

A finite dimension k-algebra A is called basic provided $A/\mathrm{rad}\, A$ is a product of copies of k. It is known that, given a finite dimensional k-algebra A, it is Morita equivalent to a basic algebra A', and A' is isomorphic to $k\Delta/\langle\rho\rangle$ for some finite quiver Δ and an admissible ideal $\langle\rho\rangle$ of $k\Delta$ (see [3–5]).

Therefore to know the global dimension of a given finite dimensional k-algebra is just to know the global dimension of some basic algebra by Morita equivalent theory, furthermore just to know the global dimension of some algebra $k\Delta/\langle\rho\rangle$ with Δ a quiver, $k\Delta$ the path algebra of Δ, $\langle\rho\rangle$ an admissible ideal of $k\Delta$. Thus using quiver methods to determine global dimensions of some particular algebras seems feasible and valuable.

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In this paper we aim to describe global dimensions of all A_n-type finite dimensional k-algebras inspired by the above observation. A quiver is called A_n-type if its underlying graph is of the form

$$1 \text{------} 2 \text{------} \cdots \text{------} n-1 \text{------} n.$$

Given an A_n-type quiver Λ with relations, first we need to classify these relations on Λ. By special structure of the quiver Λ, any relations on quiver Λ are of the form $\alpha_j\alpha_{j-1}\cdots\alpha_{i+1}\alpha_i$ with $i<j$ or $i>j$. Given an ideal $\langle\rho\rangle$ generated by relations ρ_i of Λ, we consider its minimal generators $\rho_i, i\in I$. That means each ρ_i cannot be generated by others ρ_j with $j\neq i, j\in I$. This can be done since Λ is a finite quiver without oriented cycles, and relations $\{\rho_i\}$ of Λ are finite sets.

Definition 1.1 *If two relations ρ_1 and ρ_2 on Λ have $\rho_1 = pp_1$, $\rho_2 = p_2p$ with p, p_1, p_2 non-trivial paths of Λ, we say that ρ_1 is successive to ρ_2. If ρ_1 is successive to ρ_2, we denote it by $\rho_1\sim\rho_2$. If $\rho_1\sim\rho_2$, we say ρ_i is successive with ρ_j each other for $i=1,2$.*

Definition 1.2 *A set T of relations $\{\rho_i|i\in I\}$ on Λ is named to be successive if there exists a well order on I such that any $\rho_i, i\in I$ is successive to its direct successor. A relation set T on Λ is called ultimately successive if T is a successive set on Λ which is not properly contained in other successive relation sets on Λ.*

Definition 1.3 *A successive set T of relations $\{\rho_i|i\in I\}$ on Λ is called perfectly successive if we can arrange subscripts of all elements of T as $1\prec 2\prec 3\prec\cdots m-1\prec m$, subject to $\rho_j\sim\rho_{j+1}$ for $j=1,\dots,m-1$, and if $\rho_j=\alpha_{(t_j,j)}\cdots\alpha_{(2,j)}\alpha_{(1,j)}$ for $j=1,2,\dots,m-2$, then there exists a non-trivial path starting from $\alpha_{(t_j,j)+1}$ such that it is contained in the ideal generated by T, where $\alpha_{(t_j,j)+1}$ is the directly successive arrow of $\alpha_{(t_j,j)}$ appearing in ρ_{j+1}. A relation set T on Λ is called ultimately perfectly successive if T is a perfectly successive set on Λ which is not properly contained in other perfectly successive relation sets on Λ.*

Lemma 1.4 *Minimal generators $\{\rho_i|i\in I\}$ of the ideal $\langle\rho\rangle$ consisting of relations on Λ can be divided into finitely ultimately successive subsets $R_1, R_2,\dots,R_m$ of $\{\rho_i|i\in I\}$ that are mutually disjoint.*

We can choose $\tilde{R}_i$ from R_i, one of longest ultimately perfectly successive subsets of R_i, for $i=1,2\dots,m$. Denote the cardinal number of the set $\tilde{R}_i$ by $|\tilde{R}_i|$.

Theorem 1.5 *If A is an A_n-type finite dimensional k-algebra that is Morita equivalent to $k\Lambda/\langle\rho\rangle$ with Λ being an A_n-type quiver and $\langle\rho\rangle$ an admissible ideal of $k\Lambda$. Then the global dimension of A is the maximal of $\{|\tilde{R}_i|+1, i\in I\}$ where $\tilde{R}_i, i\in I$ are all ultimately perfectly successive relation subsets of minimal generators $\{\rho_i|i\in I\}$ of the ideal $\langle\rho\rangle$ consisting of relations on Λ.*

Example 1.6 Let Λ be the quiver

$$1\xrightarrow{\alpha_1}2\xrightarrow{\alpha_2}3\xrightarrow{\alpha_3}4\xrightarrow{\alpha_4}5\xrightarrow{\alpha_5}6\,.$$

If $\langle\rho\rangle$ is an ideal of $k\Lambda$ generated by $\alpha_3\alpha_2\alpha_1, \alpha_4\alpha_3\alpha_2, \alpha_5\alpha_4\alpha_3$, then $T = \{\alpha_3\alpha_2\alpha_1, \alpha_4\alpha_3\alpha_2, \alpha_5\alpha_4\alpha_3\}$ is a successive set, but is not a perfectly successive set since there is no non-trivial path starting from α_4 that can be generated by T. $\tilde{T} = \{\alpha_3\alpha_2\alpha_1, \alpha_4\alpha_3\alpha_2\}$ is one of longest ultimately perfectly successive subsets of T, $|\tilde{T}| = 2$, so the global dimension of $k\Lambda/\langle\rho\rangle$ is $2 + 1 = 3$ by Theorem 1.5.

If $\langle\rho\rangle$ is an ideal of $k\Lambda$ generated by $\alpha_2\alpha_1, \alpha_3\alpha_2, \alpha_4\alpha_3$, then $T = \{\alpha_2\alpha_1, \alpha_3\alpha_2, \alpha_4\alpha_3\}$ is an ultimately perfectly successive set, $|T| = 3$, so the global dimension of $k\Lambda/\langle\rho\rangle$ is $3 + 1 = 4$ by Theorem 1.5.

Corollary 1.7 $\{1, 2, \ldots, n-1\}$ *are exactly all global dimensions of A_n-type finite dimensional k-algebras for $n > 1$. So the maximal global dimension of A_n-type finite dimension algebras is $n-1$, and the minimum is* 1.

2. Proof of the main theorem

A quiver $\Delta = (\Delta_0, \Delta_1, s, e)$ is given by two sets Δ_0, Δ_1 and two maps $s, e : \Delta_1 \to \Delta_0$; Δ_0, Δ_1 are respectively called the set of vertices and the sets of arrows of Δ, $s(\alpha)$ and $e(\alpha)$ are respectively called the head and the tail of $\alpha \in \Delta_1$. A path p in Δ of length l means a sequence of arrows $p = \alpha_l \cdots \alpha_1$ with $e(\alpha_i) = s(\alpha_{i+1})$ for $1 \le i \le l-1$. Set $s(p) = s(\alpha_1)$, $e(p) = e(\alpha_l)$ and $l(p) = l$, which are called the head, the tail and the length of p respectively. A vertex $i \in \Delta_0$ is regarded as a path of length 0 and is denoted by e_i. For any field k and any quiver Δ, let $k\Delta$ be the k-space with basis the set of all finite length paths in Δ. For any two paths $p = \alpha_m \cdots \alpha_1$ and $q = \beta_n \cdots \beta_1$ in Δ, define the multiplication

$$qp = \begin{cases} \beta_n \cdots \beta_1 \alpha_m \cdots \alpha_1, & \text{if } e(p) = s(q), \\ 0, & \text{otherwise.} \end{cases}$$

Then $k\Delta$ becomes a k-algebra, which is called the path algebra of Δ. In $k\Delta$, we denote by $k\Delta^+$ the ideal generated by all arrows. Note that $(k\Delta^+)^n$ is the ideal generated by all paths of length $\ge n$.

A relation σ on a quiver Δ over a field k is a k-linear combination of paths $\sigma = a_1p_1 + \cdots + a_np_n$ with $a_i \in k$ and $e(p_1) = \cdots = e(p_n)$ and $s(p_1) = \cdots = s(p_n)$. If $\rho = \{\sigma_t\}_{t\in T}$ is a set of relations on Δ over k, the pair (Δ, ρ) is called a quiver with relations over k. Associated with (Δ, ρ) is the k-algebra $k(\Delta, \rho) = k\Delta/\langle\rho\rangle$, where $\langle\rho\rangle$ denotes the ideal in $k\Delta$ generated by the set of relations ρ. An ideal $\langle\rho\rangle$ of $k\Delta$ generated by the set of relations ρ in $k\Delta$ with $(k\Delta^+)^n \subseteq \langle\rho\rangle \subseteq (k\Delta^+)^2$ for some $n \ge 2$ is called an admissible ideal of $k\Delta$.

Lemma 2.1 *Minimal generators $\{\rho_i | i \in I\}$ of the ideal $\langle\rho\rangle$ consisting of relations on Λ can be divided into finitely ultimately successive subsets $R_1, R_2, \ldots, R_m$ of $\{\rho_i | i \in I\}$ that are mutually disjoint.*

Proof The proof is given by induction on $|I|$. If non-elements in $\{\rho_i | i \in I\}$ are successive with ρ_1, we set $R_1 = \{\rho_1\}$. Otherwise we find all elements $\{\rho_{i1}, \ldots, \rho_{it_1}\}$ in $\{\rho_i | i \in I\}$ which are successive with ρ_1. Putting them together with ρ_1, we get a set $\{\rho_1, \rho_{i1}, \ldots, \rho_{it_1}\}$. If non-elements in $\{\rho_i|i\in I\}$ are successive with $\{\rho_1,\rho_{i1},\dots,\rho_{it_1}\}$, we set $R_1=\{\rho_1,\rho_{i1},\dots,\rho_{it_1}\}$. Otherwise we continue to find all elements in $\{\rho_i|i\in I\}$ which are successive with $\{\rho_1,\rho_{i1},\dots,\rho_{it_1}\}$, and these elements together with elements $\{\rho_1,\rho_{i1},\dots,\rho_{it_1}\}$ constitute a bigger set than $\{\rho_1,\rho_{i1},\dots,\rho_{it_1}\}$. Repeat former steps until we get a set R_1 whose elements are non-successive with elements in $\{\rho_i|i\in I\}/R_1$. Certainly R_1 is an ultimately successive set. Consider the set $\{\rho_i|i\in I\}/R_1$, its cardinal number is smaller than $|I|$. By induction on $|I|$ we can divide $\{\rho_i|i\in I\}/R_1$ into mutually non-successive subsets $R_2,\dots,R_m$ where each $R_i, i=2,\dots,m$ is ultimately successive. Therefore $\{\rho_i|i\in I\}$ is divided into mutually non-successive sets $R_1,R_2,\dots,R_m$ where each $R_i, i=1,2,\dots,m$ is ultimately successive. □

Denote by S_i the i-th simple module of algebra $k\Lambda/\langle\rho\rangle$ corresponding to the i-th vertex of the graph Λ. Denote by pro.dim.S_i the projective dimension of $k\Lambda/\langle\rho\rangle$-module S_i.

A representation (V,f) of a quiver Δ over a field k is a set of vector spaces $\{V_i|i\in\Delta_0\}$ together with k-linear maps $f_\alpha: V_i\to V_j$ for each arrow $\alpha: i\to j$. If $V=(V_i,f_\alpha)$ and $W=(W_i,g_\alpha)$ are two representations, a morphism $\psi=(\psi_1,\dots,\psi_n): V\to W$ is given by $\psi_i\in \mathrm{Hom}(V_i,W_i)$ such that $\psi_t(\alpha)f_\alpha=g_\alpha\psi_s(\alpha): V_s(\alpha)\to W_t(\alpha)$. This defines the category $\mathrm{rep}_k\Delta$ of representations of Δ. If $w=\alpha_l\cdots\alpha_1$ is a path in Δ, we may denote by V_w the composition $V_{\alpha_l}\cdots V_{\alpha_1}$. We say that V satisfies the relation $\rho=\sum_w c_w w$, provided $\sum_w c_w V_w=0$.

It is well known that the category of finite dimension representations of a quiver Δ over k satisfying relations $\langle\rho\rangle$ is equivalent to the category of finite dimensional $k\Delta/\langle\rho\rangle$-modules [3,4]. So we usually investigate modules through corresponding quiver representations.

To treat with general cases with the quiver Λ being

$$\cdots\xrightarrow{\alpha_{i_1}} i_1+1 \xleftarrow{\alpha_{i_1+1}} \cdots \xleftarrow{\alpha_{j_1}} j_1+1 \xrightarrow{\alpha_{j_1+1}} \cdots\ , \tag{$*$}$$

we first consider the special case with the quiver Γ being

$$1\xrightarrow{\alpha_1}2\xrightarrow{\alpha_2}3\longrightarrow\cdots\longrightarrow n-1\xrightarrow{\alpha_{n-1}}n. \tag{$**$}$$

Lemma 2.2 *Let $\{\rho_i|i\in I\}$ be a set of relations on Γ which are minimal generators of the ideal $\langle\rho\rangle$ in $k\Gamma$, $\rho_i=\alpha_{i_t}\cdots\alpha_{i_1+1}\alpha_{i_1}\in\{\rho_i|i\in I\}$. Then the projective $k\Gamma/\langle\rho\rangle$-module $P(i_1)$ is*

$$\underset{(1)}{0}\xrightarrow{0}\underset{(2)}{0}\xrightarrow{0}\cdots\underset{(i_1)}{k}\xrightarrow{1_{id}}\underset{(i_1+1)}{k}\xrightarrow{1_{id}}\cdots\underset{(i_t-1)}{k}\xrightarrow{0}\underset{(i_t)}{0}\xrightarrow{0}\cdots\xrightarrow{0}\underset{(n)}{0}$$

Proof Since $P(i_1)\cong k\Gamma e_{i_1}/\langle\rho\rangle$, and one basis of $k\Gamma e_{i_1}/\langle\rho\rangle$ is $\overline{e_{i_1}}$, $\overline{\alpha_{i_1}}$, $\overline{\alpha_{i_1+1}\alpha_{i_1}}$, …, $\overline{\alpha_{i_t-1}\cdots\alpha_{i_1+1}\alpha_{i_1}}$, the assertion follows. □

Lemma 2.3 *If α_i does not occur in any ρ_j of minimal generators $\{\rho_i|i\in I\}$ of an ideal $\langle\rho\rangle$ consisting of relations on Γ, then the projective dimension of $k\Gamma/\langle\rho\rangle$-simple module S_i is 1.*

Proof It is easy to see that $0\longrightarrow P_{i+1}\longrightarrow P_i\longrightarrow S_i\longrightarrow 0$ is a minimal projective presentation of S_i. So the projective dimension of $k\Gamma/\langle\rho\rangle$-simple module S_i is 1. □

Lemma 2.4 *If a perfectly successive relation set R_i in $k\Gamma$ consists of $\rho_{i_m}=\alpha_{i_mt}\cdots\alpha_{i_m+1}\alpha_{i_m}$, $\rho_{i_{m-1}}=\alpha_{i_{m-1}t}\cdots\alpha_{i_{m-1}+1}\alpha_{i_{m-1}},\dots,\rho_{i_1}=\alpha_{i_1t}\cdots\alpha_{i_1+1}\alpha_{i_1}$ with $i_m<i_{m-1}<\cdots<i_1$, then the projective dimension of S_{i_b} is $b+1$ for $b=1,2,\dots,m$, respectively. While the projective*

dimension of S_j is 1 when j belongs to other vertexes appearing in the relation set R_i except i_b for $b = 1, 2, \dots, m$.

Proof For S_{i_b} with $b = 1, 2, \dots, m$ we have a minimal projective presentation of S_{i_b}

$$0 \longrightarrow P_{i_1 t-1} \longrightarrow P_{i_1 t} \cdots \longrightarrow P_{i_{b-2} t} \longrightarrow P_{i_{b-1} t} \longrightarrow P_{i_b+1} \longrightarrow P_{i_b} \longrightarrow S_{i_b} \longrightarrow 0\cdot$$

For S_j with $j \neq i_b$ we have a minimal projective presentation of S_j

$$0 \longrightarrow P_{j+1} \longrightarrow P_j \longrightarrow S_j \longrightarrow 0\cdot$$

Therefore the projective dimension of S_{i_b} is $b+1$, the projective dimension of S_j is 1. □

Theorem 2.5 *If A is a special A_n-type finite dimensional k-algebra that is Morita equivalent to $k\Gamma/\langle\rho\rangle$ for the special A_n-type quiver Γ and an admissible ideal $\langle\rho\rangle$ of $k\Gamma$. Then the global dimension of A is the maximal of $\{|R_i| + 1, i \in I\}$ where $R_i, i \in I$ are all ultimately perfectly successive relation subsets of minimal generators $\{\rho_i | i \in I\}$ of the ideal $\langle\rho\rangle$ consisting of relations on Γ.*

Proof Since the global dimension of $k\Gamma/\langle\rho\rangle$ is max $\{\text{pro.dim.}S_i | i = 1, 2, \dots, n\}$, the theorem follows from Lemmas 2.3 and 2.4. □

Example 2.6 Let Γ be the following A_7-type quiver

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} 4 \xrightarrow{\alpha_4} 5 \xrightarrow{\alpha_5} 6 \xrightarrow{\alpha_6} 7$$

with the following relations on Γ: $\rho_1 = \alpha_2\alpha_1$, $\rho_2 = \alpha_4\alpha_3$, $\rho_3 = \alpha_6\alpha_5\alpha_4$. Let $\langle\rho\rangle$ be an admissible ideal of $k\Gamma$ generated by ρ_1, ρ_2 and ρ_3. Then the global dimension of $k\Gamma/\langle\rho\rangle$ is 3 by Theorem 2.5, since $R_1 = \{\rho_1\}$, $R_2 = \{\rho_2, \rho_3\}$.

Corollary 2.7 *The global dimension of $k\Gamma/\langle\rho\rangle$ is $n-1$ if and only if $\langle\rho\rangle$ is generated by $\alpha_2\alpha_1, \alpha_3\alpha_2, \dots, \alpha_{n-1}\alpha_{n-2}$.*

Proof On one hand, if $\langle\rho\rangle$ is generated by $\alpha_2\alpha_1, \alpha_3\alpha_2, \dots, \alpha_{n-1}\alpha_{n-2}$, then certainly $\alpha_2\alpha_1$, $\alpha_3\alpha_2$, $\dots$, $\alpha_{n-1}\alpha_{n-2}$ constitutes only one perfectly successive set of $\langle\rho\rangle$, its cardinal number is $n-2$, so by Theorem 2.5, the global dimension of $k\Gamma/\langle\rho\rangle$ is $n-1$.

On the other hand, if the global dimension of $k\Gamma/\langle\rho\rangle$ is $n-1$, then by Theorem 2.5 there is an ultimately perfectly successive set of $\langle\rho\rangle$ the cardinal number of which is $n-2$. In this case, $\langle\rho\rangle$ has to be generated by $\alpha_2\alpha_1, \alpha_3\alpha_2, \dots, \alpha_{n-1}\alpha_{n-2}$. □

We treat with general cases in the following theorem.

Theorem 2.8 *If A is an A_n-type finite dimensional k-algebra that is Morita equivalent to $k\Lambda/\langle\rho\rangle$ for an A_n-type quiver Λ and an admissible ideal $\langle\rho\rangle$ of $k\Lambda$. Then the global dimension of A is the maximal of $\{|R_i| + 1, i \in I\}$ where $R_i, i \in I$ are all ultimately perfectly successive relation subsets of minimal generators $\{\rho_i | i \in I\}$ of the ideal $\langle\rho\rangle$ consisting of relations on Λ.*

Proof The special vertexes of the quiver Λ which are different from the quiver Γ's are those like i_1+1 and j_1+1 by comparing $(*)$ and $(**)$. Since i_1+1 is a sink vertex, S_{i_1+1} is clearly a projective $k\Lambda/\langle\rho\rangle$-module. So we only need to consider j_1+1. Since j_1+1 is a source vertex, the relations where j_1+1 is involved can be divided into two successive sets. One is $\rho_{j_1}=\alpha_{j_1t}\cdots\alpha_{j_1-1}\alpha_{j_1}$, $\rho_{j_2}=\alpha_{j_2t}\cdots\alpha_{j_2-1}\alpha_{j_2}$, …, $\rho_{j_m}=\alpha_{j_mt}\cdots\alpha_{j_m-1}\alpha_{j_m}$ with $j_m<j_{m-1}<\cdots<j_1$. The other is $\rho_{l_1}=\alpha_{l_1t}\cdots\alpha_{j_1+2}\alpha_{j_1+1}$, $\rho_{l_2}=\alpha_{l_2t}\cdots\alpha_{l_2+1}\alpha_{l_2},\ldots,\rho_{l_u}=\alpha_{l_ut}\cdots\alpha_{l_u+1}\alpha_{l_u}$ with $j_1+1<l_2<\cdots<l_u$. So a minimal projective presentation of S_{j_1+1} can be expressed in several cases. One is

$$0\longrightarrow P_{d-1}\longrightarrow P_d\ \cdots\longrightarrow P_{j_2t}\oplus P_{l_2t}\longrightarrow P_{j_1}\oplus P_{j_1+2}\longrightarrow P_{j_1+1}\longrightarrow S_{j_1+1}\longrightarrow 0$$

where $d=j_{mt}$ if $m>u\geq 1$, $d=l_{ut}$ if $u>m\geq 1$.

If $m=u$, then a minimal projective presentation of S_{j_1+1} can be expressed as

$$0\longrightarrow P_{j_{mt}-1}\oplus P_{l_{ut}+1}\longrightarrow P_{j_{mt}}\oplus P_{l_{ut}}\ \cdots$$
$$\cdots\longrightarrow P_{j_2t}\oplus P_{l_2t}\longrightarrow P_{j_1}\oplus P_{j_1+2}\longrightarrow P_{j_1+1}\longrightarrow S_{j_1+1}\longrightarrow 0$$

So we draw a conclusion that the projective dimension of S_{j_1+1} is $\max\{m+1,u+1\}$. We can treat with other vertexes except i_1+1s and j_1+1s similarly as Theorem 2.5. This completes the proof. □

Corollary 2.9 $\{1,2,\ldots,n-1\}$ *are exactly all global dimensions of all A_n-type finite dimensional k-algebras for $n>1$. So the maximal global dimension of all A_n-type finite dimension algebras is $n-1$, and the minimum one is* 1.

Proof Since $\{0,1,\ldots,n-2\}$ are all possible lengths of perfectly successive subsets in A_n-type quivers, and it is easy to see that any of these lengths can actually occur in an A_n-type quiver, the assertion follows from Theorem 2.8. □

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References

[1] C. A. WEIBEL. *An Introduction to Homological Algebra.* Cambridge University Press, Cambridge, 2004.
[2] D. HAPPEL, C. M. RINGEL. *Tilted algebras.* Trans. Amer. Math.Soc., 1982, **274**: 399–443.
[3] I. ASSEM, D. SIMSON, A. SKOWRONSKI. *Elements of the Representation Theory of Associative Algebras.* Vol.1. Cambridge University Press, Cambridge, 2006.
[4] M. AUSLANDER, I. REITEN, S. SMALϕ. *Representation Theory of Artin Algebras.* Cambridge University Press, Cambridge, 1995.
[5] C. M. RINGEL. *Tame Algebras and Integral Quadratic Forms.* Springer-Verlag, Berlin, 1984.
[6] M. AUSLANDER. *"Representation Dimension of Artin Algebras", Queen Mary College Mathematics Notes.* Queen Mary College, London, 1971.
[7] K. IGUSA, G. TODOROV. *On the finistic global dimension conjecture for Artin algebra.* Amer. Math. Soc., 2005, 201–204,.
[8] R. ROUQUIER. *Representation dimension of exterior algebras.* Invent. Math., 2006, **165**(2): 357–367.
[9] Changchang XI. *On the representation dimension of finite dimensional algebras.* J. Algebra, 2000, **226**(1): 332–346.