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The Growth of Solutions of Higher Order Differential Equations with Coefficients Having the Same Order

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Abstract In this paper, we consider the growth of solutions of some homogeneous and non-homogeneous higher order differential equations. It is proved that under some conditions for entire functions F, A_{ji} and polynomials $P_j(z), Q_j(z)$ $(j = 0, 1, \dots, k-1; i = 1, 2)$ with degree $n \geq 1$, the equation $f^{(k)} + (A_{k-1,1}(z)e^{P_{k-1}(z)} + A_{k-1,2}(z)e^{Q_{k-1}(z)})f^{(k-1)} + \cdots + (A_{0,1}(z)e^{P_0(z)} + A_{0,2}(z)e^{Q_0(z)})f = F$, where $k \geq 2$, satisfies the properties: When $F \equiv 0$, all the non-zero solutions are of infinite order; when $F \not\equiv 0$, there exists at most one exceptional solution f_0 with finite order, and all other solutions satisfy $\overline{\lambda}(f) = \lambda(f) = \sigma(f) = \infty$.

Keywords order of growth; hyper-order; exponent of convergence of zero sequence; differential equation

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1. Introduction

It is supposed that the reader is familiar with the standard fundamental results and the notations of the Nevanlinna theory and the Wiman-Valiron theory [1,2]. We use $\lambda(f)$ and $\overline{\lambda}(f)$ to denote the exponent of convergence of the zero-sequence and the distinct zero-sequence of f, $\sigma(f)$ and $\sigma_2(f)$ to denote the order of growth and the hyper-order of f, where $f(z)$ is a nonconstant meromorphic function in the complex plane. We define

$$\lambda_2(f) = \overline{\lim_{r\to\infty}} \frac{\log\log N(r, 1/f)}{\log r}, \quad \overline{\lambda}_2(f) = \overline{\lim_{r\to\infty}} \frac{\log\log \overline{N}(r, 1/f)}{\log r}.$$

Consider the differential equation

$$f^{(k)} + A_{k-1}(z)e^{a_{k-1}z}f^{(k-1)} + \cdots + A_0(z)e^{a_0 z}f = 0. \tag{1.1}$$

Chen [3], Li and Huang [4], Chen and Shon [5] studied the growth of solution when the angles of principal value of a_j are equal and not all equal. A similar problem is considered in [6]. The following result is proved in [4].

Theorem 1.1 ([4]) *Suppose that* $A_j(z)(\not\equiv 0)$ *are entire functions with* $\sigma(A_j) < 1, j = 0, 1, \dots, k-$

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$1 (k \geq 2), a_j$ are nonzero complex numbers such that $\arg a_0 \neq \arg a_1, a_j = \alpha_j a_1, \alpha_j > 0$ $(j = 2, \ldots, k-1)$. Then every solution $f (\not\equiv 0)$ of equation (1.1) satisfies $\sigma(f) = \infty$.

In this paper, we consider the homogeneous and nonhomogeneous higher order differential equations and obtain the following results.

Theorem 1.2 *Suppose that $A_{ji} (\not\equiv 0)$ are entire functions with $\sigma(A_{ji}) < n$ (where $n \geq 1$ is a positive integer), $j = 0, 1, \ldots, k-1; i = 1, 2$. Let $P_j(z) = a_{jn}z^n + \cdots + a_{j0}$ and $Q_j(z) = b_{jn}z^n + \cdots + b_{j0}$ be polynomials, where a_{jq}, b_{jq} $(j = 0, 1, \ldots, k-1; q = 0, 1, \ldots, n)$ are complex numbers such that $a_{jn}b_{jn} \neq 0, a_{0n} \neq b_{0n}$ and $a_{jn} = c_j a_{0n}, b_{jn} = c_j b_{0n}, c_j > 1, j = 1, 2, \ldots, k-1$ are distinct numbers. Then every solution $f (\not\equiv 0)$ of the equation*

$$\begin{aligned} f^{(k)} + &(A_{k-1,1}(z)e^{P_{k-1}(z)} + A_{k-1,2}(z)e^{Q_{k-1}(z)})f^{(k-1)} + \cdots + \\ &(A_{0,1}(z)e^{P_0(z)} + A_{0,2}(z)e^{Q_0(z)})f = 0, \end{aligned} \tag{1.2}$$

is of infinite order.

Theorem 1.3 *Let $A_{ji} (\not\equiv 0)$ be entire functions with $\sigma(A_{ji}) < n$ (where $n \geq 1$ is a positive integer), $j = 0, 1, \ldots, k-1; i = 1, 2$. Let $P_j(z) = a_{jn}z^n + \cdots + a_{j0}$ and $Q_j(z) = b_{jn}z^n + \cdots + b_{j0}$ be polynomials, where a_{jq}, b_{jq} $(j = 0, 1, \ldots, k-1; q = 0, 1, \ldots, n)$ are complex numbers such that $a_{jn}b_{jn} \neq 0, a_{0n} \neq b_{0n}$ and $a_{jn} = c_j a_{0n}, b_{jn} = c_j b_{0n}, c_j > 1, j = 1, 2, \ldots, k-1$ are distinct numbers. $F (\not\equiv 0)$ is an entire function with finite order. Then the following equation satisfies*

$$\begin{aligned} f^{(k)} + &(A_{k-1,1}(z)e^{P_{k-1}(z)} + A_{k-1,2}(z)e^{Q_{k-1}(z)})f^{(k-1)} + \cdots + \\ &(A_{0,1}(z)e^{P_0(z)} + A_{0,2}(z)e^{Q_0(z)})f = F, \end{aligned} \tag{1.3}$$

(i) There exists at most one exceptional solution f_0 with finite order, and all other solutions satisfy $\overline{\lambda}(f) = \lambda(f) = \sigma(f) = \infty$ and $\overline{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) \leq \max\{n, \sigma(F)\}$.

(ii) If there exists an f_0 with finite order, then $\sigma(f_0) \leq \max\{n, \overline{\lambda}(f_0), \sigma(F)\}$.

(iii) If $F(z)$ is an entire function of order less than n and $\arg a_{0n} \neq \arg b_{0n}$, then every solution of (1.3) is of infinite order.

2. Lemmas

Lemma 2.1 ([7]) *Suppose that $f(z)$ is an entire function and $|f^{(k)}(z)|$ is unbounded on some ray $\arg z = \theta$. Then there exists an infinite sequence of points $z_n = r_n e^{i\theta}$ $(n = 1, 2, \ldots)$, where $r_n \to \infty$, such that $f^{(k)}(z_n) \to \infty$ and*

$$\Big|\frac{f^{(j)}(z_n)}{f^{(k)}(z_n)}\Big| \leq |z_n|^{k-j}(1 + o(1)), \quad j = 0, 1, \ldots, k-1.$$

Lemma 2.2 ([8]) *Suppose that $f(z)$ is a transcendental meromorphic function with $\sigma(f) = \sigma < \infty$. Let $\varepsilon > 0$ be a given constant, $\Gamma = \{(k_1, j_1), (k_2, j_2), \ldots, (k_n, j_n)\}$ be a finite set of distinct pairs of integers satisfying $k_i > j_i \geq 0, i = 1, \ldots, n$. Then there is a set $E_1 \subset [0, 2\pi)$, with linear measure zero, such that if $\theta_0 \in [0, 2\pi) \backslash E_1$, then there exists $R_0 = R_0(\theta_0) > 0$ such that for all z*

satisfying $\arg z = \theta_0$ *and* $|z| \geq R_0$ *and for all* $(k, j) \in \Gamma$, *we have*

$$|\frac{f^{(k)}(z)}{f^{(j)}(z)}| \leq |z|^{(k-j)(\sigma(f)-1+\varepsilon)}.$$

Lemma 2.3 ([3,9]) *Let* $B_0, B_1, \dots, B_{k-1}, F$ *be entire functions of finite order and set* $\sigma = \max\{\sigma(B_0), \dots, \sigma(B_{k-1}), \sigma(F)\}, k \geq 2$. *If* $f(z)$ *is a solution of the differential equation*

$$f^{(k)} + B_{k-1}f^{(k-1)} + \cdots + B_0 f = F.$$

Then $\sigma_2(f) \leq \sigma$.

Lemma 2.4 ([10]) *Let* $P(z) = (\alpha + i\beta)z^n + \cdots$ $(\alpha, \beta \in R, |\alpha| + |\beta| \neq 0)$ *be a polynomial with* $\deg P = n \geq 1$. *Suppose that* $A(z)(\not\equiv 0)$ *is a meromorphic function with* $\sigma(A) < n$. *Set* $g(z) = A(z)e^{P(z)}, z = re^{i\theta}, \delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. *Then for any given* $\varepsilon > 0$, *there is a set* $H_1 \subset [0, 2\pi)$ *with linear measure zero, such that for any* $\theta \in [0, 2\pi)\backslash(H_1 \bigcup H_2)$, *where* $H_2 = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\}$ *is a finite set, there exists a constant* $R > 1$ *such that for* $|z| = r > R$, *the following statements hold,*

(i) if $\delta(P, \theta) > 0$, *then* $\exp\{(1-\varepsilon)\delta(P, \theta)r^n\} \leq |g(re^{i\theta})| \leq \exp\{(1+\varepsilon)\delta(P, \theta)r^n\}$;

(ii) if $\delta(P, \theta) < 0$, *then* $\exp\{(1+\varepsilon)\delta(P, \theta)r^n\} \leq |g(re^{i\theta})| \leq \exp\{(1-\varepsilon)\delta(P, \theta)r^n\}$.

Lemma 2.5 ([11]) *Suppose that* $f(z)$ *is analytic in the region* $D = \{z : \alpha < \arg z < \beta, r_0 < |z| < \infty\}$, *and is continuous on* $\overline{D} = D \cup C$ *(*C *is the boundary of* D*). If for any given an* $\varepsilon > 0$, *there is* $r_1(\varepsilon) > 0$, *such that* $|f(z)| < \exp\{\varepsilon|z|^{\frac{\pi}{\beta-\alpha}}\}$, *for* $|z| \geq r_1(\varepsilon), z \in D$, *and for* $z \in C$ *we have* $|f(z)| \leq M$. *Then* $|f(z)| \leq M$ *holds for all* $z \in D$.

Lemma 2.6 ([3]) *Let* $f(z)$ *be an entire function with* $\sigma(f) = \sigma < \infty$. *If there exists a set* $E \subset [0, 2\pi)$ *with linear measure zero, such that for any ray* $\arg z = \theta_0 \in [0, 2\pi)\backslash E, |f(re^{i\theta_0})| \leq Mr^k$ *(*$k > 0$ *is a constant independent of* $\theta_0, M = M\ (\theta_0) > 0$ *is a constant). Then* $f(z)$ *is a polynomial with* $\deg f \leq k$.

Lemma 2.7 ([8]) *Suppose that* $f(z)$ *is a transcendental meromorphic function. Let* $\alpha > 1$ *be a given constant. Then there is a set* $E_2 \subset (1, \infty)$ *with finite logarithmic measure and a constant* $C = C(\alpha) > 0$, *such that for any* $|z| = r \notin [0, 1] \cup E_2$, *we have*

$$|\frac{f^{(j)}(z)}{f(z)}| \leq C(\frac{T(\alpha r, f)}{r} \log^\alpha r \log T(\alpha r, f))^k, \quad j = 1, \dots, k.$$

Lemma 2.8 ([3]) *Suppose that* $f(z)$ *is a transcendental entire function. Then there exits a set* $E \subset (1, +\infty)$ *with finite logarithmic measure such that when we select a point* z *satisfying* $|z| = r \notin [0, 1] \cup E$ *and* $|f(z)| = M(r, f)$, *we have* $|\frac{f(z)}{f^{(k)}(z)}| \leq 2r^k$.

Lemma 2.9 ([12]) *Suppsose that* $f_1(z), f_2(z), \dots, f_n(z)$ $(n \geq 2)$ *are meromorphic functions and* $g_1(z), g_2(z), \dots, g_n(z)$ *are entire functions satisfying the following conditions:*

(i) $\sum_{j=1}^n f_j(z)e^{g_j(z)} \equiv 0$,

(ii) $g_j(z) - g_k(z)$ *are not constants for* $1 \leq j < k \leq n$,

(iii) for $1 \leq j \leq n, 1 \leq h < k \leq n$, $T(r, f_j) = o\{T(r, e^{g_h - g_k})\}, \quad r \to \infty, r \notin E$,

where E is a set with finite linear measure. Then $f_j \equiv 0$ ($j = 1, 2, \ldots, n$).

Lemma 2.10 ([12]) *Let $f_1(z), f_2(z), \ldots, f_n(z)$ ($n \geq 2$) be linearly independent meromorphic functions which satisfies the following condition:*

$$\sum_{j=1}^{n} f_j \equiv 1.$$

Then for $1 \leq j \leq n$, we have

$$T(r, f_j) \leq \sum_{k=1}^{n} N(r, \frac{1}{f_k}) + N(r, f_j) + N(r, D) - \sum_{k=1}^{n} N(r, f_k) - N(r, \frac{1}{D}) + S(r),$$

where D is the Wronskian determinant $W(f_1, f_2, \ldots, f_n)$,

$$S(r) = o(\max_{1 \leq k \leq n} \{T(r, f_k)\}), \quad r \to \infty, r \notin E,$$

E is a set with finite linear measure.

Lemma 2.11 ([13]) *Suppose that $f(z)$ is an entire function and suppose further that*

$$G(z) = \frac{\log^+ |f^{(k)}(z)|}{|z|^\rho}$$

is unbounded on some ray $\arg z = \theta$ with constant $\rho > 0$. Then there is an infinite sequence of points $z_n = r_n e^{i\theta}$ ($n = 1, 2, \ldots$), where $r_n \to \infty$, such that $G(z_n) \to \infty$ and

$$|\frac{f^{(j)}(z_n)}{f^{(k)}(z_n)}| \leq \frac{1}{(k-j)!}(1 + o(1)) r_n^{k-j}, \quad j = 0, 1, \ldots, k-1.$$

Lemma 2.12 ([13]) *Suppose that $f(z)$ is an entire function with $\sigma(f) = \sigma < \infty$. There is a set $E \subset [0, 2\pi)$ with linear measure zero, such that $\log^+ |f(re^{i\theta})| \leq Mr^\rho$ for any ray $\arg z = \theta \in [0, 2\pi) \backslash E$, where M is a positive constant depending on θ, and ρ is a positive constant independent of θ. Then $\sigma(f) \leq \rho$.*

3. Proof of Theorem 1.2

Suppose that $f(z)$ is a transcendental solution of (1.2) with $\sigma(f) = \sigma < \infty$. By Lemma 2.2, for any given $\varepsilon > 0$, there is a set $E_1 \subset [0, 2\pi)$ which has linear measure zero, such that if $\theta \in [0, 2\pi) \backslash E_1$, then there is a constant $R_0 = R_0(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| \geq R_0$, we have

$$|\frac{f^{(j)}(z)}{f^{(i)}(z)}| \leq |z|^{k\sigma}, \quad 0 \leq i < j \leq k. \tag{3.1}$$

Set $z = re^{i\theta}, a_{0n} = |a_{0n}| e^{i\theta_1}, b_{0n} = |b_{0n}| e^{i\theta_2}, \theta_1, \theta_2 \in [0, 2\pi)$. Then

$$\delta(P_0, \theta) = |a_{0n}| \cos(n\theta + \theta_1), \quad \delta(Q_0, \theta) = |b_{0n}| \cos(n\theta + \theta_2). \tag{3.2}$$

Since $a_{jn} = c_j a_{0n}, b_{jn} = c_j b_{0n}, c_j > 1, j = 1, 2, \ldots, k-1$, and c_j are distinct numbers. Hence

$$\delta(P_j, \theta) = c_j \delta(P_0, \theta), \quad \delta(Q_j, \theta) = c_j \delta(Q_0, \theta), \tag{3.3}$$

and there exists exactly one c_s such that $c_s = \max\{c_j; j = 0, 1, 2, \ldots, k-1\}$. Set $c_0 = 1$. We divide our proof into two cases: $\theta_1 = \theta_2$ and $\theta_1 \neq \theta_2$.

Case 1 When $\theta_1 = \theta_2$, because of $a_{0n} \neq b_{0n}$, we suppose $|a_{0n}| < |b_{0n}|$ without loss of generality. By Lemma 2.4, for any given $\varepsilon > 0$, there exists a set $H_1 \subset [0, 2\pi)$, which has linear measure zero, such that for any $\theta \in [0, 2\pi)\backslash(H_1 \bigcup H_2)(H_2 = \{\theta \in [0, 2\pi) : \delta(P_0, \theta) = 0, \delta(Q_0, \theta) = 0\})$, there is a constant $R_1 > 1$ such that for all $z = r > R_1$, we have

(i) if $\delta(P_0, \theta) > 0$, by (3.3), one has $\delta(Q_j, \theta) > \delta(Q_0, \theta) > 0, \delta(Q_j, \theta) > \delta(P_j, \theta) > \delta(P_0, \theta) > 0$. Thus by Lemma 2.4, we obtain

$$\begin{aligned}|A_{s,1}e^{P_s(z)} + A_{s,2}e^{Q_s(z)}| &\geq |A_{s,2}e^{Q_s(z)}| - |A_{s,1}e^{P_s(z)}| \\ &\geq \exp\{(1-\varepsilon)c_s\delta(Q_0, \theta)r^n\} - \exp\{(1+\varepsilon)c_s\delta(P_0, \theta)r^n\} \\ &\geq \frac{1}{2}\exp\{(1-\varepsilon)c_s\delta(Q_0, \theta)r^n\}, \qquad (3.4)\\ |A_{j,1}e^{P_j(z)} + A_{j,2}e^{Q_j(z)}| &\leq \Big|A_{j,1}e^{P_j(z)}| + |A_{j,2}e^{Q_j(z)}| \\ &\leq \exp\{(1+\varepsilon)c_j\delta(P_0, \theta)r^n\} + \exp\{(1+\varepsilon)c_j\delta(Q_0, \theta)r^n\} \\ &\leq 2\exp\{(1+\varepsilon)c_j\delta(Q_0, \theta)r^n\}, \quad j = 0, \ldots, k-1, j \neq s. \qquad (3.5)\end{aligned}$$

If $|f^{(s)}(re^{i\theta})|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 1.1, there exists an infinite sequence of points $z_m = r_m e^{i\theta}$ $(m = 1, 2, \ldots)$ such that $r_m \to \infty, f^{(s)}(z_m) \to \infty$ and

$$|\frac{f^{(j)}(z_m)}{f^{(s)}(z_m)}| \leq |z_m|^{s-j}(1 + o(1)), \quad j = 0, 1, \ldots, s-1. \qquad (3.6)$$

From (1.2), we have

$$\begin{aligned}|A_{s,1}e^{P_s(z)} + A_{s,2}e^{Q_s(z)}| \leq &|\frac{f^{(k)}}{f^{(s)}}| + \cdots + |A_{s+1,1}e^{P_{s+1}(z)} + A_{s+1,2}e^{Q_{s+1}(z)}||\frac{f^{(s+1)}}{f^{(s)}}| + \\ &|A_{s-1,1}e^{P_{s-1}(z)} + A_{s-1,2}e^{Q_{s-1}(z)}||\frac{f^{(s-1)}}{f^{(s)}}| + \cdots + \\ &|A_{0,1}e^{P_0(z)} + A_{0,2}e^{Q_0(z)}||\frac{f}{f^{(s)}}|. \qquad (3.7)\end{aligned}$$

Substituting (3.1), (3.4)–(3.6) into (3.7), we obtain

$$\begin{aligned}\frac{1}{2}\exp\{(1-\varepsilon)c_s\delta(Q_0, \theta)r_m^n\} \leq &r_m^{k\sigma} + \cdots + 2\exp\{(1+\varepsilon)c_{s+1}\delta(Q_0, \theta)r_m^n\}r_m^{k\sigma} + \\ &2\exp\{(1+\varepsilon)c_{s-1}\delta(Q_0, \theta)r_m^n\}(1+o(1))r_m + \cdots + \\ &2\exp\{(1+\varepsilon)c_0\delta(Q_0, \theta)r_m^n\}(1+o(1))r_m^s \\ \leq &\sum_{j\neq s} 4\exp\{(1+\varepsilon)c_j\delta(Q_0, \theta)r_m^n\}r_m^{k\sigma+s}. \qquad (3.8)\end{aligned}$$

Take $0 < \varepsilon < \frac{1}{2}\min\{\frac{c_s - c_j}{c_s + c_j}, j = 0, 1, \ldots, k-1, j \neq s\}$, then (3.8) is a contradiction. Hence $|f^{(s)}(re^{i\theta})| \leq M$ on the ray $\arg z = \theta$. Moreover, we obtain by s-fold iterated integration along the line segment $[0, z]$,

$$f(z) = f(0) + f'(0)z + \cdots + \frac{1}{(s-1)!}f^{(s-1)}(0)z^{s-1} + \int_0^z \cdots \int_0^z f^{(s)}(t)\mathrm{d}t \cdots \mathrm{d}t. \qquad (3.9)$$

Therefore, by an elementary inequality estimate, we get on the ray $\arg z = \theta$.

$$|f(z)| \le M|z|^s. \tag{3.10}$$

(ii) If $\delta(P_0, \theta) < 0$, by (3.2) and (3.3), one has $\delta(Q_j, \theta) < \delta(Q_0, \theta) < \delta(P_0, \theta) < 0, \delta(P_j, \theta) < \delta(P_0, \theta) < 0$. Therefore by Lemma 2.4, we get

$$\begin{aligned} |A_{j,1}e^{P_j(z)} + A_{j,2}e^{Q_j(z)}| &\le |A_{j,1}e^{P_j(z)}| + |A_{j,2}e^{Q_j(z)}| \\ &\le \exp\{(1-\varepsilon)\delta(P_j, \theta)r^n\} + \exp\{(1-\varepsilon)\delta(Q_j, \theta)r^n\} \\ &\le 2\exp\{(1-\varepsilon)\delta(P_0, \theta)r^n\}, \quad j = 0, 1, \ldots, k-1. \end{aligned} \tag{3.11}$$

By (1.2), we have

$$-1 = (A_{k-1,1}e^{P_{k-1}(z)} + A_{k-1,2}e^{Q_{k-1}(z)})\frac{f^{(k-1)}}{f^{(k)}} + \cdots + (A_{0,1}e^{P_0(z)} + A_{0,2}e^{Q_0(z)})\frac{f}{f^{(k)}}. \tag{3.12}$$

If $|f^{(k)}(re^{i\theta})|$ is unbounded on the ray $\arg z = \theta$, then by Lemma 1.1, there exists an infinite sequence of points $z_m = r_m e^{i\theta}$ $(m = 1, 2, \ldots)$ such that $r_m \to \infty, f^{(k)}(z_m) \to \infty$ and

$$\Big|\frac{f^{(j)}(z_m)}{f^{(k)}(z_m)}\Big| \le |z_m|^{k-j}(1 + o(1)), \quad j = 0, 1, \ldots, k-1. \tag{3.13}$$

By (3.11)–(3.13), we have

$$1 \le 2\exp\{(1-\varepsilon)\delta(P_0, \theta)r_m^n\}r_m(1 + o(1)) + \cdots + 2\exp\{(1-\varepsilon)\delta(P_0, \theta)r_m^n\}r_m^k(1 + o(1)). \tag{3.14}$$

Comparing the growth of two sides, we know (3.14) is a contradiction. Hence $|f^{(k)}(re^{i\theta})| \le M$ on the ray $\arg z = \theta$. Moreover, we obtain by k-fold iterated integration along the line segment $[0, z]$,

$$f(z) = f(0) + f'(0)z + \cdots + \frac{1}{(k-1)!}f^{(k-1)}(0)z^{k-1} + \int_0^z \cdots \int_0^z f^{(k)}(t)\mathrm{d}t \cdots \mathrm{d}t. \tag{3.15}$$

Therefore, by an elementary inequality estimate, we get on the ray $\arg z = \theta$

$$|f(z)| \le M|z|^k. \tag{3.16}$$

Therefore, for any ray $\arg z = \theta \in [0, 2\pi)\backslash(H_1 \cup H_2)$, we have $|f(z)| \le M|z|^k$. By Lemma 2.6, we know that $f(z)$ is a polynomial which contradicts our assumption.

Case 2 When $\theta_1 \ne \theta_2$. By Lemma 2.4, for any given $\varepsilon > 0$, there exists a set $H_1 \subset [0, 2\pi)$, which has linear measure zero, such that for any $\theta \in [0, 2\pi)\backslash(H_1 \bigcup H_3)$, where $H_3 = \{\theta \in [0, 2\pi) : \delta(P_0, \theta) = 0, \delta(Q_0, \theta) = 0, \delta(P_0, \theta) = \delta(Q_0, \theta)\}$ has linear measure zero, there is a constant $R_1 > 1$ such that for all $z = r > R_1$, we have

(I) If $\delta(P_0, \theta) > 0, \delta(Q_0, \theta) < 0$. By (3.3), we get $\delta(P_j, \theta) > \delta(P_0, \theta) > 0, \delta(Q_j, \theta) < \delta(Q_0, \theta) < 0$. Thus by Lemma 2.4, we obtain

$$\begin{aligned} |A_{s,1}e^{P_s(z)} + A_{s,2}e^{Q_s(z)}| &\ge |A_{s,1}e^{P_s(z)}| - |A_{s,2}e^{Q_s(z)}| \\ &\ge \exp\{(1-\varepsilon)c_s\delta(P_0, \theta)r^n\} - \exp\{(1-\varepsilon)c_s\delta(Q_0, \theta)r^n\} \\ &\ge \frac{1}{2}\exp\{(1-\varepsilon)c_s\delta(P_0, \theta)r^n\}, \end{aligned} \tag{3.17}$$

$$\begin{aligned}|A_{j,1}e^{P_j(z)}+A_{j,2}e^{Q_j(z)}| &\le |A_{j,1}e^{P_j(z)}|+|A_{j,2}e^{Q_j(z)}|\\ &\le \exp\{(1+\varepsilon)c_j\delta(P_0,\theta)r^n\}+\exp\{(1-\varepsilon)c_j\delta(Q_0,\theta)r^n\}\\ &\le 2\exp\{(1+\varepsilon)c_j\delta(P_0,\theta)r^n\},\quad j=0,1,\dots,k-1, j\neq s. \qquad (3.18)\end{aligned}$$

If $|f^{(s)}(re^{i\theta})|$ is unbounded on the ray $\arg z=\theta$, then by Lemma 1.1, there exists an infinite sequence of points $z_m=r_me^{i\theta}$ $(m=1,2,\dots)$ such that $r_m\to\infty, f^{(s)}(z_m)\to\infty$ and

$$|\frac{f^{(j)}(z_m)}{f^{(s)}(z_m)}|\le |z_m|^{s-j}(1+o(1)),\quad j=0,1,\dots,s-1. \qquad (3.19)$$

By (3.1), (3.17)–(3.19), (3.7), we have

$$\begin{aligned}\frac{1}{2}\exp\{(1-\varepsilon)c_s\delta(P_0,\theta)r_m^n\} \le & r_m^{k\sigma}+\cdots+2\exp\{(1+\varepsilon)c_{s+1}\delta(P_0,\theta)r_m^n\}r_m^{k\sigma}+\\ & 2\exp\{(1+\varepsilon)c_{s-1}\delta(P_0,\theta)r_m^n\}(1+o(1))r_m+\cdots+\\ & 2\exp\{(1+\varepsilon)c_0\delta(P_0,\theta)r_m^n\}(1+o(1))r_m^s\\ \le & \sum_{j\neq s}4\exp\{(1+\varepsilon)c_j\delta(P_0,\theta)r_m^n\}r_m^{k\sigma+s}. \qquad (3.20)\end{aligned}$$

Take $0<\varepsilon<\frac{1}{2}\min\{\frac{c_s-c_j}{c_s+c_j}, j=0,1,\dots,k-1, j\neq s\}$, then (3.20) is a contradiction. Hence $|f^{(s)}(re^{i\theta})|\le M$ on the ray $\arg z=\theta$. Moreover, we obtain by s-fold iterated integration along the line segment $[0,z]$,

$$f(z)=f(0)+f'(0)z+\cdots+\frac{1}{(s-1)!}f^{(s-1)}(0)z^{s-1}+\int_0^z\cdots\int_0^z f^{(s)}(t)\mathrm{d}t\cdots\mathrm{d}t. \qquad (3.21)$$

Therefore, by an elementary inequality estimate, we get on the ray $\arg z=\theta$

$$|f(z)|\le M|z|^s. \qquad (3.22)$$

(II) If $\delta(P_0,\theta)<0, \delta(Q_0,\theta)>0$, by (3.3), we can get $\delta(Q_j,\theta)>\delta(Q_0,\theta)>0, \delta(P_j,\theta)<\delta(P_0,\theta)<0$. Using the similar proof as that of (I), we can obtain

$$\begin{aligned}\frac{1}{2}\exp\{(1-\varepsilon)c_s\delta(Q_0,\theta)r_m^n\} \le & r_m^{k\sigma}+\cdots+2\exp\{(1+\varepsilon)c_{s+1}\delta(Q_0,\theta)r_m^n\}r_m^{k\sigma}+\\ & 2\exp\{(1+\varepsilon)c_{s-1}\delta(Q_0,\theta)r_m^n\}(1+o(1))r_m+\cdots+\\ & 2\exp\{(1+\varepsilon)c_0\delta(Q_0,\theta)r_m^n\}(1+o(1))r_m^s\\ \le & \sum_{j\neq s}4\exp\{(1+\varepsilon)c_j\delta(Q_0,\theta)r_m^n\}r_m^{k\sigma+s}. \qquad (3.23)\end{aligned}$$

Take $0<\varepsilon<\frac{1}{2}\min\{\frac{c_s-c_j}{c_s+c_j}, j=0,1,\dots,k-1, j\neq s\}$, then (3.23) is a contradiction. Hence $|f^{(s)}(re^{i\theta})|\le M$ on the ray $\arg z=\theta$. Moreover, we obtain by s-fold iterated integration along the line segment $[0,z]$,

$$f(z)=f(0)+f'(0)z+\cdots+\frac{1}{(s-1)!}f^{(s-1)}(0)z^{s-1}+\int_0^z\cdots\int_0^z f^{(s)}(t)\mathrm{d}t\cdots\mathrm{d}t. \qquad (3.24)$$

Therefore, by an elementary inequality estimate, we get on the ray $\arg z=\theta$

$$|f(z)|\le M|z|^s. \qquad (3.25)$$

(III) If $\delta(P_0,\theta)>0, \delta(Q_0,\theta)>0$, by (3.3), one has $\delta(P_j,\theta)>\delta(P_0,\theta)>0, \delta(Q_j,\theta)>$

$\delta(Q_0,\theta)>0$. We suppose $\delta(P_0,\theta)>\delta(Q_0,\theta)$ without loss of generality. Thus by Lemma 2.4, we obtain

$$\begin{aligned}|A_{s,1}e^{P_s(z)}+A_{s,2}e^{Q_s(z)}| &\ge |A_{s,1}e^{P_s(z)}|-|A_{s,2}e^{Q_s(z)}|\\ &\ge \exp\{(1-\varepsilon)c_s\delta(P_0,\theta)r^n\}-\exp\{(1-\varepsilon)c_s\delta(Q_0,\theta)r^n\}\\ &\ge \frac{1}{2}\exp\{(1-\varepsilon)c_s\delta(P_0,\theta)r^n\}, \quad (3.26)\\ |A_{j,1}e^{P_j(z)}+A_{j,2}e^{Q_j(z)}| &\le |A_{j,1}e^{P_j(z)}|+|A_{j,2}e^{Q_j(z)}|\\ &\le \exp\{(1+\varepsilon)c_j\delta(P_0,\theta)r^n\}+\exp\{(1+\varepsilon)c_j\delta(Q_0,\theta)r^n\}\\ &\le 2\exp\{(1+\varepsilon)c_j\delta(P_0,\theta)r^n\},\quad j=0,1,\ldots,k-1, j\ne s. \quad (3.27)\end{aligned}$$

If $|f^{(s)}(re^{i\theta})|$ is unbounded on the ray $\arg z=\theta$, then by Lemma 1.1, there exists an infinite sequence of points $z_m=r_me^{i\theta}$ $(m=1,2,\ldots)$ such that $r_m\to\infty$, $f^{(s)}(z_m)\to\infty$ and

$$|\frac{f^{(j)}(z_m)}{f^{(s)}(z_m)}|\le |z_m|^{s-j}(1+o(1)),\quad j=0,1,\ldots,s-1. \tag{3.28}$$

By (3.1), (3.26)–(3.28), (3.7), we have

$$\begin{aligned}\frac{1}{2}\exp\{(1-\varepsilon)c_s\delta(P_0,\theta)r_m^n\} \le & r_m^{k\sigma}+\cdots+2\exp\{(1+\varepsilon)c_{s+1}\delta(P_0,\theta)r_m^n\}r_m^{k\sigma}+\\ & 2\exp\{(1+\varepsilon)c_{s-1}\delta(P_0,\theta)r_m^n\}(1+o(1))r_m+\cdots+\\ & 2\exp\{(1+\varepsilon)c_0\delta(P_0,\theta)r_m^n\}(1+o(1))r_m^s\\ \le & \sum_{j\ne s}4\exp\{(1+\varepsilon)c_j\delta(P_0,\theta)r_m^n\}r_m^{k\sigma+s}. \quad (3.29)\end{aligned}$$

Take $0<\varepsilon<\frac{1}{2}\min\{\frac{c_s-c_j}{c_s+c_j}, j=0,1,\ldots,k-1, j\ne s\}$, then (3.29) is a contradiction. Hence $|f^{(s)}(re^{i\theta})|\le M$ on the ray $\arg z=\theta$. Moreover, we obtain by s-fold iterated integration along the line segment $[0,z]$,

$$f(z)=f(0)+f'(0)z+\cdots+\frac{1}{(s-1)!}f^{(s-1)}(0)z^{s-1}+\int_0^z\cdots\int_0^z f^{(s)}(t)\mathrm{d}t\cdots\mathrm{d}t. \tag{3.30}$$

Therefore, by an elementary inequality estimate, we get on the ray $\arg z=\theta$.

$$|f(z)|\le M|z|^s. \tag{3.31}$$

(IV) If $\delta(P_0,\theta)<0,\delta(Q_0,\theta)<0$, by (3.3), one has $\delta(Q_j,\theta)<\delta(Q_0,\theta)<0,\delta(P_j,\theta)<\delta(P_0,\theta)<0$. Set $\delta=\max\{\delta(Q_0,\theta),\delta(P_0,\theta)\}$. Hence by Lemma 2.4, we get

$$\begin{aligned}|A_{j,1}e^{P_j(z)}+A_{j,2}e^{Q_j(z)}| &\le |A_{j,1}e^{P_j(z)}|+|A_{j,2}e^{Q_j(z)}|\\ &\le \exp\{(1-\varepsilon)c_j\delta(P_0,\theta)r^n\}+\exp\{(1-\varepsilon)c_j\delta(Q_0,\theta)r^n\}\\ &\le 2\exp\{(1-\varepsilon)c_j\delta r^n\},\quad j=0,1,\ldots,k-1. \quad (3.32)\end{aligned}$$

By equation (1.2), we have

$$-1=(A_{k-1,1}e^{P_{k-1}(z)}+A_{k-1,2}e^{Q_{k-1}(z)})\frac{f^{(k-1)}}{f^{(k)}}+\cdots+(A_{0,1}e^{P_0(z)}+A_{0,2}e^{Q_0(z)})\frac{f}{f^{(k)}}. \tag{3.33}$$

If $|f^{(k)}(re^{i\theta})|$ is unbounded on the ray $\arg z=\theta$, then by Lemma 1.1, there exists an infinite sequence of points $z_m = r_m e^{i\theta}$ $(m = 1, 2, \ldots)$ such that $r_m \to \infty$, $f^{(k)}(z_m) \to \infty$ and

$$|\frac{f^{(j)}(z_m)}{f^{(k)}(z_m)}| \le |z_m|^{k-j}(1 + o(1)), \quad j = 0, 1, \ldots, k-1. \tag{3.34}$$

By (3.32)–(3.34), we have

$$1 \le 2\exp\{(1-\varepsilon)\delta r_m^n\} r_m(1+o(1)) + \cdots + 2\exp\{(1-\varepsilon)\delta r_m^n\} r_m^k(1+o(1)). \tag{3.35}$$

Comparing the growth of two sides, we know (3.35) is a contradiction. Hence $|f^{(k)}(re^{i\theta})| \le M$ on the ray $\arg z = \theta$. Moreover, we obtain by k-fold iterated integration along the line segment $[0, z]$,

$$f(z) = f(0) + f'(0)z + \cdots + \frac{1}{(k-1)!} f^{(k-1)}(0) z^{k-1} + \int_0^z \cdots \int_0^z f^{(k)}(t) \mathrm{d}t \cdots \mathrm{d}t. \tag{3.36}$$

Therefore, by an elementary inequality estimate, we get on the ray $\arg z = \theta$

$$|f(z)| \le M|z|^k. \tag{3.37}$$

In Case 2, for any ray $\arg z = \theta \in [0, 2\pi) \backslash (H_1 \cup H_3)$, we have $|f(z)| \le M|z|^k$. By Lemma 2.6, we know that $|f(z)|$ is a polynomial which contradicts our assumption.

Furthermore, assume f is a polynomial solution of (1.2) with the degree $\deg f = m$. When $m \ge s$, we take $\arg z = \theta \in [0, 2\pi) \backslash (H_1 \cup H_3)$ satisfying $\delta(Q_0, \theta) > \delta(P_0, \theta) > 0$. For any given $\varepsilon (0 < 2\varepsilon < \min\{\frac{c_s - c_j}{c_s + c_j}, \frac{\delta(Q_0,\theta) - \delta(P_0,\theta)}{\delta(Q_0,\theta) + \delta(P_0,\theta)}\})$, by (1.2) and Lemma 2.4, we have

$$\begin{aligned}
&\frac{1}{2}\exp\{(1-\varepsilon)c_s\delta(Q_0,\theta)r^n\} r^{m-s} \le |(A_{s,1}e^{P_s(z)} + A_{s,2}e^{Q_s(z)}) f^{(s)}(re^{i\theta})| \\
&\le |f^{(k)}(re^{i\theta})| + \cdots + |(A_{s+1,1}e^{P_{s+1}(z)} + A_{s+1,2}e^{Q_{s+1}(z)}) f^{(s+1)}(re^{i\theta})| + \\
&\quad |(A_{s-1,1}e^{P_{s-1}(z)} + A_{s-1,2}e^{Q_{s-1}(z)}) f^{(s-1)}(re^{i\theta})| + \cdots + |(A_{0,1}e^{P_0(z)} + A_{0,2}e^{Q_0(z)}) f(re^{i\theta})| \\
&\le \sum_{j \ne s} 2r^m \exp\{(1+\varepsilon)c_j\delta(Q_0,\theta)r^n\}.
\end{aligned}$$

Considering the value of ε, we obtain that the above formula is a contradiction.

When $m < s$, we take $\arg z = \theta \in [0, 2\pi) \backslash (H_1 \cup H_3)$ satisfying $\delta(Q_0, \theta) > \delta(P_0, \theta) > 0$. Since c_j are distinct numbers, there exists exactly one $c_d = \max\{c_j, j = 0, 1, \ldots, m\}$. For any given ε $(0 < 2\varepsilon < \min\{\frac{c_d - c_j}{c_d + c_j}, \frac{\delta(Q_0,\theta) - \delta(P_0,\theta)}{\delta(Q_0,\theta) + \delta(P_0,\theta)}\})$, by (1.2) and Lemma 2.4, we have

$$\begin{aligned}
&\frac{1}{2}\exp\{(1-\varepsilon)c_d\delta(Q_0,\theta)r^n\} r^{m-d} \le |(A_{d,1}e^{P_d(z)} + A_{d,2}e^{Q_d(z)}) f^{(d)}(re^{i\theta})| \\
&\le |(A_{m,1}e^{P_m(z)} + A_{m,2}e^{Q_m(z)}) f^{(m)}(re^{i\theta})| + \cdots + \\
&\quad |(A_{d+1,1}e^{P_{d+1}(z)} + A_{d+1,2}e^{Q_{d+1}(z)}) f^{(d+1)}(re^{i\theta})| + \\
&\quad |(A_{d-1,1}e^{P_{d-1}(z)} + A_{d-1,2}e^{Q_{d-1}(z)}) f^{(d-1)}(re^{i\theta})| + \cdots + \\
&\quad |(A_{0,1}e^{P_0(z)} + A_{0,2}e^{Q_0(z)}) f(re^{i\theta})| \\
&\le \sum_{j \ne s} 2r^m \exp\{(1+\varepsilon)c_j\delta(Q_0,\theta)r^n\}.
\end{aligned}$$

Considering the value of ε, we obtain that the above formula is a contradiction.

We can conclude from the above discussion that every solution f ($\not\equiv 0$) of (1.2) has infinite order. □

4. Proof of Theorem 1.3

(i) Suppose f_0 is a solution of the equation (1.3) with finite order. If f_1 ($\not\equiv f_0$) is an another solution of the equation (1.3) with finite order, then $f_1 - f_0$ is a nonzero solution of the equation (1.2) with $\sigma(f_1 - f_0) < \infty$. This contradicts Theorem 1.2. Hence equation (1.3) at most has one solution with finite order.

We assume $f(z)$ is a solution of (1.3) with infinite order. We rewrite the equation (1.3) to

$$\frac{1}{f} = \frac{1}{F}(\frac{f^{(k)}}{f} + \frac{B_{k-1}(z)f^{(k-1)}}{f} + \cdots + B_0(z)), \tag{4.1}$$

where $B_j(z) = A_{j1}(z)e^{P_j(z)} + A_{j2}(z)e^{Q_j(z)}$, $j = 0, 1, \ldots, k-1$. If f has $\alpha(\alpha > k)$ order zeros in z_0, by (4.1), we know z_0 is $\alpha - k$ order zeros of F. Hence

$$N(r, \frac{1}{f}) \le k\overline{N}(r, \frac{1}{f}) + N(r, \frac{1}{F}). \tag{4.2}$$

By Logarithmic Derivative Lemma and (4.1), there exists a set $E \subset (0, \infty)$ that has finite linear measure such that for $|z| = r \notin E$, we have

$$m(r, \frac{1}{f}) \le m(r, \frac{1}{F}) + \sum_{j=0}^{k-1} m(r, B_j) + O\{\log(rT(r, f))\}. \tag{4.3}$$

By (4.2), (4.3), we get

$$T(r, f) = T(r, \frac{1}{f}) + O(1) \le T(r, F) + \sum_{j=0}^{k-1} T(r, B_j) + O\{\log(rT(r, f))\} + k\overline{N}(r, \frac{1}{f}). \tag{4.4}$$

When r is sufficiently large, we obtain

$$O\{\log(rT(r, f))\} \le \frac{1}{2}T(r, f). \tag{4.5}$$

Set $\sigma_1 = \max\{\sigma(F), n\}$. Then for any given $\varepsilon > 0$ and sufficiently large r, we have

$$T(r, F) \le r^{\sigma_1+\varepsilon}, \quad T(r, B_j) \le r^{\sigma_1+\varepsilon}. \tag{4.6}$$

By (4.4)–(4.6), when $r \notin E$ and is sufficiently large, we obtain

$$T(r, f) \le 2k\overline{N}(r, \frac{1}{f}) + 2(k+1)r^{\sigma_1+\varepsilon}. \tag{4.7}$$

By (4.7), we obtain $\overline{\lambda}(f) \ge \sigma(f), \overline{\lambda}_2(f) \ge \sigma_2(f)$. Thus $\sigma(f) = \lambda(f) = \overline{\lambda}(f) = \infty, \overline{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f)$. Combining $\sigma(B_j) = n$ ($j = 0, 1, \ldots, k-1$) with Lemma 2.3, we have $\sigma_2(f) \le \max\{n, \sigma(F)\}$.

(ii) Suppose f_0 is a solution of the equation (1.3) with finite order. Then $m(r, \frac{f_0^{(j)}}{f_0}) = O(\log r)$, $j = 1, \ldots, k-1$. Using the similar proof as (i), we can get

$$T(r, \frac{1}{f_0}) \le T(r, \frac{1}{F}) + \sum_{j=0}^{k-1} m(r, B_j) + O(\log r) + k\overline{N}(r, \frac{1}{f_0}). \tag{4.8}$$

By (4.6) and (4.8), we get

$$T(r,f_0)\le k\overline{N}(r,\frac{1}{f_0})+(k+1)r^{\sigma_1+\varepsilon}+O(\log r). \tag{4.9}$$

Hence by (4.9), we obtain $\sigma(f_0)\le\max\{\overline{\lambda}(f_0),n,\sigma(F)\}$.

(iii) Suppose that $f(z)$ is a solution of (1.3) with $\sigma(f)=\sigma<\infty$. Then $n\le\sigma$.

If $\sigma<n$ and $f^{(k)}(z)\equiv F(z)$, it follows from (1.3) that

$$\sum_{j=0}^{k-1}f^{(j)}A_{j,1}(z)e^{P_j(z)}+\sum_{j=0}^{k-1}f^{(j)}A_{j,2}(z)e^{Q_j(z)}\equiv 0. \tag{4.10}$$

We may apply Lemma 2.9 to conclude that $A_{s,1}f^{(s)}\equiv 0$ for some $s, 0\le s\le k-1$. Since $A_{s,1}\not\equiv 0$, f has to be a polynomial of degree less than s, so $F(z)\equiv 0$, leading to a contradiction. Therefore, we assume that $f^{(k)}\not\equiv F$. By Lemma 2.10, it is easy to see that $n\le\sigma$, since the $e^{P_j(z)}, e^{Q_j(z)}$ $(j=0,1,\dots,k-1)$ are linearly independent.

By Lemma 2.2, for any given $\varepsilon>0$, there exists a set $E_2\subset[0,2\pi)$ which has linear measure zero, such that if $\theta\in[0,2\pi)\backslash(E_2\cup E_3)(E_3=\{\theta\in[0,2\pi):\delta(P_0,\theta)=\delta(Q_0,\theta)\})$, then there is a constant $R_1=R_1(\theta)>1$ such that for all z satisfying $\arg z=\theta$ and $|z|\ge R_1$, we have

$$|\frac{f^{(j)}(z)}{f^{(i)}(z)}|\le|z|^{k\sigma},\quad 0\le i<j\le k. \tag{4.11}$$

Set $z=re^{i\theta}, a_{0n}=|a_{0n}|e^{i\theta_1}, b_{0n}=|b_{0n}|e^{i\theta_2}, \theta_1,\theta_2\in[0,2\pi)$. Then

$$\delta(P_0,\theta)=|a_{0n}|\cos(n\theta+\theta_1),\quad \delta(Q_0,\theta)=|b_{0n}|\cos(n\theta+\theta_2). \tag{4.12}$$

Since $a_{jn}=c_ja_{0n}, b_{jn}=c_jb_{0n}, c_j>1$, $j=1,2,\dots,k-1$, and c_j are distinct numbers. Hence

$$\delta(P_j,\theta)=c_j\delta(P_0,\theta),\quad \delta(Q_j,\theta)=c_j\delta(Q_0,\theta), \tag{4.13}$$

and there exists exactly one c_s such that $c_s=\max\{c_j; j=0,1,2,\dots,k-1\}$. Set $c_0=1$, $\delta_1=\max\{\delta(P_0,\theta),\delta(Q_0,\theta)\}$. We now discuss two cases separately.

Case 1 Assume that $\delta_1>0$. By Lemma 2.4, for any given ε with $0<3\varepsilon_1<\min\{n-\sigma(F),\frac{c_s-c_j}{c_s+c_j}, j=0,1,\dots,k-1, j\ne s\}$, we obtain

$$|A_{s,1}e^{P_s(z)}+A_{s,2}e^{Q_s(z)}|\ge\frac{1}{2}\exp\{(1-\varepsilon_1)c_s\delta_1r^n\}, \tag{4.14}$$

$$\begin{aligned}|A_{j,1}e^{P_j(z)}+A_{j,2}e^{Q_j(z)}|&\le|A_{j,1}e^{P_j(z)}|+|A_{j,2}e^{Q_j(z)}|\\&\le 2\exp\{(1+\varepsilon_1)c_j\delta_1r^n\},\quad j=0,1,\dots,k-1, j\ne s.\end{aligned} \tag{4.15}$$

We now proceed to show that

$$\frac{\log^+|f^{(s)}(z)|}{|z|^{\sigma(F)+\varepsilon_1}} \tag{4.16}$$

is bounded on the ray $\arg z=\theta$. If that is not the case, then by Lemma 2.11, there is a sequence of points $z_m=r_me^{i\theta}$, such that $r_m\to\infty$ and that

$$\frac{\log^+|f^{(s)}(z_m)|}{r_m^{\sigma(F)+\varepsilon_1}}\to\infty, \tag{4.17}$$

$$|\frac{f^{(j)}(z_m)}{f^{(s)}(z_m)}| \le (1+o(1))r_m^{s-j}, \quad j=0,\ldots,s-1. \tag{4.18}$$

From (4.17) and the definition of order, it is easy to see that

$$|\frac{F(z_m)}{f^{(s)}(z_m)}| \to 0. \tag{4.19}$$

From (1.3), we obtain

$$\begin{aligned}|A_{s,1}e^{P_s(z)}+A_{s,2}e^{Q_s(z)}| \le & |\frac{f^{(k)}}{f^{(s)}}|+\cdots+|A_{s+1,1}e^{P_{s+1}(z)}+A_{s+1,2}e^{Q_{s+1}(z)}||\frac{f^{(s+1)}}{f^{(s)}}|+\\ & |A_{s-1,1}e^{P_{s-1}(z)}+A_{s-1,2}e^{Q_{s-1}(z)}||\frac{f^(s-1)}{f^{(s)}}|+\cdots+\\ & |A_{0,1}e^{P_0(z)}+A_{0,2}e^{Q_0(z)}||\frac{f}{f^{(s)}}|+|\frac{F}{f^{(s)}}|. \end{aligned}\tag{4.20}$$

Using inequalities (4.11), (4.14), (4.15), (4.18) and the limit (4.19), we have

$$\frac{1}{2}\exp\{(1-\varepsilon_1)c_s\delta_1 r_m^n\} \le \sum_{j\ne s} 4\exp\{(1+\varepsilon_1)c_j\delta_1 r_m^n\}r_m^{k\sigma+s}, \tag{4.21}$$

which is a contradiction. Therefore $\frac{\log^+|f^{(s)}(z)|}{|z|^{\sigma(F)+\varepsilon_1}}$ is bounded, and one has $|f^{(s)}(z)| \le M\exp\{r^{\sigma(F)+\varepsilon_1}\}$ on the ray $\arg z=\theta$. Moreover, we obtain by s-fold iterated integration along the line segment $[0,z]$ and by an elementary inequality estimate, we get on the ray $\arg z=\theta$

$$|f(z)| \le M\exp\{r^{\sigma(F)+2\varepsilon_1}\}. \tag{4.22}$$

Case 2 If $\delta_1<0$, by Lemma 2.4, we get

$$\begin{aligned}|A_{j,1}e^{P_j(z)}+A_{j,2}e^{Q_j(z)}| &\le |A_{j,1}e^{P_j(z)}|+|A_{j,2}e^{Q_j(z)}|\\ &\le \exp\{(1-\varepsilon_1)\delta(P_j,\theta)r^n\}+\exp\{(1-\varepsilon_1)\delta(Q_j,\theta)r^n\}\\ &\le 2\exp\{(1-\varepsilon_1)\delta_1 r^n\}, \quad j=0,1,\ldots,k-1.\end{aligned}\tag{4.23}$$

By (1.3), we have

$$-1=(A_{k-1,1}e^{P_{k-1}(z)}+A_{k-1,2}e^{Q_{k-1}(z)})\frac{f^{(k-1)}}{f^{(k)}}+\cdots+(A_{0,1}e^{P_0(z)}+A_{0,2}e^{Q_0(z)})\frac{f}{f^{(k)}}-\frac{F}{f^{(k)}}. \tag{4.24}$$

As in Case 1, it is easy to prove that

$$\frac{\log^+|f^{(k)}(z)|}{|z|^{\sigma(F)+\varepsilon_1}} \tag{4.25}$$

is bounded on the ray $\arg z=\theta$. Hence, we have $|f^{(k)}| \le M\exp\{r^{\sigma(F)+\varepsilon_1}\}$ on the ray $\arg z=\theta$. This implies, as in Case 1, that

$$|f(z)| \le M\exp\{r^{\sigma(F)+2\varepsilon_1}\}. \tag{4.26}$$

Therefore for any given $\theta\in[0,2\pi)\backslash(E_2\cup E_3)$, where $E_2\cup E_3$ has linear measure zero, we have (4.26). Then by Lemma 2.12, we have $\sigma(f)\le\sigma(F)+2\varepsilon_1<n$. It is a contradiction. So every solution of (1.3) must be of infinite order. □

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