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Behavior at Infinity for Nonnegative Superfuctions in a Cone

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Abstract In this paper we show that a positive superfunction on a cone behaves regularly at infinity outside a minimally thin set associated with the stationary Schrödinger operator.

Keywords stationary Schrödinger operator; superfunction; minimally thin sets; cone

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1. Introduction

Let $\mathbf{R}^n (n \geq 2)$ be the n-dimensional Euclidean space and $\mathbf{S}$ its an open set. The boundary and the closure of $\mathbf{S}$ are denoted by $\partial\mathbf{S}$ and $\overline{\mathbf{S}}$, respectively. In cartesian coordinate a point P is denoted by (X, x_n), where $X = (x_1, x_2, \ldots, x_{n-1})$, hence the upper half space $\mathbf{T}_n = \{P = (X, x_n) \in \mathbf{R}^n; x_n > 0\}$. Let $|P|$ be the Euclidean norm of P and $|P-Q|$ the Euclidean distance of two points P and Q in $\mathbf{R}^n$. The unit sphere and the upper half unit sphere are denoted by $\mathbf{S}^{n-1}$ and $\mathbf{S}^{n-1}_+$, respectively. For $P \in \mathbf{R}^n$ and $r > 0$, let $B(P, r)$ be the open ball of radius r centered at P in $\mathbf{R}^n$. Then $S_r = \partial B(O, r)$. Furthermore, we denote by dS_r the $(n-1)$-dimensional volume elements induced by the Euclidean metric on S_r.

In this paper we are concerned with some properties for the generalized subharmonic function associated with the stationary Schrödinger operator (i.e., subfunction or superfunction [1]). Lelong-Ferrand [2], Essén and Jackson [3] obtained some properties for minimally thin sets and rarefied sets at ∞ with respect to $\mathbf{T}_n$. Aikawa [4] introduced the definition of $\mathcal{L}_0^b$-minimally thinness $(0 \leq b \leq 1)$ and gave the following theorem.

Theorem A *Let $0 \leq b \leq 1$. If u is a non-negative superharmonic function on $\mathbf{T}_n$, then there exists a set E in $\mathbf{T}_n$ which is $\mathcal{L}_0^b$-minimally thin at ∞ such that*

$$\lim_{|P|\to\infty, P\in\mathbf{T}_n\setminus E} \frac{u(P) - c(u)x_n}{x_n^b \mid P \mid^{1-b}} = 0.$$

Conversely, if E is unbounded and $\mathcal{L}_0^b$-minimally thin at ∞ with respect to $C_n(\Omega)$, then there

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exists a non-negative superharmonic function $u(P)$ on $\mathbf{T}_n$ such that

$$\lim_{|P|\to\infty, P\in E} \frac{u(P)-c(u)x_n}{x_n^b|P|^{1-b}} = \infty.$$

Remark 1.1 Here $c(u) = \inf_{P=(X,x_n)\in\mathbf{T}_n} \frac{u(P)}{x_n}$. When $b = 0$ and $b = 1$, Theorem A is due to [3] and [2], respectively.

In [5] Miyamoto and Yoshida generalized the above results in [2] and [3] from $\mathbf{T}_n$ to a cone. By modifying the methods of [3] and [4], Yanagishita [6] proved that Theorem A holds in a cone. In addition, Qiao and Deng [7,8] considered some problems for the stationary Schrödinger operator at ∞ with respect to a cone as well as Levin and Kheyfits [1,9]. Similarly, in [10] the first author joined the results from [5] to the stationary Schrödinger operator. Hence, when we introduce the notion of the $\mathcal{L}_a^b$-minimally thin at ∞ with respect to a cone, we may extend Theorem A through [6] to Theorem 2.2 of Section 2 in this paper. To state our results, we will need some notations and background materials below.

Relative to the system of spherical coordinates, the Laplace operator Δ may be written by

$$\Delta = \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{\Delta^*}{r^2},$$

where the explicit form of the Beltrami operator Δ^* is given by Azarin [11].

Let D be an arbitrary domain in $\mathbf{R}^n$ and $\mathscr{A}_a$ denote the class of nonnegative radial potentials $a(P)$, i.e., $0 \le a(P) = a(r)$, $P = (r,\Theta) \in D$, such that $a \in L^b_{\rm loc}(D)$ with some $b > n/2$ if $n \ge 4$ and with $b = 2$ if $n = 2$ or $n = 3$.

If $a \in \mathscr{A}_a$, then the stationary Schrödinger operator with a potential $a(\cdot)$

$$\mathcal{L}_a = -\Delta + a(\cdot)\mathrm{I} \tag{1.1}$$

can be extended in the usual way from the space $C_0^\infty(D)$ to an essentially self-adjoint operator on $L^2(D)$, where Δ is the Laplace operator and I the identical operator [12, Chap.13]. Then $\mathcal{L}_a$ has a Green a-function $G_D^a(\cdot,Q)$ which is positive on D and whose inner normal derivative $\partial G_D^a(\cdot,Q)/\partial n_Q$ is not negative, here $\partial/\partial n_Q$ denotes the differentiation at Q along the inward normal into D. We write this derivative by $PI_D^a(\cdot,Q)$, which is called the Poisson a-kernel with respect to D, and denote by $G_D^0(\cdot,Q)$ the Green function of Laplacian. For simplicity, a point $(1,\Theta)$ on $\mathbf{S}^{n-1}$ and the set $\{\Theta;(1,\Theta)\in\Omega\}$ for a set Ω ($\Omega\subset\mathbf{S}^{n-1}$) are often identified with Θ and Ω, respectively. For two sets $\Xi\subset\mathbf{R}_+$ and $\Omega\subset\mathbf{S}^{n-1}$, the set $\{(r,\Theta)\in\mathbf{R}^n; r\in\Xi,(1,\Theta)\in\Omega\}$ in $\mathbf{R}^n$ is simply denoted by $\Xi\times\Omega$. In particular, the upper half space $\mathbf{T}_n = \mathbf{R}_+\times\mathbf{S}_+^{n-1}$. By $C_n(\Omega)$ we denote the set $\mathbf{R}_+\times\Omega$ in $\mathbf{R}^n$ with the domain Ω on $\mathbf{S}^{n-1}$ and call it a cone. We mean the sets $I\times\Omega$ and $I\times\partial\Omega$ with an interval on $\mathbf{R}_+$ by $C_n(\Omega;I)$ and $S_n(\Omega;I)$, and $C_n(\Omega)\cap S_r$ by $C_n(\Omega;r)$. By $S_n(\Omega)$ we denote $S_n(\Omega;(0,+\infty))$, which is $\partial C_n(\Omega)\setminus\{O\}$. From now on, we always assume $D = C_n(\Omega)$ and write $G_\Omega^a(\cdot,Q)$ instead of $G_{C_n(\Omega)}^a(\cdot,Q)$.

Let Ω be a domain on $\mathbf{S}^{n-1}$ with smooth boundary and λ the least positive eigenvalue for $-\Delta^*$ on Ω (see [13, p.41])

$$(\Delta^*+\lambda)\varphi(\Theta) = 0 \quad \text{on } \Omega, \tag{1.2}$$

$$\varphi(\Theta) = 0 \quad \text{on } \partial\Omega.$$

The corresponding eigenfunction is denoted by $\varphi(\Theta)$ satisfying $\int_\Omega \varphi^2(\Theta)\mathrm{d}S_1 = 1$. In order to ensure the existence of λ and $\varphi(\Theta)$, we put a rather strong assumption on Ω: if $n \geq 3$, then Ω is a $C^{2,\alpha}$-domain ($0 < \alpha < 1$) on $\mathbf{S}^{n-1}$ surrounded by a finite number of mutually disjoint closed hypersurfaces ([14, p.88,89] for the definition of $C^{2,\alpha}$-domain).

Solutions of an ordinary differential equation

$$-Q''(r) - \frac{n-1}{r}Q'(r) + (\frac{\lambda}{r^2} + a(r))Q(r) = 0, \quad \text{for} \quad 0 < r < \infty \tag{1.3}$$

are known (see [15] for more references) if the potential $a \in \mathscr{A}_a$. We know the equation (1.3) has a fundamental system of positive solutions $\{V, W\}$ such that V is nondecreasing with

$$0 \leq V(0+) \leq V(r) \;\text{ as }\; r \to +\infty \tag{1.4}$$

and W is monotonically decreasing with

$$+\infty = W(0+) > W(r) \searrow 0 \;\text{ as }\; r \to +\infty. \tag{1.5}$$

We remark that both $V(r)\varphi(\Theta)$ and $W(r)\varphi(\Theta)$ are a-harmonic on $C_n(\Omega)$ and vanish continuously on $S_n(\Omega)$.

We will also consider the class $\mathscr{B}_a$, consisting of the potentials $a \in \mathscr{A}_a$ such that there exists the finite limit $\lim_{r\to\infty} r^2 a(r) = \kappa \in [0, \infty)$, moreover, $r^{-1}|r^2 a(r) - \kappa| \in L(1, \infty)$. If $a \in \mathscr{B}_a$, then the (super)subfunctions are continuous [16]. For simplicity, in the rest of the paper we assume that $a \in \mathscr{B}_a$.

Denote

$$\iota_\kappa^{\pm} = \frac{2 - n \pm \sqrt{(n-2)^2 + 4(\kappa + \lambda)}}{2},$$

then the solutions $V(r)$ and $W(r)$ to the equation (1.3) normalized by $V(1) = W(1) = 1$ have the asymptotic [14]

$$V(r) \approx r^{\iota_\kappa^+}, \quad W(r) \approx r^{\iota_\kappa^-}, \quad \text{as} \quad r \to \infty \tag{1.6}$$

and

$$\chi = \iota_\kappa^+ - \iota_\kappa^- = \sqrt{(n-2)^2 + 4(\kappa + \lambda)}, \quad \chi' = \omega(V(r), W(r))|_{r=1}, \tag{1.7}$$

where χ' is their Wronskian at $r = 1$.

Remark 1.2 If $a = 0$ and $\Omega = \mathbf{S}_+^{n-1}$, then $\iota_0^+ = 1$, $\iota_0^- = 1 - n$ and $\varphi(\Theta) = (2ns_n^{-1})^{1/2}\cos\theta_1$, where s_n is the surface area $2\pi^{n/2}\{\Gamma(n/2)\}^{-1}$ of $\mathbf{S}^{n-1}$.

It is known that the Martin boundary $\triangle$ of $C_n(\Omega)$ is the set $\partial C_n(\Omega) \cup \{\infty\}$. Define the generalized Martin type kernel $M_\Omega^a(P, Q)$ $(P = (r, \Theta) \in C_n(\Omega), Q = (t, \Phi) \in \overline{C_n(\Omega)} \cup \{\infty\})$ as follows:

$$M_\Omega^a(P, Q) = \begin{cases} \frac{G_\Omega^a(P,Q)}{V(t)\varphi(\Phi)} & \text{on } C_n(\Omega) \times C_n(\Omega), \\ \frac{\partial G_\Omega^a(P,Q)}{\partial n_Q}\{V(t)t^{-1}\frac{\partial \varphi(\Phi)}{\partial n_\Phi}\}^{-1} & \text{on } C_n(\Omega) \times S_n(\Omega), \\ V(r)\varphi(\Theta) & \text{on } C_n(\Omega) \times \{\infty\}, \\ \kappa W(r)\varphi(\Theta) & \text{on } C_n(\Omega) \times \{o\}, \end{cases}$$

where $\partial/\partial n_Q$ denotes the differentiation at Q along the inward normal into $C_n(\Omega)$. Note that $M^a_\Omega(P,Q)$ is continuous in the extending sense on $C_n(\Omega)\cup S_n(\Omega)$. Set $\ddot{M}^a_\Omega(P,Q) = M^a_\Omega(Q,P)$ $(P=(r,\Theta)\in C_n(\Omega), Q=(t,\Phi)\in \overline{C_n(\Omega)}\cup\{\infty\})$ and $\ddot{M}^a_\Omega\nu(P)=\int_{C_n(\Omega)}\ddot{M}^a_\Omega(P,Q)\mathrm{d}\nu(Q)$.

For $b\in[0,1]$, we define the positive superfunction g^b_a by $g^b_a(P)=M^a_\Omega(P,Q)^b$ for $P\in C_n(\Omega)$. For a bounded subset E of $C_n(\Omega)$, $\mathcal{L}^b_a$-mass of E is defined by $\lambda^a_{E,b}(C_n(\Omega))$ for $0\le b\le 1$, where $\lambda^a_{E,b}$ is the measure on $\overline{C_n(\Omega)}$ such that

$$M^a_\Omega\lambda^a_{E,b}=\widehat{R}^E_{g^b_a} \tag{1.8}$$

and

$$\lambda^a_{E,b}(\overline{C_n(\Omega)})=\int_{C_n(\Omega)} g^b_a\mathrm{d}\lambda_E. \tag{1.9}$$

A subset E of $C_n(\Omega)$ is $\mathcal{L}^b_a$-minimally thin at ∞ in $C_n(\Omega)$ if

$$\sum_{k=0}^{\infty}\lambda^a_{E_k,b}(\overline{C_n(\Omega)})V^{-\eta}(2^k)W(2^k)<\infty, \tag{1.10}$$

where $E_k=E\cap I_k$ and

$$I_k=\{P\in C_n(\Omega); 2^k\le r\le 2^{k+1}\},\quad k=0,1,2,\ldots.$$

The rest of the paper is organized as follows. In Section 2, we will give our main theorems. In Section 3, some necessary lemmas are given. In Section 4, we will prove the main results.

2. Statements of main results

Let E be a bounded subset of $C_n(\Omega)$. Since $g^b_a(P)$ is a positive superfunction on $C_n(\Omega)$ vanishing on $\partial C_n(\Omega)$ and $\widehat{R}^E_{g^b_a}(P)$ is bounded on $C_n(\Omega)$, the greatest a-harmonic minorant of $\widehat{R}^E_{g^b_a}(P)$ is zero. By the Riesz decomposition theorem there exists a unique positive measure $\lambda^a_{E,b}$ on $C_n(\Omega)$ such that

$$\widehat{R}^E_{g^b_a}(P)=G^a_\Omega\lambda^a_{E,b}(P) \tag{2.1}$$

for any $P\in C_n(\Omega)$ and $\lambda^a_{E,b}$ is concentrated on B_E, where

$$B_E=\{P\in C_n(\Omega): E \text{ is not } \mathcal{L}^b_a-\text{thin at } P\}.$$

Let η be a real number satisfying

$$\lim_{r\to\infty}\frac{\log W(r)}{\log V(r)}<\eta\le 1. \tag{2.2}$$

Define the positive superfunction $h^a_\eta(P)=M^a_\Omega(P,\infty)W(r)V^{-\eta}(r)$. Since $M^a_\Omega(P,\infty)$ is a minimal a-harmonic function on $C_n(\Omega)$, there exists a measure ν_η on $C_n(\Omega)$ such that $G^a_\Omega\nu_\eta(P)=\min\{M^a_\Omega(P,\infty),h^a_\eta(P)\}$.

Set $c_\infty(u,a)=\inf_{P\in C_n(\Omega)}\frac{u(P)}{M^a_\Omega(P,\infty)}$. Let $\mathfrak{F}^a_\eta$ be the class of all non-negative superfunction u on $C_n(\Omega)$ such that $c_\infty(u,a)=0$ and

$$\int_{C_n(\Omega;(1,\infty))\cup S_n(\Omega;(1,\infty))} W(t)V^{-\eta}(t)\mathrm{d}\mu_u(Q)<\infty. \tag{2.3}$$

Remark 2.1 If $P \in C_n(\Omega)$, then $\ddot{M}^a_\Omega \nu_\eta(P) = \frac{G^a_\Omega \nu_\eta(P)}{M^a_\Omega(P,\infty)}$. Following the method from [10] for $\mathcal{L}^1_a$, we know that subset E of $C_n(\Omega)$ is $\mathcal{L}^b_a$-minimally thin at $P \in C_n(\Omega)$ such that

$$\ddot{M}^a_\Omega \nu_\eta(P) \leq \liminf_{Q \to P, Q \in C_n(\Omega)} \ddot{M}^a_\Omega \nu_\eta(Q).$$

Therefore, if $P \in S_n(\Omega)$, then

$$\ddot{M}^a_\Omega \nu_\eta(P) = \liminf_{Q \to P, Q \in C_n(\Omega)} \ddot{M}^a_\Omega \nu_\eta(Q).$$

Hence for $P \in C_n(\Omega) \cup S_n(\Omega)$, we have

$$\ddot{M}^a_\Omega \nu_\eta(P) = \begin{cases} 1 & \text{for } 0 < r < 1, \\ W(r)V^{-\eta}(r) & \text{for } r \geq 1. \end{cases}$$

Set $h^a_{\eta,b}(P) = M^a_\Omega(P,\infty)^b V(r)^{\eta-b}$. Next we state our results as follows.

Theorem 2.2 *If $u(P) \in \mathfrak{F}^a_\eta$, then there exists a subset E of $C_n(\Omega)$ which is $\mathcal{L}^b_a$-minimally thin at ∞ with respect to $C_n(\Omega)$ such that*

$$\lim_{|P| \to \infty, P \in C_n(\Omega) \setminus E} \frac{u(P)}{h^a_{\eta,b}} = 0. \tag{2.4}$$

Conversely, if E is unbounded and $\mathcal{L}^b_a$-minimally thin at ∞ with respect to $C_n(\Omega)$, then there exists $u(P) \in \mathfrak{F}^a_\eta$ such that

$$\lim_{|P| \to \infty, P \in E} \frac{u(P)}{h^a_{\eta,b}} = \infty. \tag{2.5}$$

Corollary 2.3 *Let $u(P)$ be a non-negative superfunction on $C_n(\Omega)$. Then there exists a subset E of $C_n(\Omega)$ which is $\mathcal{L}^b_a$-minimally thin at ∞ with respect to $C_n(\Omega)$ such that*

$$\lim_{|P| \to \infty, P \in C_n(\Omega) \setminus E} \frac{u(P) - c_\infty(u,a) M^a_\Omega(P,\infty)}{M^a_\Omega(P,\infty)^b V(r)^{1-b}} = 0. \tag{2.6}$$

Conversely, if E is unbounded and $\mathcal{L}^b_a$-minimally thin at ∞ with respect to $C_n(\Omega)$, then there exists a non-negative superfunction $u(P)$ such that

$$\lim_{|P| \to \infty, P \in E} \frac{u(P) - c_\infty(u,a) M^a_\Omega(P,\infty)}{M^a_\Omega(P,\infty)^b V(r)^{1-b}} = \infty. \tag{2.7}$$

Remark 2.4 When $a = 0$, Theorem 2.2 and Corollary 2.3 are from Yanagishita [6]. Further, if $a = 0$ and S^{n-1}_+, Theorem 2.2 is the result of Aikawa [4, Theorem 3.2]. In addition, when $a = 0$ and $b = 0$, and $b = 0$ in Corollary 2.3, we refer to [5] and [10], respectively. Since $\widetilde{u}(P) = u(P) - c_\infty(u,a) M^a_\Omega(P,\infty) \in \mathfrak{F}^a_1$ in Corollary 2.3, we may follow the same method as the proof of Theorem 2.2.

3. Some lemmas

To state our results better, in the arguments we need the following results.

Lemma 3.1 ([7]) *For any $P = (r,\Theta) \in C_n(\Omega)$ and any $Q = (t,\Phi) \in S_n(\Omega)$ satisfying $0 < \frac{t}{r} \leq$*

$\frac{4}{5}$ (*resp.*, $0<\frac{r}{t}\le\frac{4}{5}$),

$$\frac{\partial G^a_\Omega(P,Q)}{\partial n_Q}\le Ct^{-1}V(t)W(r)\varphi(\Theta)\frac{\partial\varphi(\Phi)}{\partial n_\Phi} \tag{3.1}$$

$$(\text{resp.}, \frac{\partial G^a_\Omega(P,Q)}{\partial n_Q}\le CV(r)t^{-1}W(t)\varphi(\Theta)\frac{\partial\varphi(\Phi)}{\partial n_\Phi}). \tag{3.2}$$

Further,

$$\frac{\partial G^0_\Omega(P,Q)}{\partial n_Q}\lesssim\frac{\varphi(\Theta)}{t^{n-1}}\frac{\partial\varphi(\Phi)}{\partial n_\Phi}+\frac{r\varphi(\Theta)}{|P-Q|^n}\frac{\partial\varphi(\Phi)}{\partial n_\Phi} \tag{3.3}$$

for any $P=(r,\Theta)\in C_n(\Omega)$ *and any* $Q=(t,\Phi)\in S_n(\Omega;(\frac{4}{5}r,\frac{5}{4}r))$.

Lemma 3.2 ([7]) *For any* $P=(r,\Theta)\in C_n(\Omega)$ *and any* $Q=(t,\Phi)\in C_n(\Omega)$ *satisfying* $0<\frac{t}{r}\le\frac{4}{5}$ (*resp.*, $0<\frac{r}{t}\le\frac{4}{5}$),

$$G^a_\Omega(P,Q)\le Ct^{-1}V(t)W(r)\varphi(\Theta)\varphi(\Phi) \tag{3.4}$$

$$(\text{resp.}, G^a_\Omega(P,Q)\le CV(r)t^{-1}W(t)\varphi(\Theta)\varphi(\Phi)) \tag{3.5}$$

Further,

$$G^0_\Omega(P,Q)\lesssim\frac{\varphi(\Theta)\varphi(\Phi)}{t^{n-2}}+U_\Omega(P,Q) \tag{3.6}$$

for any $P=(r,\Theta)\in C_n(\Omega)$ *and any* $Q=(t,\Phi)\in S_n(\Omega;(\frac{4}{5}r,\frac{5}{4}r))$, *where*

$$U_\Omega(P,Q)=\min\{\frac{1}{|P-Q|^{n-2}},\frac{rt\varphi(\Theta)\varphi(\Phi)}{|P-Q|^n}\}.$$

Lemma 3.3 *The set* $E\subset C_n(\Omega;(1,\infty))$ *is* $\mathcal{L}^b_a$*-minimally thin at* ∞ *if and only if*

$$\sum_{k=0}^{\infty}\widehat{R}^{E_k}_{h^a_{\eta,b}}\in\mathfrak{F}^a_\eta.$$

Proof Note that for every $k=0,1,2,\dots$

$$\widehat{R}^{E_k}_{g^b_a}\approx V(2^k)^{b-\eta}\widehat{R}^{E_k}_{h^a_{\eta,b}},$$

$$\lambda^a_{E_k,b}(\overline{C_n(\Omega)})\approx V^\eta(2^k)W^{-1}(2^k)\int_{C_n(\Omega)\cup S_n(\Omega)}V^{-\eta}(t)W(t)\mathrm{d}\lambda^a_{E_k,b}(Q),$$

where the constants of comparison are independent of k. Since

$$\begin{aligned}\int_{C_n(\Omega)}\widehat{R}^{E_k}_{g^b_a}\mathrm{d}\nu_\eta(P)&=\int_{C_n(\Omega)}M^a_\Omega\lambda^a_{E_k,b}(P)\mathrm{d}\nu_\eta(P)=\int_{C_n(\Omega)\cup S_n(\Omega)}\ddot{M}^a_\Omega\nu_\eta(Q)\mathrm{d}\lambda^a_{E_k,b}(Q)\\&=\int_{C_n(\Omega)\cup S_n(\Omega)}V^{-\eta}(t)W(t)\mathrm{d}\lambda^a_{E_k,b}(Q),\end{aligned}$$

we have

$$V^{-b}(2^k)W(2^k)\lambda^a_{E_k,b}(\overline{C_n(\Omega)})\approx\int_{C_n(\Omega)}\widehat{R}^{E_k}_{h^a_{\eta,b}}(P)\mathrm{d}\nu_\eta(P),$$

where the constants of comparison are independent of k, which gives the conclusion. □

Lemma 3.4 *Let* $E\subset C_n(\Omega;(1,\infty))$. *If* $\widehat{R}^E_{h^a_{\eta,b}}\in\mathfrak{F}^a_\eta$, *then* E *is* $\mathcal{L}^b_a$*-minimally thin at* ∞.

Proof Since $h^a_{\eta,b}(P)$ satisfies

$$\liminf_{|P|\to\infty}\frac{h^a_{\eta,b}(P)}{M^a_\Omega(P,\infty)V(r)^{\eta-1}}>0,$$

we find a positive constant C' and a natural number N_1 such that

$$h^a_{\eta,b}(P)\ge C'M^a_\Omega(P,\infty)V(r)^{\eta-1}$$

for $r>2^{N_1}$. Let $\widehat{R}^E_{h^a_{\eta,b}}=M^a_\Omega\mu$, where μ satisfies (2.3) and

$$A=\int_{C_n(\Omega;(1,\infty))\cup S_n(\Omega;(1,\infty))}V^{-\eta}(t)W(t)\mathrm{d}\mu(t,\Phi)<\infty.$$

Set $\widetilde{C}=\frac{C}{C'}$. By (1.4–1.7) we may take a natural number N_2 such that

$$4A\widetilde{C}V^{-\eta}(2^{N_2}r)W(2^{N_2}r)<V^{-\eta}(r)W(r).$$

Then there exists a natural number N_0 such that

$$\widetilde{C}\int_{\{Q\in C_n(\Omega)\cup S_n(\Omega):t\ge 2^{k+N_2+1}\}}V^{-\eta}(t)W(t)\mathrm{d}\mu(t,\Phi)<\frac{1}{4}$$

for $k\ge N_0$. Let $N=\max\{N_0,N_1,N_2\}$. Since

$$\sum_{k=0}^{N}\widehat{R}^{E_k}_{h^a_{\eta,b}}\le(N+1)\widehat{R}^E_{h^a_{\eta,b}}\in\mathfrak{F}^a_\eta,$$

it suffices to prove $\sum_{k>N}^{\infty}\widehat{R}^{E_k}_{h^a_{\eta,b}}\in\mathfrak{F}^a_\eta$. Set $J_k=I_{k-N_2}\cup\cdots\cup I_k\cup\cdots\cup I_{k+N_2}$. Let $k>N$ and $P=(r,\Theta)\in E_k$. If $Q\in C_n(\Omega)$ and $t<2^{k-N_2}$, then from the inequality (3.4) in Lemma 3.2 we obtain

$$M^a_\Omega(P,Q)=\frac{G^a_\Omega(P,Q)}{V(t)\varphi(\Phi)}\le CW(r)\varphi(\Theta).$$

Hence

$$\begin{aligned}\int_{\{Q\in C_n(\Omega):t\le 2^{k-N_2}\}}M^a_\Omega(P,Q)\mathrm{d}\mu(t,\Phi)&\le\widetilde{C}h^a_{\eta,b}(P)V^{-\eta}(r)W(r)\int_{1\le t\le 2^{k-N_2}}\mathrm{d}\mu(Q)\\&\le\widetilde{C}h^a_{\eta,b}(P)\int_{1\le t\le 2^{k-N_2}}V^{-\eta}(r)W(r)\mathrm{d}\mu(Q).\end{aligned}$$

On the other hand, if $Q\in C_n(\Omega)$ and $t\ge 2^{k+N_2+1}$, then we get

$$\begin{aligned}\int_{\{Q\in C_n(\Omega):t\ge 2^{k+N_2+1}\}}M^a_\Omega(P,Q)\mathrm{d}\mu(t,\Phi)&\le\widetilde{C}h^a_{\eta,b}(P)V^{1-\eta}(r)\int_{t\ge 2^{k+N_2+1}}V^{-1}(t)W(t)\mathrm{d}\mu(Q)\\&\le\widetilde{C}h^a_{\eta,b}(P)\int_{t\ge 2^{k+N_2+1}}V^{-\eta}(t)W(t)\mathrm{d}\mu(Q).\end{aligned}$$

If $Q\in S_n(\Omega)$ and $t\le 2^{k-N_2}$ or $Q\in S_n(\Omega)$ and $t\ge 2^{k+N_2+1}$, then from Lemma 3.1 we have similar inequalities. According to these inequalities we obtain

$$\begin{aligned}\widetilde{C}^{-1}\int_{\overline{C_n(\Omega)}\setminus J_k}M^a_\Omega(P,Q)\mathrm{d}\mu(Q)\le&h^a_{\eta,b}(P)\int_{t\le 2^{k-N_2}}V^{-\eta}(t)W(t)\mathrm{d}\mu(Q)+\\&h^a_{\eta,b}(P)\int_{t\ge 2^{k+N_2+1}}V^{-\eta}(t)W(t)\mathrm{d}\mu(Q).\end{aligned}$$

Since $4A\widetilde{C}V^{-\eta}(2^{N_2}r)W(2^{N_2}r) < V^{-\eta}(r)W(r)$, we see that

$$\widetilde{C}\int_{t\le 2^{k-N_2}} V^{-\eta}(r)W(r)\mathrm{d}\mu(t,\Phi) \le \frac{1}{4A}\int_{t\le 2^{k-N_2}} V^{-\eta}(2^{-N_2}r)W(2^{-N_2}r)\mathrm{d}\mu(t,\Phi)$$
$$\le \frac{1}{4A}\int_{t\le 2^{k-N_2}} V^{-\eta}(t)W(t)\mathrm{d}\mu(t,\Phi) \le \frac{1}{4}.$$

So we have

$$\int_{\overline{C_n(\Omega)}\setminus \overline{J}_k} M^a_\Omega(P,Q)\mathrm{d}\mu(Q) \le \frac{1}{2}h^a_{\eta,b}(P) \quad \text{on } E_k,$$

which implies that

$$h^a_{\eta,b}(P) \le \widehat{R}^{E_k}_{h^a_{\eta,b}}(P) \le \int_{\overline{J}_k} M^a_\Omega(P,Q)\mathrm{d}\mu(Q) + \frac{1}{2}h^a_{\eta,b}(P)$$

q.e on E_k. Hence

$$h^a_{\eta,b}(P) \le 2\int_{\overline{J}_k} M^a_\Omega(P,Q)\mathrm{d}\mu(Q)$$

q.e on E_k. By the definition of $\widehat{R}^{E_k}_{h^a_{\eta,b}}$, $\widehat{R}^{E_k}_{h^a_{\eta,b}} \le 2\int_{\overline{J}_k} M^a_\Omega(P,Q)\mathrm{d}\mu(Q)$ on $C_n(\Omega)$. If we sum up $\widehat{R}^{E_k}_{h^a_{\eta,b}}$ over $k > N$, we obtain

$$\sum_{k=N}^{\infty} \widehat{R}^{E_k}_{h^a_{\eta,b}} \le c\widehat{R}^{E}_{h^a_{\eta,b}}.$$

Since $\sum_{k>N} \widehat{R}^{E_k}_{h^a_{\eta,b}} \in \mathfrak{F}^a_\eta$, we get Lemma 3.4 from Lemma 3.3. $\Box$

4. Proofs of Theorems

In the section we will mainly give the proofs of theorems in the paper.

Proof of Theorem 2.2 Let $u_1(P) = u(P) - c_o(u,a)M^a_\Omega(P,Q)$ for $P = (r,\Theta) \in C_n(\Omega)$, where $c_o(u) = \inf_{P\in C_n(\Omega)} \frac{u(P)}{M(P,O)}$. Then $u_1 \in \mathfrak{F}^a_\eta$. For each nonnegative integer j, set $A_j = \{P = (r,\Theta) \in C_n(\Omega); \frac{u_1(P)}{h^a_{\eta,b}(P)} \ge (j+1)^{-1}\}$. Since $\widehat{R}^{A_j}_{h^a_{\eta,b}} \le (j+1)u_1 \in \mathfrak{F}^a_\eta$, $\widehat{R}^{A_j}_{h^a_{\eta,b}} \in \mathfrak{F}^a_\eta$, and so A_j is $\mathcal{L}^b_a$-minimally thin by Lemma 3.4. We can find an increasing sequence $\{m(j)\}$ of natural numbers such that

$$\sum_j \widehat{R}^{\cup_{k\ge j}(A_j\cap I_k)}_{h^a_{\eta,b}} \in \mathfrak{F}^a_\eta.$$

Set $E = \cup_{j=0}^\infty \cup_{k\ge m(j)} (A_j \cap I_k)$. Since

$$\widehat{R}^{E}_{h^a_{\eta,b}} \le \sum_j \widehat{R}^{\cup_{k\ge j}(A_j\cap I_k)}_{h^a_{\eta,b}},$$

by Lemma 3.4 E is $\mathcal{L}^b_a$-minimally thin. If $P \in C_n(\Omega)\setminus E$, then $P \in C_n(\Omega)\setminus \cup_{k\ge j}(A_j\cap I_k)$ for every j. It follows that if $r \ge 2^{m(j)}$, then $P \in C_n(\Omega)\setminus A_j$. This implies that $\frac{u_1(P)}{h^a_{\eta,b}(P)} \le (j+1)^{-1}$. Hence we have $\frac{u_1(P)}{h^a_{\eta,b}(P)} \to 0$ as $r\to\infty$ for $P = (r,\Theta) \in C_n(\Omega)\setminus E$. On the other hand, $\frac{M^a_\Omega(P,O)}{h^a_{\eta,b}(P)} = \kappa W(r)V^\eta(r)\varphi(\Theta)^{1-b} \to 0$ as $r\to\infty$. Thus we obtain

$$\frac{u(P)}{h^a_{\eta,b}(P)} = \frac{u_1(P) + c_o(u,a)M^a_\Omega(P,Q)}{h^a_{\eta,b}(P)} \to 0 \quad \text{as } r\to\infty, \quad P = (r,\Theta) \in C_n(\Omega)\setminus E.$$

Conversely we take an unbounded and $\mathcal{L}_a^b$-minimally thin set E. It is well known that if U is bounded, then

$$\lambda_{U,b}^a(\overline{C_n(\Omega)}) = \inf\{\lambda_{O,b}^a(\overline{C_n(\Omega)}); U \subset O, O \text{ is open}\}.$$

By applying the above property to E_k ($k = 0, 1, 2, \ldots$), we get an open set $O \supset E$ such that O is $\mathcal{L}_a^b$-minimally thin. According to Lemma 3.3 we have

$$\sum_{k=0}^{\infty} \widehat{R}_{h_{\eta,b}^a}^{O_k} \in \mathfrak{F}_\eta^a,$$

where $O_k = O \cap I_k$, which implies

$$\sum_{k=0}^{\infty} \int \widehat{R}_{h_{\eta,b}^a}^{O_k}(P)\mathrm{d}\nu_\eta(P) < \infty.$$

We can find an increasing sequence c_k of positive numbers such that $c_k \uparrow \infty$ and

$$\sum_{k=0}^{\infty} c_k \int \widehat{R}_{h_{\eta,b}^a}^{O_k}(P)\mathrm{d}\nu_\eta(P) < \infty.$$

Put

$$u(P) = \sum_{k=0}^{\infty} \widehat{R}_{h_{\eta,b}^a}^{O_k}(P).$$

By Lebesgue's monotone convergence theorem, we see that $u \in \mathfrak{F}_\eta^a$. Since $O_k \subsetneq O_k \cup O_{k-1}$, we have

$$\widehat{R}_{h_{\eta,b}^a}^{O_{k-1}}(P) + \widehat{R}_{h_{\eta,b}^a}^{O_k}(P) \geq \widehat{R}_{h_{\eta,b}^a}^{O_k}(P) \geq h_{\eta,b}^a(P)$$

for $P \in O_k$. Hence, if $P = (r, \Theta) \in E_k \subset O_k$, then

$$u(P) \geq c_{k-1}\widehat{R}_{h_{\eta,b}^a}^{O_{k-1}}(P) + c_k\widehat{R}_{h_{\eta,b}^a}^{O_k}(P) \geq c_{k-1}h_{\eta,b}^a(P).$$

Therefore

$$\lim_{r\to\infty, P\in E} \frac{u(P)}{h_{\eta,b}^a(P)} = \infty$$

holds. □

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