

# Inversion Formula for the Dunkl-Wigner Transform and Compactness Property for the Dunkl-Weyl Transforms

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**Abstract** We define and study the Fourier-Wigner transform associated with the Dunkl operators, and we prove for this transform an inversion formula. Next, we introduce and study the Weyl transforms  $W_\sigma$  associated with the Dunkl operators, where  $\sigma$  is a symbol in the Schwartz space  $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ . An integral relation between the precedent Weyl and Wigner transforms is given. At last, we give criteria in terms of  $\sigma$  for boundedness and compactness of the transform  $W_\sigma$ .

**Keywords** Dunkl transform; Dunkl-Wigner transform; Dunkl-Weyl transforms; inversion formula; boundedness and compactness

**MR(2010) Subject Classification** 42B10; 44A20; 46F12

## 1. Introduction

In this paper, we consider  $\mathbb{R}^d$  with the Euclidean inner product  $\langle \cdot, \cdot \rangle$  and norm  $|y| := \sqrt{\langle y, y \rangle}$ . For  $\alpha \in \mathbb{R}^d \setminus \{0\}$ , let  $\sigma_\alpha$  be the reflection in the hyperplane  $H_\alpha \subset \mathbb{R}^d$  orthogonal to  $\alpha$ :

$$\sigma_\alpha y := y - \frac{2\langle \alpha, y \rangle}{|\alpha|^2} \alpha.$$

A finite set  $\mathfrak{R} \subset \mathbb{R}^d \setminus \{0\}$  is called a root system, if  $\mathfrak{R} \cap \mathbb{R}\alpha = \{-\alpha, \alpha\}$  and  $\sigma_\alpha \mathfrak{R} = \mathfrak{R}$  for all  $\alpha \in \mathfrak{R}$ . We assume that it is normalized by  $|\alpha|^2 = 2$  for all  $\alpha \in \mathfrak{R}$ . For a root system  $\mathfrak{R}$ , the reflections  $\sigma_\alpha$ ,  $\alpha \in \mathfrak{R}$ , generate a finite group  $G$ . The Coxeter group  $G$  is a subgroup of the orthogonal group  $O(d)$ . All reflections in  $G$ , correspond to suitable pairs of roots. For a given  $\beta \in \mathbb{R}^d \setminus \bigcup_{\alpha \in \mathfrak{R}} H_\alpha$ , we fix the positive subsystem  $\mathfrak{R}_+ := \{\alpha \in \mathfrak{R} : \langle \alpha, \beta \rangle > 0\}$ . Then for each  $\alpha \in \mathfrak{R}$  either  $\alpha \in \mathfrak{R}_+$  or  $-\alpha \in \mathfrak{R}_+$ .

Let  $k : \mathfrak{R} \rightarrow \mathbb{C}$  be a multiplicity function on  $\mathfrak{R}$  (a function which is constant on the orbits under the action of  $G$ ). As an abbreviation, we introduce the index  $\gamma = \gamma_k := \sum_{\alpha \in \mathfrak{R}_+} k(\alpha)$ .

Throughout this paper, we will assume that  $k(\alpha) \geq 0$  for all  $\alpha \in \mathfrak{R}$ . Moreover, let  $w_k$  denote the weight function  $w_k(y) := \prod_{\alpha \in \mathfrak{R}_+} |\langle \alpha, y \rangle|^{2k(\alpha)}$ , for all  $y \in \mathbb{R}^d$ , which is  $G$ -invariant and homogeneous of degree  $2\gamma$ .

Let  $c_k$  be the Mehta-type constant given by  $c_k := (\int_{\mathbb{R}^d} e^{-|y|^2/2} w_k(y) dy)^{-1}$ . We denote by  $\mu_k$  the measure on  $\mathbb{R}^d$  given by  $d\mu_k(y) := c_k w_k(y) dy$ ; and by  $L^p(\mu_k)$ ,  $1 \leq p \leq \infty$ , the space of

Received January 21, 2015; Accepted April 27, 2015

Supported by the DGRST Research Project LR11ES11 and CMCU Program 10G/1503.

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measurable functions  $f$  on  $\mathbb{R}^d$ , such that

$$\|f\|_{L^p(\mu_k)} := \left( \int_{\mathbb{R}^d} |f(y)|^p d\mu_k(y) \right)^{1/p} < \infty, \quad 1 \leq p < \infty,$$

$$\|f\|_{L^\infty(\mu_k)} := \operatorname{ess\,sup}_{y \in \mathbb{R}^d} |f(y)| < \infty,$$

and by  $L^p_{\text{rad}}(\mu_k)$  the subspace of  $L^p(\mu_k)$  consisting of radial functions.

For  $f \in L^1(\mu_k)$  the Dunkl transform of  $f$  is defined by [3]

$$\mathcal{F}_k(f)(x) := \int_{\mathbb{R}^d} E_k(-ix, y) f(y) d\mu_k(y), \quad x \in \mathbb{R}^d,$$

where  $E_k(-ix, y)$  denotes the Dunkl kernel (For more details see the next section).

The Dunkl translation operators  $\tau_x$ ,  $x \in \mathbb{R}^d$ , [10] are defined on  $L^2(\mu_k)$  by

$$\mathcal{F}_k(\tau_x f)(y) = E_k(ix, y) \mathcal{F}_k(f)(y), \quad y \in \mathbb{R}^d.$$

Using these results, we define the Dunkl-Wigner transform  $V$ , by

$$V(f, g)(x, y) := \int_{\mathbb{R}^d} f(t) \tau_x g(-t) E_k(-iy, t) d\mu_k(t), \quad f, g \in L^2(\mu_k).$$

Next, we study some of its properties, and we prove an inversion formula for this transform.

Next, we introduce the Dunkl-Weyl transform  $W_\sigma$ , by

$$W_\sigma(f)(x) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sigma(y, z) E_k(ix, z) \tau_x f(-y) d\mu_k(y) d\mu_k(z), \quad f \in L^2(\mu_k),$$

with  $\sigma$  in the Schwartz space  $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ , and we give its connection with the Dunkl-Wigner transform  $V$ . Furthermore, we prove that for  $\sigma$  in  $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ , the transform  $W_\sigma$  is a compact operator from  $L^2(\mu_k)$  into itself. At last, we define  $W_\sigma$  for  $\sigma$  in the spaces  $L^p(\mu_k \otimes \mu_k)$ , with  $p \in [1, 2]$ , and we establish that  $W_\sigma$  is again a compact operator.

In the classical case, the Fourier-Wigner transform and the Weyl transform were studied by Weyl [12] and Wong [13]. In the Bessel-Kingman hypergroups, these operators were studied by Dachraoui [1].

This paper is organized as follows. In Section 2, we recall some properties of harmonic analysis for the Dunkl operators. In Section 3, we define the Fourier-Wigner transform  $V$  in the Dunkl setting, and we have established for it an inversion formula. In Section 4, we introduce and study the Dunkl-Weyl transforms  $W_\sigma$  for  $\sigma$  in  $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ ; and we prove that these transforms are compact operators from  $L^2(\mu_k)$  into itself. In Section 5, we define  $W_\sigma$  for  $\sigma$  in  $L^p(\mu_k \otimes \mu_k)$ , with  $p \in [1, 2]$ , and we prove the boundedness and compactness of these transforms on these spaces. In Section 6, we define  $W_\sigma$  for  $\sigma$  in  $\mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ .

## 2. The Dunkl analysis on $\mathbb{R}^d$

The Dunkl operators  $\mathcal{D}_j$ ;  $j = 1, \dots, d$ , on  $\mathbb{R}^d$  associated with the finite reflection group  $G$  and multiplicity function  $k$  are given, for a function  $f$  of class  $C^1$  on  $\mathbb{R}^d$ , by

$$\mathcal{D}_j f(y) := \frac{\partial}{\partial y_j} f(y) + \sum_{\alpha \in \mathfrak{R}_+} k(\alpha) \alpha_j \frac{f(y) - f(\sigma_\alpha y)}{\langle \alpha, y \rangle}.$$

For  $y \in \mathbb{R}^d$ , the initial problem  $\mathcal{D}_j u(\cdot, y)(x) = y_j u(x, y)$ ,  $j = 1, \dots, d$ , with  $u(0, y) = 1$  admits a unique analytic solution on  $\mathbb{R}^d$ , which will be denoted by  $E_k(x, y)$  and called Dunkl kernel [2,5]. This kernel has a unique analytic extension to  $\mathbb{C}^d \times \mathbb{C}^d$ . The Dunkl kernel has the Laplace-type representation [6]

$$E_k(x, y) = \int_{\mathbb{R}^d} e^{\langle y, z \rangle} d\Gamma_x(z), \quad x \in \mathbb{R}^d, \quad y \in \mathbb{C}^d, \tag{2.1}$$

where  $\langle y, z \rangle := \sum_{i=1}^d y_i z_i$  and  $\Gamma_x$  is a probability measure on  $\mathbb{R}^d$ , such that  $\text{supp}(\Gamma_x) \subset \{z \in \mathbb{R}^d : |z| \leq |x|\}$ . In our case,

$$|E_k(ix, y)| \leq 1, \quad x, y \in \mathbb{R}^d. \tag{2.2}$$

The Dunkl kernel gives rise to an integral transform, which is called Dunkl transform on  $\mathbb{R}^d$ , and was introduced by Dunkl in [3], where already many basic properties were established. Dunkl's results were completed and extended later by De Jeu [5]. The Dunkl transform of a function  $f$  in  $L^1(\mu_k)$ , is defined by

$$\mathcal{F}_k(f)(x) := \int_{\mathbb{R}^d} E_k(-ix, y) f(y) d\mu_k(y), \quad x \in \mathbb{R}^d.$$

We notice that  $\mathcal{F}_0$  agrees with the Fourier transform  $\mathcal{F}$  that is given by

$$\mathcal{F}(f)(x) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\langle x, y \rangle} f(y) dy, \quad x \in \mathbb{R}^d.$$

Some of the properties of Dunkl transform  $\mathcal{F}_k$  are collected bellow [3,5].

**Theorem 2.1** (i)  $L^1 - L^\infty$ -boundedness. For all  $f \in L^1(\mu_k)$ ,  $\mathcal{F}_k(f) \in L^\infty(\mu_k)$ , and

$$\|\mathcal{F}_k(f)\|_{L^\infty(\mu_k)} \leq \|f\|_{L^1(\mu_k)}.$$

(ii) The Dunkl transform  $\mathcal{F}_k$  is a topological isomorphism from  $\mathcal{S}(\mathbb{R}^d)$  onto itself.

(iii) Inversion theorem. Let  $f \in L^1(\mu_k)$ , such that  $\mathcal{F}_k(f) \in L^1(\mu_k)$ . Then

$$f(x) = \mathcal{F}(\mathcal{F}_k(f))(-x), \quad \text{a.e. } x \in \mathbb{R}^d. \tag{2.3}$$

(iv) Plancherel theorem. The Dunkl transform  $\mathcal{F}_k$  extends uniquely to an isometric isomorphism of  $L^2(\mu_k)$  onto itself. In particular,

$$\|f\|_{L^2(\mu_k)} = \|\mathcal{F}_k(f)\|_{L^2(\mu_k)}. \tag{2.4}$$

The Dunkl transform  $\mathcal{F}_k$  allows us to define a generalized translation operators on  $L^2(\mu_k)$  by setting

$$\mathcal{F}_k(\tau_x f)(y) = E_k(ix, y) \mathcal{F}_k(f)(y), \quad y \in \mathbb{R}^d. \tag{2.5}$$

It is the definition of Thangavelu and Xu given in [10]. It plays the role of the ordinary translation  $\tau_x f = f(x + \cdot)$  in  $\mathbb{R}^d$ , since the Euclidean Fourier transform satisfies  $\mathcal{F}(\tau_x f)(y) = e^{ixy} \mathcal{F}(f)(y)$ . Note that from (2.2) and (2.4), the definition (2.5) makes sense, and

$$\|\tau_x f\|_{L^2(\mu_k)} \leq \|f\|_{L^2(\mu_k)}, \quad f \in L^2(\mu_k). \tag{2.6}$$

Rösler [7] introduced the Dunkl translation operators for radial functions. If  $f$  are radial functions,  $f(x) = F(|x|)$ , then

$$\tau_x f(y) = \int_{\mathbb{R}^d} F\left(\sqrt{|x|^2 + |y|^2 + 2\langle y, z \rangle}\right) d\Gamma_x(z); \quad x, y \in \mathbb{R}^d,$$

where  $\Gamma_x$  is the representing measure given by (2.1).

This formula allows us to establish the following results [10,11].

**Proposition 2.2** (i) For all  $p \in [1, 2]$  and for all  $x \in \mathbb{R}^d$ , the Dunkl translation  $\tau_x : L^p_{\text{rad}}(\mu_k) \rightarrow L^p(\mu_k)$  is a bounded operator, and for  $f \in L^p_{\text{rad}}(\mu_k)$ ,

$$\|\tau_x f\|_{L^p(\mu_k)} \leq \|f\|_{L^p_{\text{rad}}(\mu_k)}. \tag{2.7}$$

(ii) Let  $f \in L^1_{\text{rad}}(\mu_k)$ . Then, for all  $x \in \mathbb{R}^d$ ,

$$\int_{\mathbb{R}^d} \tau_x f(y) d\mu_k(y) = \int_{\mathbb{R}^d} f(y) d\mu_k(y). \tag{2.8}$$

### 3. The Dunkl-Wigner transform

The Fourier-Wigner transform associated to the Dunkl operators, is the mapping  $V$  defined on  $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$  by

$$V(f, g)(x, y) := \int_{\mathbb{R}^d} f(t) \tau_x g(-t) E_k(-iy, t) d\mu_k(t), \quad x, y \in \mathbb{R}^d. \tag{3.1}$$

The transform  $V$  can also be written in the form

$$V(f, g)(x, y) = \mathcal{F}_k(f \widetilde{\tau_x g})(y), \quad \widetilde{f}(x) = f(-x). \tag{3.2}$$

**Proposition 3.1** (i) The Dunkl-Wigner transform  $V$  is a bilinear, continuous mapping from  $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$  into  $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ .

(ii) For  $f, g \in L^2(\mu_k)$ , then  $V(f, g) \in L^\infty \cap L^2(\mu_k \otimes \mu_k)$ , and

$$\|V(f, g)\|_{L^\infty(\mu_k \otimes \mu_k)} \leq \|f\|_{L^2(\mu_k)} \|g\|_{L^2(\mu_k)}, \tag{3.3}$$

$$\|V(f, g)\|_{L^2(\mu_k \otimes \mu_k)} \leq \|f\|_{L^2(\mu_k)} \|g\|_{L^2(\mu_k)}. \tag{3.4}$$

(iii) Let  $p \in [1, 2]$  and  $q$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ . For  $(f, g) \in L^q(\mu_k) \times L^p_{\text{rad}}(\mu_k)$ , then  $V(f, g) \in L^\infty(\mu_k \otimes \mu_k)$ , and

$$\|V(f, g)\|_{L^\infty(\mu_k \otimes \mu_k)} \leq \|f\|_{L^q(\mu_k)} \|g\|_{L^p_{\text{rad}}(\mu_k)}. \tag{3.5}$$

**Proof** (i) Let  $f, g \in \mathcal{S}(\mathbb{R}^d)$ , and let  $F$  be the function defined on  $\mathbb{R}^d \times \mathbb{R}^d$  by

$$F(x, y) := f(y) \tau_x g(-y).$$

Then  $V(f, g)(x, y) = (I \otimes \mathcal{F}_k)(F)(x, y)$ , where  $I$  is the identity operator. This with Theorem 2.1 (ii) gives (i).

(ii) We get (3.3) from (3.1), Hölder’s inequality and relation (2.6).

We obtain (3.4) from (3.2), (2.4), Minkowski’s inequality for integrals [4, p.186], and from (2.6).

(iii) We deduce (3.5) from (3.1), Hölder's inequality and relation (2.7).  $\square$

**Proposition 3.2** *Let  $f, g \in \mathcal{S}(\mathbb{R}^d)$ . Then for all  $\xi, \lambda \in \mathbb{R}^d$ ,*

$$\mathcal{F}_k \otimes \mathcal{F}_k^{-1}[V(f, g)](\xi, \lambda) = E_k(-i\lambda, \xi)f(\lambda)\mathcal{F}_k(g)(\xi).$$

**Proof** Let  $f, g \in \mathcal{S}(\mathbb{R}^d)$ . From (3.1), (3.2) and Fubini's theorem,

$$\begin{aligned} \mathcal{F}_k \otimes \mathcal{F}_k^{-1}[V(f, g)](\xi, \lambda) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} V(f, g)(x, y)E_k(-i\xi, x)E_k(i\lambda, y)d\mu_k(x)d\mu_k(y) \\ &= \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}^d} \mathcal{F}_k(f\widehat{\tau_x g})(y)E_k(i\lambda, y)d\mu_k(y) \right] E_k(-i\xi, x)d\mu_k(x) \\ &= f(\lambda) \int_{\mathbb{R}^d} \tau_x g(-\lambda)E_k(-i\xi, x)d\mu_k(x). \end{aligned}$$

Then, by (2.5),

$$\mathcal{F}_k \otimes \mathcal{F}_k^{-1}[V(f, g)](\xi, \lambda) = f(\lambda)\mathcal{F}_k(\tau_{-\lambda}g)(\xi) = E_k(-i\lambda, \xi)f(\lambda)\mathcal{F}_k(g)(\xi),$$

which completes the proof.  $\square$

**Corollary 3.3** *Let  $f, g \in \mathcal{S}(\mathbb{R}^d)$ . Then*

- (i)  $\int_{\mathbb{R}^d} \mathcal{F}_k \otimes \mathcal{F}_k^{-1}[V(f, g)](\xi, \lambda)d\mu_k(\lambda) = \mathcal{F}_k(f)(\xi)\mathcal{F}_k(g)(\xi), \xi \in \mathbb{R}^d,$
- (ii)  $\int_{\mathbb{R}^d} \mathcal{F}_k \otimes \mathcal{F}_k^{-1}[V(f, g)](\xi, \lambda)d\mu_k(\xi) = f(\lambda)g(-\lambda), \lambda \in \mathbb{R}^d.$

**Theorem 3.4** *Let  $g \in L^1_{\text{rad}} \cap L^2(\mu_k)$  such that  $c = \int_{\mathbb{R}^d} g(x)d\mu_k(x) \neq 0$ . Then for all  $f \in L^1 \cap L^2(\mu_k)$ ,*

$$\mathcal{F}_k(f)(y) = \frac{1}{c} \int_{\mathbb{R}^d} V(f, g)(x, y)d\mu_k(x).$$

**Proof** Using (3.1), Fubini's theorem and (2.8),

$$\begin{aligned} \int_{\mathbb{R}^d} V(f, g)(x, y)d\mu_k(x) &= \int_{\mathbb{R}^d} E_k(-iy, t)f(t) \left[ \int_{\mathbb{R}^d} \tau_x g(-t)d\mu_k(x) \right] d\mu_k(t) \\ &= c\mathcal{F}_k(f)(y), \end{aligned}$$

where  $c = \int_{\mathbb{R}^d} g(x)d\mu_k(x)$ .  $\square$

**Corollary 3.5** *Let  $g \in L^1_{\text{rad}} \cap L^2(\mu_k)$  such that  $c = \int_{\mathbb{R}^d} g(x)d\mu_k(x), c \neq 0$ . Then*

- (i) *For all  $f \in L^1 \cap L^2(\mu_k)$  such that  $\mathcal{F}_k(f) \in L^1(\mu_k)$ ,*

$$f(z) = \frac{1}{c} \int_{\mathbb{R}^d} E_k(iy, z) \left[ \int_{\mathbb{R}^d} V(f, g)(x, y)d\mu_k(x) \right] d\mu_k(y).$$
- (ii) *For all  $f \in L^1 \cap L^2(\mu_k)$ ,*

$$\|f\|_{L^2(\mu_k)}^2 = \frac{1}{c^2} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} V(f, g)(x, y)d\mu_k(x) \right|^2 d\mu_k(y).$$

#### 4. The Dunkl-Weyl transforms

In this section, we introduce and study the Weyl transform associated to the Dunkl operators. Let  $\sigma \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ , we define the Weyl transform  $W_\sigma$  associated to the Dunkl operators

on  $\mathcal{S}(\mathbb{R}^d)$ , by

$$W_\sigma(f)(x) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sigma(y, z) E_k(-ix, z) \tau_x f(-y) d\mu_k(y) d\mu_k(z), \quad x \in \mathbb{R}^d. \quad (4.1)$$

**Proposition 4.1** *Let  $\sigma \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ . Then  $W_\sigma$  is continuous from  $\mathcal{S}(\mathbb{R}^d)$  into itself.*

**Proof** Let  $f \in \mathcal{S}(\mathbb{R}^d)$ . From Theorem 2.1 (ii) and (2.3),

$$\tau_x f(-y) = \int_{\mathbb{R}^d} E_k(ix, t) E_k(-iy, t) \mathcal{F}_k(f)(t) d\mu_k(t), \quad x, y \in \mathbb{R}^d.$$

Then, by (4.1) and Fubini's theorem,

$$\begin{aligned} W_\sigma(f)(x) &= \int_{\mathbb{R}^d} E_k(-ix, z) \left[ \int_{\mathbb{R}^d} E_k(ix, t) \mathcal{F}_k(f)(t) \right. \\ &\quad \left. \left\{ \int_{\mathbb{R}^d} \sigma(y, z) E_k(-iy, t) d\mu_k(y) \right\} d\mu_k(t) \right] d\mu_k(z) \\ &= \int_{\mathbb{R}^d} E_k(-ix, z) \left[ \int_{\mathbb{R}^d} E_k(ix, t) \mathcal{F}_k(f)(t) \right. \\ &\quad \left. \mathcal{F}_k(\sigma(\cdot, z))(t) d\mu_k(t) \right] d\mu_k(z). \end{aligned}$$

Now the function  $(t, z) \rightarrow \mathcal{F}_k(\sigma(\cdot, z))(t)$  belongs to  $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ .

On the other hand, the mapping  $f \rightarrow G_f$ , given by

$$G_f(t, z) = \mathcal{F}_k(f)(t) \mathcal{F}_k(\sigma(\cdot, z))(t), \quad t, z \in \mathbb{R}^d,$$

is continuous from  $\mathcal{S}(\mathbb{R}^d)$  into  $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ , and for all  $x \in \mathbb{R}^d$ ,

$$\begin{aligned} W_\sigma(f)(x) &= \int_{\mathbb{R}^d} E_k(-ix, z) \left[ \int_{\mathbb{R}^d} E_k(ix, t) G_f(t, z) d\mu_k(t) \right] d\mu_k(z) \\ &= \mathcal{F}_k^{-1} \otimes \mathcal{F}_k(G_f)(x, x). \end{aligned}$$

We deduce the result from the fact that  $\mathcal{F}_k^{-1} \otimes \mathcal{F}_k$  is an isomorphism from  $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$  onto itself.

□

**Lemma 4.2** *Let  $\sigma \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ . Then, the function  $h$  defined on  $\mathbb{R}^d \times \mathbb{R}^d$  by*

$$h(x, y) := \int_{\mathbb{R}^d} E_k(-ix, z) \tau_x[\sigma(\cdot, z)](-y) d\mu_k(z) \quad (4.2)$$

*belongs to  $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ .*

**Proof** The function  $h$  can be written in the form

$$h(x, y) = \tau_x[\mathcal{F}_k^{-1}(G(\cdot, x))](y) = \tau_x[(I \otimes \mathcal{F}_k^{-1})(G)(\cdot, x)](-y),$$

where  $G(t, x) = \int_{\mathbb{R}^d} E_k(-ix, z) \mathcal{F}_k(\sigma(\cdot, z))(t) d\mu_k(z)$ . Now the function  $(t, x) \rightarrow G(t, x)$  belongs to  $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ . Thus, by Theorem 2.1 (ii), we deduce that the function  $(I \otimes \mathcal{F}_k^{-1})(G)$  belongs to  $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ . Then the result follows from the fact that for all  $g \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ , the function  $(x, y) \rightarrow \tau_x[g(\cdot, x)](-y)$  belongs to  $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ . □

**Theorem 4.3** *Let  $\sigma \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ .*

(i) For all  $f \in \mathcal{S}(\mathbb{R}^d)$ ,

$$W_\sigma(f)(x) = \int_{\mathbb{R}^d} h(x, y)f(y)d\mu_k(y),$$

where  $h(x, y)$  is the kernel given by (4.2).

(ii) For all  $f \in \mathcal{S}(\mathbb{R}^d)$  and  $p, q \in [1, \infty]$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ ,

$$\|W_\sigma(f)\|_{L^q(\mu_k)} \leq \|h\|_{L^q(\mu_k \otimes \mu_k)} \|f\|_{L^p(\mu_k)}.$$

(iii) For  $p, q \in [1, \infty[$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ , the operator  $W_\sigma$  can be extended to a bounded operator from  $L^p(\mu_k)$  into  $L^q(\mu_k)$ . In particular  $W_\sigma : L^2(\mu_k) \rightarrow L^2(\mu_k)$  is a Hilbert-Schmidt operator, and consequently it is compact.

**Proof** (i) Let  $f \in \mathcal{S}(\mathbb{R}^d)$ . From (4.1),

$$W_\sigma(f)(x) = \int_{\mathbb{R}^d} E_k(-ix, z) \left[ \int_{\mathbb{R}^d} \tau_x f(-y)\sigma(y, z)d\mu_k(y) \right] d\mu_k(z).$$

Using Fubini's theorem, and the equality

$$\int_{\mathbb{R}^d} \tau_x f(-y)\sigma(y, z)d\mu_k(y) = \int_{\mathbb{R}^d} f(y)\tau_x[\sigma(\cdot, z)](-y)d\mu_k(y),$$

we deduce that

$$W_\sigma(f)(x) = \int_{\mathbb{R}^d} h(x, y)f(y)d\mu_k(y),$$

where  $h(x, y) = \int_{\mathbb{R}^d} E_k(-ix, z)\tau_x[\sigma(\cdot, z)](-y)d\mu_k(z)$ .

(ii) Follows from (i), Hölder's inequality, and Lemma 4.2.

(iii) From (ii) and the fact that the space  $\mathcal{S}(\mathbb{R}^d)$  is dense in  $L^p(\mu_k)$ ,  $p \in [1, \infty[$ , we deduce that  $W_\sigma$  can be extended to a continuous mapping from  $L^p(\mu_k)$  into  $L^q(\mu_k)$ .

By Lemma 4.2, the kernel  $h$  belongs to  $L^2(\mu_k \otimes \mu_k)$ , hence  $W_\sigma$  is a Hilbert-Schmidt operator. In particular, it is compact.  $\square$

### 5. The transforms $W_\sigma$ with $\sigma \in L^p(\mu_k \otimes \mu_k)$ , $p \in [1, 2]$

In this section, we prove that the Weyl transform with symbol in  $L^p(\mu_k \otimes \mu_k)$ ,  $p \in [1, 2]$ , is a compact operator.

We denote by  $\mathcal{B}(L^2(\mu_k))$  the  $\mathbb{C}^*$ -algebra of bounded operators  $\Psi$  from  $L^2(\mu_k)$  into itself, equipped with the norm

$$\|\Psi\| := \sup_{\|f\|_{L^2(\mu_k)}=1} \|\Psi(f)\|_{L^2(\mu_k)}.$$

Let  $\sigma \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ . Define the operator  $H_\sigma$  on  $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ , by

$$H_\sigma(f, g)(z) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sigma(x, y)E_k(-iz, y)V(f, g)(x, y)d\mu_k(x)d\mu_k(y), \quad z \in \mathbb{R}^d. \quad (5.1)$$

**Lemma 5.1** Let  $\sigma \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ . For all  $f, g \in \mathcal{S}(\mathbb{R}^d)$ ,

$$H_\sigma(f, g)(0) = \langle W_\sigma(\tilde{g}), \tilde{f} \rangle_{L^2(\mu_k)},$$

where  $\langle \cdot, \cdot \rangle_{L^2(\mu_k)}$  is the inner product of  $L^2(\mu_k)$ .

**Proof** From (3.1) and (5.1),

$$\begin{aligned} H_\sigma(f, g)(0) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sigma(x, y) V(f, g)(x, y) d\mu_k(x) d\mu_k(y) \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sigma(x, y) \left[ \int_{\mathbb{R}^d} f(t) \tau_x g(-t) E_k(-iy, t) d\mu_k(t) \right] d\mu_k(x) d\mu_k(y). \end{aligned}$$

From Fubini's theorem, we get

$$H_\sigma(f, g)(0) = \int_{\mathbb{R}^d} f(t) \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sigma(x, y) \tau_x g(-t) E_k(-iy, t) d\mu_k(x) d\mu_k(y) \right] d\mu_k(t).$$

Using the fact  $\tau_x g(-t) = \tau_t \tilde{g}(-x)$ , then by (4.1) we obtain

$$H_\sigma(f, g)(0) = \int_{\mathbb{R}^d} f(t) W_\sigma(\tilde{g})(t) d\mu_k(t) = \langle W_\sigma(\tilde{g}), \bar{f} \rangle_{L^2(\mu_k)},$$

which completes the proof.  $\square$

**Theorem 5.2** For  $p \in [1, 2]$ , there exists a unique bounded operator

$$Q : L^p(\mu_k \otimes \mu_k) \rightarrow \mathcal{B}(L^2(\mu_k)),$$

whose action is denoted by  $\sigma \rightarrow Q_\sigma$ , such that for all  $f, g \in \mathcal{S}(\mathbb{R}^d)$ ,

$$\begin{aligned} \langle Q_\sigma(g), \bar{f} \rangle_{L^2(\mu_k)} &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sigma(x, y) V(f, \tilde{g})(x, y) d\mu_k(x) d\mu_k(y), \\ \|Q_\sigma\| &\leq \|\sigma\|_{L^p(\mu_k \otimes \mu_k)}. \end{aligned}$$

**Proof** (i) The case  $p = 2$ . Let  $\sigma \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ . For  $g \in \mathcal{S}(\mathbb{R}^d)$ , we put

$$Q_\sigma(g) = W_\sigma(g). \tag{5.2}$$

From Lemma 5.1, we obtain

$$\begin{aligned} \langle Q_\sigma(g), \bar{f} \rangle_{L^2(\mu_k)} &= \langle W_\sigma(g), \bar{f} \rangle_{L^2(\mu_k)} = H_\sigma(f, \tilde{g})(0) \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sigma(x, y) V(f, \tilde{g})(x, y) d\mu_k(x) d\mu_k(y). \end{aligned}$$

On the other hand, from Hölder's inequality and (3.4),

$$|\langle Q_\sigma(g), \bar{f} \rangle_{L^2(\mu_k)}| \leq \|\sigma\|_{L^2(\mu_k \otimes \mu_k)} \|f\|_{L^2(\mu_k)} \|g\|_{L^2(\mu_k)}.$$

This implies that  $Q_\sigma \in \mathcal{B}(L^2(\mu_k))$  and

$$\|Q_\sigma\| \leq \|\sigma\|_{L^2(\mu_k \otimes \mu_k)}. \tag{5.3}$$

Now, we consider  $\sigma \in L^2(\mu_k \otimes \mu_k)$ . Let  $(\sigma_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$  such that  $\|\sigma_n - \sigma\|_{L^2(\mu_k \otimes \mu_k)} \rightarrow 0$  as  $n \rightarrow \infty$ . From (5.3) we have, for all  $m, n \in \mathbb{N}$ ,

$$\|Q_{\sigma_m} - Q_{\sigma_n}\| \leq \|\sigma_m - \sigma_n\|_{L^2(\mu_k \otimes \mu_k)} \leq \|\sigma_m - \sigma\|_{L^2(\mu_k \otimes \mu_k)} + \|\sigma_n - \sigma\|_{L^2(\mu_k \otimes \mu_k)}.$$

Thus  $(Q_{\sigma_n})_{n \in \mathbb{N}}$  is a Cauchy sequence in  $\mathcal{B}(L^2(\mu_k))$ . Let it converge to  $Q_\sigma$ . Again by relation (5.3), the limit  $Q_\sigma$  is independent of the choice of  $(\sigma_n)_{n \in \mathbb{N}}$  and

$$\|Q_\sigma\| = \lim_{n \rightarrow \infty} \|Q_{\sigma_n}\| \leq \lim_{n \rightarrow \infty} \|\sigma_n\|_{L^2(\mu_k \otimes \mu_k)} \leq \|\sigma\|_{L^2(\mu_k \otimes \mu_k)}.$$



On the other hand, for  $f, g \in \mathcal{S}(\mathbb{R}^d)$ ,

$$\begin{aligned} \langle Q_\sigma(g), \bar{f} \rangle_{L^2(\mu_k)} &= \lim_{n \rightarrow \infty} \langle Q_{\sigma_n}(g), \bar{f} \rangle_{L^2(\mu_k)} \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sigma_n(x, y) V(f, \tilde{g})(x, y) d\mu_k(x) d\mu_k(y) \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sigma(x, y) V(f, \tilde{g})(x, y) d\mu_k(x) d\mu_k(y). \end{aligned}$$

(ii) The case  $p = 1$ . For  $\sigma \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ , we consider the the operator  $Q_\sigma$  defined by (5.2). Then from Hölder's inequality and (3.3), for  $f, g \in \mathcal{S}(\mathbb{R}^d)$ ,

$$\begin{aligned} |\langle Q_\sigma(g), \bar{f} \rangle_{L^2(\mu_k)}| &\leq \|\sigma\|_{L^1(\mu_k \otimes \mu_k)} \|V(f, \tilde{g})\|_{L^\infty(\mu_k \otimes \mu_k)} \\ &\leq \|\sigma\|_{L^1(\mu_k \otimes \mu_k)} \|f\|_{L^2(\mu_k)} \|g\|_{L^2(\mu_k)}. \end{aligned}$$

This implies that  $Q_\sigma \in \mathcal{B}(L^2(\mu_k))$  and  $\|Q_\sigma\| \leq \|\sigma\|_{L^1(\mu_k \otimes \mu_k)}$ . Using the same proof as of (i), we obtain for all  $\sigma \in L^1(\mu_k \otimes \mu_k)$ ,  $\|Q_\sigma\| \leq \|\sigma\|_{L^1(\mu_k \otimes \mu_k)}$ .

(iii) Using the cases  $p = 1$ ,  $p = 2$ , and the Riesz-Thorin theorem [8,9], we complete the proof for all  $p \in [1, 2]$ .  $\square$

**Remark 5.3** It is natural to denote  $Q_\sigma$  by  $W_\sigma$  for any  $\sigma \in L^p(\mu_k \otimes \mu_k)$ ,  $p \in [1, 2]$ .

**Theorem 5.4** For  $\sigma \in L^p(\mu_k \otimes \mu_k)$ ,  $p \in [1, 2]$ , the operator  $Q_\sigma$  from  $L^2(\mu_k)$  into itself is a compact operator.

**Proof** Let  $\sigma \in L^p(\mu_k \otimes \mu_k)$ ,  $p \in [1, 2]$ , and let  $(\sigma_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ , such that  $\lim_{n \rightarrow \infty} \|\sigma_n - \sigma\|_{L^p(\mu_k \otimes \mu_k)} = 0$ . From Theorem 5.2, we have  $\|Q_{\sigma_n} - Q_\sigma\| \leq \|\sigma_n - \sigma\|_{L^p(\mu_k \otimes \mu_k)}$ . This implies that  $\lim_{n \rightarrow \infty} Q_{\sigma_n} = Q_\sigma$ , in  $\mathcal{B}(L^2(\mu_k))$ . But from Theorem 4.3 (iii), we know that for all  $n \in \mathbb{N}$ , the operator  $W_{\sigma_n}$  is compact, then the result of the theorem follows from the fact that the subspace  $\mathcal{K}(L^2(\mu_k))$  of  $\mathcal{B}(L^2(\mu_k))$  consisting of compact operators is a closed ideal of  $\mathcal{B}(L^2(\mu_k))$ .  $\square$

### 6. The transforms $W_\sigma$ with $\sigma \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$

We denote by

- (i)  $\mathcal{S}'(\mathbb{R}^d)$  the space of tempered distributions on  $\mathbb{R}^d$ . It is the topological dual of  $\mathcal{S}(\mathbb{R}^d)$ .
- (ii)  $\mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$  the space of tempered distributions on  $\mathbb{R}^d \times \mathbb{R}^d$ . It is the topological dual of  $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ .

For  $\sigma \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$  and  $g \in \mathcal{S}(\mathbb{R}^d)$ , define the operator  $W_\sigma(g)$  on  $\mathcal{S}(\mathbb{R}^d)$ , by

$$[W_\sigma(g)](f) = \sigma(V(f, g)), \quad f \in \mathcal{S}(\mathbb{R}^d), \tag{6.1}$$

where  $V$  is the mapping given by (3.1).

From Proposition 3.1 (i), it is clear that  $W_\sigma(g)$  given by (6.1) belongs to  $\mathcal{S}'(\mathbb{R}^d)$ .

For a slowly increasing function  $h$  on  $\mathbb{R}^d \times \mathbb{R}^d$ , we denote by  $\sigma_h$  the element of  $\mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$

defined by

$$\sigma_h(F) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F(x, y) h(x, y) d\mu_k(x) d\mu_k(y). \quad (6.2)$$

Then, we have the following.

**Proposition 6.1** *Let  $\sigma_1 \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ , given by the function equal to 1. For  $g$  radial function in  $\mathcal{S}(\mathbb{R}^d)$ , we have  $W_{\sigma_1}(g) = c\delta$ , where  $c = \int_{\mathbb{R}^d} g(x) d\mu_k(x)$  and  $\delta$  is the Dirac distribution at 0.*

**Proof** By relations (6.1) and (6.2), we have for all  $f \in \mathcal{S}(\mathbb{R}^d)$ ,

$$[W_{\sigma_1}(g)](f) = \sigma_1(V(f, g)) = \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}^d} V(f, g)(x, y) d\mu_k(x) \right] d\mu_k(y),$$

and by Theorem 3.4,

$$[W_{\sigma_1}(g)](f) := \sigma_1(V(f, g)) = c \int_{\mathbb{R}^d} \mathcal{F}_k(f)(y) d\mu_k(y).$$

We complete the proof by using relation (2.3).  $\square$

**Remark 6.2** From Proposition 6.1, we deduce that there exists  $\sigma \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$  given by a function in  $L^\infty(\mu_k \otimes \mu_k)$ , such that for all  $g$  radial function in  $\mathcal{S}(\mathbb{R}^d)$  satisfying  $c = \int_{\mathbb{R}^d} g(x) d\mu_k(x) \neq 0$ , the distribution  $W_\sigma(g)$  is not given by a function of  $L^2(\mu_k)$ .

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