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Hamilton's Gradient Estimate for a Nonlinear Parabolic Equation on Riemannian Manifolds

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Abstract In this paper, we give a local Hamilton's gradient estimate for a nonlinear parabolic equation on Riemannian manifolds. As its application, a Harnack-type inequality and a Liouville-type theorem are obtained.

Keywords Hamilton's gradient estimate; nonlinear parabolic equation; Riemannian manifold

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1. Introduction

Let (M^n,g) be a complete Riemannian manifold. We consider the following equation on (M^n,g),

$$\frac{\partial v^{p-1}}{\partial t}=(p-1)^{p-1}\mathrm{div}(|\nabla v|^{p-2}\nabla v),\quad p>1, \tag{1.1}$$

where div and ∇ are, respectively, the divergence operator and the gradient operator of the metric g.

In 2006, Souplet and Zhang [7] got a local Hamilton's gradient estimate for heat equation on Riemannian manifolds. That is,

Theorem 1.1 ([7]) *Let M be a Riemannian manifold with dimension $n\ge 2$,* Ricci $\ge -k$, $k\ge 0$. *Suppose v is any positive solution to the heat equation in $Q_{R,T}\equiv B(x_0,R)\times[t_0-T,t_0]\subset M\times(-\infty,+\infty)$. Suppose also $v\le A$ in $Q_{R,T}$. Then there exists a dimensional constant c such that*

$$\frac{|\nabla v(x,t)|}{v(x,t)}\le c(\frac{1}{R}+\frac{1}{T^{1/2}}+\sqrt{k})(1+\ln\frac{A}{v(x,t)}),$$

in $Q_{R/2,T/2}$. Moreover, if M has nonnegative Ricci curvature and v is any positive solution of the heat equation on $M\times(0,\infty)$, then there exist dimensional constants c_1,c_2 such that

$$\frac{|\nabla v(x,t)|}{v(x,t)}\le c_1\frac{1}{t^{1/2}}(c_2+\ln\frac{v(x,2t)}{v(x,t)}),$$

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for all $x \in M$ *and* $t > 0$.

Hamilton firstly got this gradient estimates for the heat equation on compact manifold in [2]. Recently, there are some interesting results on the Hamilton's gradient estimates for nonlinear parabolic equation on Riemannian manifold in [5,8,10–12] and references therein.

As we know the heat equation is $p = 2$ in (1.1). When $1 < p < 2$ in (1.1), Hamilton's gradient estimates for positive continuous weak solutions to the equation (1.1) have been obtained by Wang in [8] (see also [9]). In this paper, we consider the positive smoothing solution to the equation (1.1) and $p > 2$. We derive a similar Hamilton's gradient estimate, when $2 < p < 2 + \frac{1+\sqrt{2n+3}}{2n+2}$.

Theorem 1.2 *Let* M^n *be a Riemannian manifold and sectional curvature* $K_M \geq -k^2, k \geq 0$. *Suppose that* v *is a positive and bounded solution to the equation (1.1) and* $2 < p < 2+\frac{1+\sqrt{2n+3}}{2n+2}$, *that is,* $0 < v \leq A$, *in* $Q_{R,T} = B(x_0, R) \times [t_0 - T, t_0] \subset M \times (-\infty, +\infty)$. *Then we obtain*

$$\frac{|\nabla v(x,t)|}{v(x,t)} \leq C(n,p)(\frac{1}{R} + \frac{1}{T^{\frac{1}{p}}} + k)(1 + (p-1)\ln\frac{A}{v(x,t)}) \tag{1.2}$$

in $Q_{R/2,T/2}$, *where* $C(n,p)$ *depends on* n *and* p.

Using the theorem, we get two corollaries. The first application is the following Harnack-type inequality:

Corollary 1.3 *Let* M *be a complete noncompact Riemannian manifold and sectional curvature* $K_M \geq -k^2, k \geq 0$. *Suppose that* v *is a positive and bounded solution to the equation (1.1) and* $2 < p < 2 + \frac{1+\sqrt{2n+3}}{2n+2}$, *that is,* $0 < v(x,t) \leq A$, $(x,t) \in M \times (0, +\infty)$. *Then for any* $x_1, x_2 \in M, t \in (0, +\infty)$ *there exists*

$$v(x_1, t) \leq v^{\gamma}(x_2, t)e^{\frac{\alpha}{p-1}(1-\gamma)}$$

where $\alpha = 1+(p-1)\ln A$, $\gamma = \exp\{-C(n,p)\rho(\frac{1}{t^{1/p}} + k)\}$, *and* $\rho = \rho(x_1, x_2)$ *denotes the geodesic distance between* x_1 *and* x_2.

The second application is the following Liouville-type theorem:

Corollary 1.4 *Let* M *be a complete noncompact Riemannian manifold and nonnegative sectional curvature, that is,* $K_M \geq 0$. *Suppose that* v *is a positive solution to the equation (1.1),* $2 < p < 2 + \frac{1+\sqrt{2n+3}}{2n+2}$, *and* $v = \exp\{o(d(x_0, x) + |t|^{\frac{1}{p}})\}$. *And also suppose* $|\nabla v| > 0$ *in* $M \times (-\infty, 0)$. *Then the equation (1.1) does not have a positive ancient solution.*

The equation (1.1) on Riemannian manifold has been studied in [3], where Li-Yau type gradient estimates and an entropy formula were obtained.

The paper is organized as follows. In Section 2 we establish some lemmas for $p > 1$. In Section 3, for $2 < p < 2 + \frac{1+\sqrt{2n+3}}{2n+2}$, and using maximum principle, we prove Theorem 1.2. And using Theorem 1.2, we prove Corollaries 1.3 and 1.4.

2. Preliminaires

Let v be a positive and bounded solution to the equation (1.1), and let $u = (p-1)\ln v$. It is easy to see that u satisfies

$$\frac{\partial u}{\partial t}=\operatorname{div}(|\nabla u|^{p-2}\nabla u)+|\nabla u|^{p}. \tag{2.1}$$

Let $f=|\nabla u|^2$, and assume that $f>0$ over some region of M. Similarly to [3] or [4], we use the linearized operator on right-hand side of the nonlinear equation (2.1).

Lemma 2.1 *Let u be a solution to (2.1) and $f=|\nabla u|^2$, and also assume that $f>0$ over some region of M. Then the linearized operator on right-hand side of the nonlinear equation (2.1) at u is*

$$\begin{aligned}\mathcal{L}(\psi)=&\frac{1}{2}(p-2)(p-4)f^{\frac{p}{2}-3}\langle\nabla u,\nabla\psi\rangle\langle\nabla f,\nabla u\rangle+\\&(p-2)f^{\frac{p}{2}-2}\langle\nabla\langle\nabla u,\nabla\psi\rangle,\nabla u\rangle+\frac{1}{2}(p-2)f^{\frac{p}{2}-2}\langle\nabla f,\nabla\psi\rangle+\\&(p-2)f^{\frac{p}{2}-2}\langle\nabla u,\nabla\psi\rangle\triangle u+f^{\frac{p}{2}-1}\triangle\psi+pf^{\frac{p}{2}-1}\langle\nabla u,\nabla\psi\rangle.\end{aligned} \tag{2.2}$$

Proof Using variational method, we can get

$$\begin{aligned}\mathcal{L}(\psi)=&\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}\{\operatorname{div}[|\nabla(u+\epsilon\psi)|^{p-2}\nabla(u+\epsilon\psi)]+|\nabla(u+\epsilon\psi)|^{p}\}\\=&\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}\{\operatorname{div}[(|\nabla(u+\epsilon\psi)|^2)^{\frac{p-2}{2}}\nabla(u+\epsilon\psi)]+(|\nabla(u+\epsilon\psi)|^2)^{\frac{p}{2}}\}\\=&\operatorname{div}\left[(p-2)|\nabla u|^{p-4}\langle\nabla u,\nabla\psi\rangle\nabla u+|\nabla u|^{p-2}\nabla\psi\right]+p|\nabla u|^{p-2}\langle\nabla u,\nabla\psi\rangle\\=&\frac{1}{2}(p-2)(p-4)f^{\frac{p}{2}-3}\langle\nabla u,\nabla\psi\rangle\langle\nabla f,\nabla u\rangle+\\&(p-2)f^{\frac{p}{2}-2}\langle\nabla\langle\nabla u,\nabla\psi\rangle,\nabla u\rangle+\frac{1}{2}(p-2)f^{\frac{p}{2}-2}\langle\nabla f,\nabla\psi\rangle+\\&(p-2)f^{\frac{p}{2}-2}\langle\nabla u,\nabla\psi\rangle\triangle u+f^{\frac{p}{2}-1}\triangle\psi+pf^{\frac{p}{2}-1}\langle\nabla u,\nabla\psi\rangle.\quad\square\end{aligned}$$

Let u,f be as above, $0<v\le A$, $\alpha=1+(p-1)\ln A$ and $\omega=|\nabla(\alpha-u)|^2=\frac{|\nabla u|^2}{(\alpha-u)^2}$. Now we will derive $\omega_t-\mathcal{L}(\omega)$. Firstly, we need the following two lemmas.

Lemma 2.2 *Let $\omega=\frac{|\nabla u|^2}{(\alpha-u)^2}$. Then*

$$\begin{aligned}\mathcal{L}(\omega)=&\frac{1}{2}(p-2)(p-4)(\alpha-u)^2|\nabla u|^{p-6}(\langle\nabla u,\nabla\omega\rangle)^2+(p-2)|\nabla u|^{p-4}\langle\nabla\langle\nabla u,\nabla\omega\rangle,\nabla u\rangle+\\&\frac{1}{2}(p-2)(\alpha-u)^2|\nabla u|^{p-4}|\nabla\omega|^2+(p-2)|\nabla u|^{p-4}\langle\nabla u,\nabla\omega\rangle\triangle u+\\&\frac{2|\nabla u|^{p-2}}{(\alpha-u)^2}u_{ij}^2+\frac{2|\nabla u|^{p-2}}{(\alpha-u)^2}u_iu_{ijj}-(p^2-5p+2)\frac{|\nabla u|^{p-2}}{(\alpha-u)}\langle\nabla u,\nabla\omega\rangle+\\&\frac{2|\nabla u|^{p}}{(\alpha-u)^3}\triangle u-2\frac{|\nabla u|^{p+2}}{(\alpha-u)^4}+p|\nabla u|^{p-2}\langle\nabla u,\nabla\omega\rangle.\end{aligned} \tag{2.3}$$

Proof Using Lemma 2.1, we obtain

$$\begin{aligned}\mathcal{L}(\omega)=&\frac{1}{2}(p-2)(p-4)f^{\frac{p}{2}-3}\langle\nabla u,\nabla\omega\rangle\langle\nabla f,\nabla u\rangle+(p-2)f^{\frac{p}{2}-2}\langle\nabla\langle\nabla u,\nabla\omega\rangle,\nabla u\rangle+\\&\frac{1}{2}(p-2)f^{\frac{p}{2}-2}\langle\nabla f,\nabla\omega\rangle+(p-2)f^{\frac{p}{2}-2}\langle\nabla u,\nabla\omega\rangle\triangle u\\&f^{\frac{p}{2}-1}\triangle\omega+pf^{\frac{p}{2}-1}\langle\nabla u,\nabla\omega\rangle=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}+\mathrm{VI}.\end{aligned} \tag{2.4}$$

Firstly, $f=\omega(\alpha-u)^2$,

$$\nabla f=(\alpha-u)^2\nabla\omega-2(\alpha-u)\omega\nabla u=(\alpha-u)^2\nabla\omega-\frac{2|\nabla u|^2}{(\alpha-u)}\nabla u,$$

$$\langle\nabla f,\nabla u\rangle=(\alpha-u)^2\langle\nabla u,\nabla\omega\rangle-\frac{2|\nabla u|^4}{(\alpha-u)},\tag{2.5}$$

$$\langle\nabla\omega,\nabla f\rangle=(\alpha-u)^2|\nabla\omega|^2-\frac{2|\nabla u|^2}{(\alpha-u)}\langle\nabla u,\nabla\omega\rangle,\tag{2.6}$$

$$\omega_j=\frac{2u_iu_{ij}}{(\alpha-u)^2}+\frac{2|\nabla u|^2}{(\alpha-u)^3}u_j,$$

$$\triangle\omega=\omega_{jj}=\frac{2u_{ij}^2}{(\alpha-u)^2}+\frac{2u_iu_{ijj}}{(\alpha-u)^2}+\frac{8u_iu_ju_{ij}}{(\alpha-u)^3}+\frac{2|\nabla u|^2}{(\alpha-u)^3}\triangle u+\frac{6|\nabla u|^4}{(\alpha-u)^4}.\tag{2.7}$$

For the first term on the right-hand side of (2.4), using (2.5) and $f=|\nabla u|^2$, we can get

$$\begin{aligned}\mathrm{I}=&\frac{1}{2}(p-2)(p-4)|\nabla u|^{p-6}\langle\nabla u,\nabla\omega\rangle[(\alpha-u)^2\langle\nabla u,\nabla\omega\rangle-\frac{2|\nabla u|^4}{(\alpha-u)}]\\=&\frac{1}{2}(p-2)(p-4)|\nabla u|^{p-6}(\alpha-u)^2(\langle\nabla u,\nabla\omega\rangle)^2-\\&(p-2)(p-4)\frac{|\nabla u|^{p-2}}{(\alpha-u)}\langle\nabla u,\nabla\omega\rangle.\end{aligned}$$

For the second term on the right-hand side of (2.4), using $f=|\nabla u|^2$, we obtain

$$\mathrm{II}=(p-2)|\nabla u|^{p-4}\langle\nabla\langle\nabla u,\nabla\omega\rangle,\nabla u\rangle.$$

For the third term on the right-hand side of (2.4), using (2.6) and $f=|\nabla u|^2$, we can get

$$\begin{aligned}\mathrm{III}&=\frac{1}{2}(p-2)|\nabla u|^{p-4}[(\alpha-u)^2|\nabla\omega|^2-\frac{2|\nabla u|^2}{(\alpha-u)}\langle\nabla u,\nabla\omega\rangle]\\&=\frac{1}{2}(p-2)(\alpha-u)^2|\nabla u|^{p-4}|\nabla\omega|^2-(p-2)\frac{|\nabla u|^{p-2}}{(\alpha-u)}\langle\nabla u,\nabla\omega\rangle.\end{aligned}$$

For the fourth term on the right-hand side of (2.4), using $f=|\nabla u|^2$, we obtain

$$\mathrm{IV}=(p-2)|\nabla u|^{p-2}\langle\nabla u,\nabla\omega\rangle\triangle u.$$

For the fifth term on the right-hand side of (2.4), using (2.7) and $f=|\nabla u|^2$, we can get

$$\begin{aligned}\mathrm{V}=&|\nabla u|^{p-2}[\frac{2u_{ij}^2}{(\alpha-u)^2}+\frac{2u_iu_{ijj}}{(\alpha-u)^2}+\frac{8u_iu_ju_{ij}}{(\alpha-u)^3}+\frac{2|\nabla u|^2}{(\alpha-u)^3}\triangle u+\frac{6|\nabla u|^4}{(\alpha-u)^4}]\\=&\frac{2|\nabla u|^{p-2}}{(\alpha-u)^2}u_{ij}^2+\frac{2|\nabla u|^{p-2}}{(\alpha-u)^2}u_iu_{ijj}+\frac{4|\nabla u|^{p-2}}{(\alpha-u)^3}\langle\nabla u,\nabla f\rangle+\frac{2|\nabla u|^p}{(\alpha-u)^3}\triangle u+\frac{6|\nabla u|^{p+2}}{(\alpha-u)^4}\\=&\frac{2|\nabla u|^{p-2}}{(\alpha-u)^2}u_{ij}^2+\frac{2|\nabla u|^{p-2}}{(\alpha-u)^2}u_iu_{ijj}+\frac{4|\nabla u|^{p-2}}{(\alpha-u)}\langle\nabla u,\nabla\omega\rangle+\frac{2|\nabla u|^p}{(\alpha-u)^3}\triangle u-\frac{2|\nabla u|^{p+2}}{(\alpha-u)^4}.\end{aligned}$$

For the sixth term on the right-hand side of (2.4), using $f=|\nabla u|^2$, we obtain

$$\mathrm{VI}=p|\nabla u|^{p-2}\langle\nabla u,\nabla\omega\rangle.$$

Combining the equations above and (2.4), we obtain (2.3). □

Lemma 2.3 *Let* $\omega = \frac{|\nabla u|^2}{(\alpha-u)^2}$. *Then*

$$\begin{aligned}\frac{\partial\omega}{\partial t} =&\frac{1}{2}(p-2)(p-4)(\alpha-u)^2|\nabla u|^{p-6}(\langle\nabla u,\nabla\omega\rangle)^2+\\&(p-2)|\nabla u|^{p-4}\langle\nabla\langle\nabla u,\nabla\omega\rangle,\nabla u\rangle+2(1-p)\frac{|\nabla u|^{p+2}}{(\alpha-u)^3}+\\&(p-2)|\nabla u|^{p-4}\langle\nabla u,\nabla\omega\rangle\triangle u-\frac{2|\nabla u|^{p-2}}{(\alpha-u)^2}R_{ij}u_iu_j+\\&\frac{2|\nabla u|^{p-2}}{(\alpha-u)^2}u_iu_{ijj}-(2p^2-7p+6)\frac{|\nabla u|^{p-2}}{(\alpha-u)}\langle\nabla u,\nabla\omega\rangle+\\&2(3-p)\frac{|\nabla u|^p}{(\alpha-u)^3}\triangle u+2(p-2)^2\frac{|\nabla u|^{p+2}}{(\alpha-u)^4}+\\&p|\nabla u|^{p-2}\langle\nabla u,\nabla\omega\rangle.\end{aligned}$$

Proof Firstly, we calculate

$$\frac{\partial w}{\partial t}=\frac{2u_iu_{ti}}{(\alpha-u)^2}+\frac{2|\nabla u|^2u_t}{(\alpha-u)^3}. \tag{2.8}$$

Using

$$\begin{aligned}\frac{\partial u}{\partial t}&=\mathrm{div}(|\nabla u|^{p-2}\nabla u)+|\nabla u|^p=\mathrm{div}[f^{\frac{p-2}{2}}\nabla u]+f^{\frac{p}{2}}\\&=\frac{p-2}{2}f^{\frac{p}{2}-2}\langle\nabla f,\nabla u\rangle+f^{\frac{p}{2}-1}\triangle u+f^{\frac{p}{2}},\end{aligned}$$

and $\langle\nabla f,\nabla u\rangle=(\alpha-u)^2\langle\nabla u,\nabla\omega\rangle-\frac{2|\nabla u|^4}{(\alpha-u)}$, we have

$$\begin{aligned}\frac{2u_iu_{ti}}{(\alpha-u)^2}=&\frac{2}{(\alpha-u)^2}u_i(\frac{p-2}{2}f^{\frac{p}{2}-2}\langle\nabla f,\nabla u\rangle+f^{\frac{p}{2}-1}\triangle u+f^{\frac{p}{2}})_i\\=&\frac{2}{(\alpha-u)^2}\Big(\frac{1}{4}(p-2)(p-4)|\nabla u|^{(p-6)}(\langle\nabla u,\nabla f\rangle)^2+\\&\frac{p-2}{2}|\nabla u|^{(p-4)}\langle\nabla u,\nabla\langle\nabla u,\nabla f\rangle\rangle+\frac{1}{2}(p-2)|\nabla u|^{(p-4)}\langle\nabla u,\nabla f\rangle\triangle u+\\&|\nabla u|^{(p-2)}\langle\nabla u,\nabla\triangle u\rangle+\frac{p}{2}|\nabla u|^{(p-2)}\langle\nabla u,\nabla f\rangle\Big)\\=&\frac{2}{(\alpha-u)^2}\Big\{\frac{1}{4}(p-2)(p-4)|\nabla u|^{(p-6)}\Big((\alpha-u)^2\langle\nabla u,\nabla w\rangle-\frac{2|\nabla u|^4}{(\alpha-u)}\Big)^2+\\&\frac{p-2}{2}|\nabla u|^{(p-4)}\langle\nabla u,\nabla\Big((\alpha-u)^2\langle\nabla u,\nabla w\rangle-\frac{2|\nabla u|^4}{(\alpha-u)}\Big)\rangle+\\&\frac{1}{2}(p-2)|\nabla u|^{(p-4)}\Big((\alpha-u)^2\langle\nabla u,\nabla w\rangle-\frac{2|\nabla u|^4}{(\alpha-u)}\Big)\triangle u+\\&|\nabla u|^{(p-2)}\langle\nabla u,\nabla\triangle u\rangle+\frac{p}{2}|\nabla u|^{(p-2)}\Big((\alpha-u)^2\langle\nabla u,\nabla w\rangle-\frac{2|\nabla u|^4}{(\alpha-u)}\Big)\Big\}\\=&\frac{1}{2}(p-2)(p-4)|\nabla u|^{(p-6)}(\alpha-u)^2(\langle\nabla u,\nabla w\rangle)^2-\\&2(p-1)(p-2)\frac{|\nabla u|^{(p-2)}}{(\alpha-u)}\langle\nabla u,\nabla w\rangle+2(p-2)(p-4)\frac{|\nabla u|^{(p+2)}}{(\alpha-u)^4}+\\&(p-2)|\nabla u|^{(p-4)}\langle\nabla u,\nabla\langle\nabla u,\nabla w\rangle\rangle+6(p-2)\frac{|\nabla u|^{(p+2)}}{(\alpha-u)^4}+\end{aligned}$$

$$(p-2)|\nabla u|^{(p-4)}\langle\nabla u,\nabla w\rangle\triangle u-2(p-2)\frac{|\nabla u|^{p}}{(\alpha-u)^{3}}\triangle u+$$
$$\frac{2|\nabla u|^{(p-2)}}{(\alpha-u)^{2}}\langle\nabla u,\nabla\triangle u\rangle+p|\nabla u|^{(p-2)}\langle\nabla u,\nabla w\rangle-2p\frac{|\nabla u|^{(p+2)}}{(\alpha-u)^{3}},\tag{2.9}$$

and

$$\begin{aligned}\frac{2|\nabla u|^{2}u_{t}}{(\alpha-u)^{3}}=&\frac{2|\nabla u|^{2}}{(\alpha-u)^{3}}(\frac{p-2}{2}f^{\frac{p}{2}-2}\langle\nabla f,\nabla u\rangle+f^{\frac{p}{2}-1}\triangle u+f^{\frac{p}{2}})\\=&(p-2)\frac{|\nabla u|^{(p-2)}}{(\alpha-u)^{3}}[(\alpha-u)^{2}\langle\nabla u,\nabla\omega\rangle-\frac{2|\nabla u|^{4}}{(\alpha-u)}]+\\&\frac{2|\nabla u|^{p}}{(\alpha-u)^{3}}\triangle u+\frac{2|\nabla u|^{(p+2)}}{(\alpha-u)^{3}}\\=&(p-2)\frac{|\nabla u|^{(p-2)}}{(\alpha-u)}\langle\nabla u,\nabla\omega\rangle-2(p-2)\frac{|\nabla u|^{(p+2)}}{(\alpha-u)^{4}}+\\&\frac{2|\nabla u|^{p}}{(\alpha-u)^{3}}\triangle u+\frac{2|\nabla u|^{(p+2)}}{(\alpha-u)^{3}}.\end{aligned}\tag{2.10}$$

Using (2.8), (2.9), (2.10) and $u_{jji}-u_{jij}=-R_{ij}u_{j}$, we can obtain the result.

Proposition 2.4 *Let* $\omega=\frac{|\nabla u|^{2}}{(\alpha-u)^{2}}$. *Then*

$$\begin{aligned}\frac{\partial\omega}{\partial t}-\mathcal{L}(\omega)=&\left[2(p-2)^{2}+2\right]\frac{|\nabla u|^{p+2}}{(\alpha-u)^{4}}+\\&2(1-p)\frac{|\nabla u|^{p+2}}{(\alpha-u)^{3}}+2(2-p)\frac{|\nabla u|^{p}}{(\alpha-u)^{3}}\triangle u-\\&(p^{2}-2p+4)\frac{|\nabla u|^{p-2}}{(\alpha-u)}\langle\nabla u,\nabla\omega\rangle-\frac{2|\nabla u|^{p-2}}{(\alpha-u)^{2}}u_{i}u_{j}R_{ij}-\\&\frac{1}{2}(p-2)(\alpha-u)^{2}|\nabla u|^{p-4}|\nabla\omega|^{2}-\frac{2|\nabla u|^{p-2}}{(\alpha-u)^{2}}u_{ij}^{2}.\end{aligned}\tag{2.11}$$

Proof Using Lemmas 2.2 and 2.3, we can get (2.11). □

3. Proofs of Theorem 1.2 and Corollaries 1.3 and 1.4

Let $\varphi=\varphi(x,t)$ be a smooth cut-off function supported in $Q_{R,T}$, satisfying the following properties:

(1) $\varphi=\varphi(d(x,x_0),t)\equiv\varphi(r,t);\varphi(x,t)=1$ in $Q_{R/2,T/2}$ and $\partial_r\varphi=0$ in $Q_{R/2,T}$; $0\le\varphi\le1$.

(2) φ is decreasing as a radial function in the spatial variables.

(3) $\frac{|\partial_r\varphi|}{\varphi^a}\le\frac{C_a}{R}$, $\frac{|\partial_r^2\varphi|}{\varphi^a}\le\frac{C_a}{R^2}$, when $0<a<1$.

(4) $\frac{|\partial_t\varphi|}{\varphi^{\frac{2}{p+2}}}\le\frac{C}{T}$.

Lemma 3.1 *If* $p>1$, *and let* $\nabla(\varphi\omega)=0$, *also* $\mathrm{Ric}(M)\ge-(n-1)k^2$, *then*

$$\begin{aligned}[\frac{\partial\omega}{\partial t}-\mathcal{L}(\omega)]\varphi\le&[(2n+2)(p-2)^{2}+1]\omega^{\frac{p+2}{2}}\varphi(\alpha-u)^{p-2}+2(1-p)\omega^{\frac{p+2}{2}}\varphi(\alpha-u)^{p-1}+\\&[3\varepsilon\omega^{\frac{p+2}{2}}\varphi+C(n,\ p,\ \varepsilon)k^{p+2}+C(p,\varepsilon)\frac{1}{R^{p+2}}](\alpha-u)^{p-2}-\end{aligned}$$

$$\frac{|\nabla u|^{p-2}}{2(\alpha-u)^2}u_{ij}^2\varphi, \tag{3.1}$$

where ε is a positive constant and will be chosen later, and $C(n,p,\varepsilon)$, $C(p,\varepsilon)$ are positive constants, depending on n,p,ε.

Proof Using Proposition 2.4 and $f=|\nabla u|^2$, we can get

$$\begin{aligned}\Big[\frac{\partial \omega}{\partial t}-\mathcal{L}(\omega)\Big]\varphi=&\big[2(p-2)^2+2\big]\frac{|\nabla u|^{p+2}}{(\alpha-u)^4}\varphi+\\ &2(1-p)\frac{|\nabla u|^{p+2}}{(\alpha-u)^3}\varphi+2(2-p)\frac{|\nabla u|^{p}}{(\alpha-u)^3}\varphi\triangle u-\\ &(p^2-2p+4)\frac{|\nabla u|^{p-2}}{(\alpha-u)}\langle\nabla u,\nabla\omega\rangle\varphi-\frac{2|\nabla u|^{p-2}}{(\alpha-u)^2}u_iu_jR_{ij}\varphi-\\ &\frac{1}{2}(p-2)(\alpha-u)^2|\nabla u|^{p-4}|\nabla\omega|^2\varphi-\frac{|\nabla u|^{p-2}}{(\alpha-u)^2}u_{ij}^2\varphi-\frac{|\nabla u|^{p-2}}{(\alpha-u)^2}u_{ij}^2\varphi.\end{aligned}$$

Using $\langle\nabla u,\nabla\omega\rangle=\frac{2u_iu_ju_{ij}}{(\alpha-u)^2}+\frac{2|\nabla u|^4}{(\alpha-u)^3}$ and

$$-\frac{|\nabla u|^{p-2}}{(\alpha-u)^2}u_{ij}^2\varphi-2\frac{|\nabla u|^{p-2}}{(\alpha-u)^3}u_iu_ju_{ij}\varphi-\frac{|\nabla u|^{p+2}}{(\alpha-u)^4}\varphi\le 0,$$

we obtain

$$\begin{aligned}\Big[\frac{\partial \omega}{\partial t}-\mathcal{L}(\omega)\Big]\varphi\le&\big[2(p-2)^2+1\big]\frac{|\nabla u|^{p+2}}{(\alpha-u)^4}\varphi+\\ &2(1-p)\frac{|\nabla u|^{p+2}}{(\alpha-u)^3}\varphi+2(2-p)\frac{|\nabla u|^{p}}{(\alpha-u)^3}\varphi\triangle u-\\ &(p^2-2p+3)\frac{|\nabla u|^{p-2}}{(\alpha-u)}\langle\nabla u,\nabla\omega\rangle\varphi-\frac{2|\nabla u|^{p-2}}{(\alpha-u)^2}u_iu_jR_{ij}\varphi-\\ &\frac{1}{2}(p-2)(\alpha-u)^2|\nabla u|^{p-4}|\nabla\omega|^2\varphi-\frac{|\nabla u|^{p-2}}{2(\alpha-u)^2}u_{ij}^2\varphi-\frac{|\nabla u|^{p-2}}{2(\alpha-u)^2}u_{ij}^2\varphi\\ =&\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}+\mathrm{VI}+\mathrm{VII}+\mathrm{VIII}.\end{aligned} \tag{3.2}$$

For the first and the second term on the right-hand side of (3.2), using $\omega=\frac{|\nabla u|^2}{(\alpha-u)^2}$, we obtain

$$\mathrm{I}=\big[2(p-2)^2+1\big]\omega^{\frac{p+2}{2}}\varphi(\alpha-u)^{p-2},\quad \mathrm{II}=2(1-p)\omega^{\frac{p+2}{2}}\varphi(\alpha-u)^{p-1}.$$

For the third and the seventh term on the right-hand side of (3.2), using Schwarz's inequality and $-x^2+bx\le\frac{b^2}{4}$, we get

$$\begin{aligned}\mathrm{III}+\mathrm{VII}=&2(2-p)\frac{|\nabla u|^{p}}{(\alpha-u)^3}\varphi\triangle u-\frac{|\nabla u|^{p-2}}{2(\alpha-u)^2}u_{ij}^2\varphi\\ \le&\frac{|\nabla u|^{p-2}}{2n(\alpha-u)^2}\Big[-(\triangle u)^2+\frac{4n|2-p||\nabla u|^2}{(\alpha-u)}|\triangle u|\Big]\varphi\\ \le&2n(2-p)^2\frac{|\nabla u|^{p+2}}{(\alpha-u)^4}\varphi=2n(2-p)^2\omega^{\frac{p+2}{2}}\varphi(\alpha-u)^{p-2}.\end{aligned}$$

For the fourth term on the right-hand side of (3.2), using Young's inequality and Schwarz's inequality, and $0=\nabla(\omega\varphi)=\varphi\nabla\omega+\omega\nabla\varphi$, we get

$$\begin{aligned}
\mathrm{IV}&=-(p^2-2p+3)\frac{|\nabla u|^{p-2}}{(\alpha-u)}\langle\nabla u,\nabla\omega\rangle\varphi\le(p^2-2p+3)\frac{|\nabla u|^{p+1}}{(\alpha-u)^3}|\nabla\varphi|\\
&=(p^2-2p+3)\left(\frac{|\nabla u|^2}{(\alpha-u)^2}\right)^{\frac{p+1}{2}}\varphi^{\frac{p+1}{p+2}}\frac{|\nabla\varphi|}{\varphi^{\frac{p+1}{p+2}}}(\alpha-u)^{p-2}\\
&\le[\varepsilon\omega^{\frac{p+2}{2}}\varphi+C(p,\ \varepsilon)\frac{|\nabla\varphi|^{p+2}}{\varphi^{p+1}}](\alpha-u)^{p-2}\\
&\le[\varepsilon\omega^{\frac{p+2}{2}}\varphi+C(p,\ \varepsilon)\frac{1}{R^{p+2}}](\alpha-u)^{p-2}.
\end{aligned}\tag{3.3}$$

For the fifth term on the right-hand side of (3.2), using Young's inequality and $\mathrm{Ric}(M)\ge-(n-1)k^2$, we get

$$\begin{aligned}
\mathrm{V}&=-\frac{2|\nabla u|^{p-2}}{(\alpha-u)^2}R_{ij}u_iu_j\varphi\le2(n-1)k^2\frac{|\nabla u|^p}{(\alpha-u)^2}\varphi\\
&=2(n-1)k^2(\frac{|\nabla u|^2}{(\alpha-u)^2})^{\frac{p}{2}}\varphi(\alpha-u)^{p-2}\\
&\le[\varepsilon\omega^{\frac{p+2}{2}}\varphi+C(n,p,\varepsilon)k^{p+2}](\alpha-u)^{p-2}.
\end{aligned}$$

For the sixth term on the right-hand side of (3.2), using Young's inequality and $0=\nabla(\omega\varphi)=\varphi\nabla\omega+\omega\nabla\varphi$, we get

$$\begin{aligned}
\mathrm{VI}&=-\frac{1}{2}(p-2)(\alpha-u)^2|\nabla u|^{p-4}|\nabla\omega|^2\varphi\le\frac{1}{2}|p-2|\frac{|\nabla u|^p}{(\alpha-u)^2}\frac{|\nabla\varphi|^2}{\varphi}\\
&=\frac{1}{2}|p-2|(\frac{|\nabla u|^2}{(\alpha-u)^2})^{\frac{p}{2}}\varphi^{\frac{p}{p+2}}\frac{|\nabla\varphi|^2}{\varphi^{\frac{2p+2}{p+2}}}(\alpha-u)^{p-2}\\
&\le[\varepsilon\omega^{\frac{p+2}{2}}\varphi+C(p,\ \varepsilon)\frac{|\nabla\varphi|^{p+2}}{\varphi^{p+1}}](\alpha-u)^{p-2}\\
&\le[\varepsilon\omega^{\frac{p+2}{2}}\varphi+C(p,\ \varepsilon)\frac{1}{R^{p+2}}](\alpha-u)^{p-2}.
\end{aligned}\tag{3.4}$$

In inequalities (3.3) and (3.4), we have used φ's properties. Combining the estimates above and equation (3.2), we get (3.1), where ε will be chosen later and $C(n,p,\varepsilon),C(p,\varepsilon)$ are constants. $\square$

Lemma 3.2 *If $p>1$, and we assume $\nabla(\varphi\omega)=0$, and sectional curvature $K_M\ge-k^2,k\ge0$, then*

$$\begin{aligned}
[\frac{\partial\varphi}{\partial t}-\mathcal{L}(\varphi)]\omega\le&\varepsilon\omega^{\frac{p+2}{2}}\varphi+\frac{1}{T^{\frac{p+2}{p}}}+[\varepsilon\omega^{\frac{p+2}{2}}\varphi+\frac{c(p,\varepsilon)}{R^{p+2}}](\alpha-u)^{p-1}+\\
&[10\varepsilon\omega^{\frac{p+2}{2}}\varphi+\frac{c(n,p,\varepsilon)}{R^{p+2}}+C(n,p,\varepsilon)k^{p+2}](\alpha-u)^{p-2}+\\
&\frac{|\nabla u|^{p-2}}{2(\alpha-u)^2}u_{ij}^2\varphi,
\end{aligned}\tag{3.5}$$

where ε is a positive constant and will be chosen later, $C(p,\varepsilon),C(n,p,\varepsilon)$ are positive constants, depending on n,p,ε.

Proof Using Lemma 2.1 and $f=|\nabla u|^2$, we get

$$\begin{aligned}[\frac{\partial\varphi}{\partial t}-\mathcal{L}(\varphi)]\omega =&\frac{\partial\varphi}{\partial t}\omega-\frac{1}{2}(p-2)(p-4)\frac{|\nabla u|^{p-4}}{(\alpha-u)^2}\langle\nabla u,\nabla\varphi\rangle\langle\nabla u,\nabla f\rangle-\\ &(p-2)\frac{|\nabla u|^{p-2}}{(\alpha-u)^2}\langle\nabla\langle\nabla u,\nabla\varphi\rangle,\nabla u\rangle-\frac{1}{2}(p-2)\frac{|\nabla u|^{p-2}}{(\alpha-u)^2}\langle\nabla f,\nabla\varphi\rangle-\\ &(p-2)\frac{|\nabla u|^{p-2}}{(\alpha-u)^2}\langle\nabla u,\nabla\varphi\rangle\triangle u-\frac{|\nabla u|^{p}}{(\alpha-u)^2}\triangle\varphi-\\ &\frac{p|\nabla u|^{p}}{(\alpha-u)^2}\langle\nabla u,\nabla\varphi\rangle\\ =&\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}+\mathrm{VI}+\mathrm{VII}. \end{aligned} \tag{3.6}$$

For the first term on the right-hand side of (3.6), using Young's inequality and φ's properties, we obtain

$$\begin{aligned}\mathrm{I}=\varphi_t\omega\le&\frac{|\nabla u|^2}{(\alpha-u)^2}\varphi^{\frac{2}{p+2}}\frac{|\varphi_t|}{\varphi^{\frac{2}{p+2}}}\le\varepsilon\omega^{\frac{p+2}{2}}\varphi+C(p,\varepsilon)\frac{|\varphi_t|^{\frac{p+2}{p}}}{\varphi^{\frac{2}{p}}}\\ \le&\varepsilon\omega^{\frac{p+2}{2}}\varphi+C(p,\varepsilon)\frac{1}{T^{\frac{p+2}{p}}}.\end{aligned}$$

For the second term on the right-hand side of (3.6), using Young's inequality and φ's properties, and the following two equalities

$$\langle\nabla f,\nabla u\rangle=(\alpha-u)^2\langle\nabla u,\nabla\omega\rangle-\frac{2|\nabla u|^4}{(\alpha-u)},\quad 0=\nabla(\omega\varphi)=\varphi\nabla\omega+\omega\nabla\varphi,$$

we obtain

$$\begin{aligned}\mathrm{II}=&-\frac{1}{2}(p-2)(p-4)\frac{|\nabla u|^{p-4}}{(\alpha-u)^{\beta}}\langle\nabla u,\nabla\varphi\rangle\langle\nabla u,\nabla f\rangle\\ \le&\frac{1}{2}|(p-2)(p-4)|\frac{|\nabla u|^{p}}{(\alpha-u)^2}\frac{|\nabla\varphi|^2}{\varphi}+|(p-2)(p-4)|\frac{|\nabla u|^{p+1}}{(\alpha-u)^3}|\nabla\varphi|\\ \le&[\varepsilon\omega^{\frac{p+2}{2}}\varphi+C(p,\ \varepsilon)\frac{|\nabla\varphi|^{p+2}}{\varphi^{p+1}}](\alpha-u)^{p-2}+\\ &[\varepsilon\omega^{\frac{p+2}{2}}\varphi+C(p,\ \varepsilon)\frac{|\nabla\varphi|^{p+2}}{\varphi^{p+1}}](\alpha-u)^{p-2}\\ \le&[2\varepsilon\omega^{\frac{p+2}{2}}\varphi+C(p,\varepsilon)\frac{1}{R^{p+2}}](\alpha-u)^{p-2}.\end{aligned}$$

For the third term on the right-hand side of (3.6), using

$$\langle\nabla f,\nabla\varphi\rangle=(\alpha-u)^2\langle\nabla\varphi,\nabla\omega\rangle-\frac{2|\nabla u|^2}{(\alpha-u)}\langle\nabla\varphi,\nabla u\rangle,$$

we obtain

$$\begin{aligned}\mathrm{III}=&-(p-2)\frac{|\nabla u|^{p-2}}{(\alpha-u)^2}\langle\nabla\langle\nabla u,\nabla\varphi\rangle,\nabla u\rangle\\ =&\frac{2-p}{2}\frac{|\nabla u|^{p-2}}{(\alpha-u)^2}\langle\nabla f,\nabla\varphi\rangle-(p-2)\frac{|\nabla u|^{p-2}}{(\alpha-u)^2}u_iu_j\varphi_{ij}\\ =&\frac{2-p}{2}|\nabla u|^{p-2}\langle\nabla\omega,\nabla\varphi\rangle+(p-2)\frac{|\nabla u|^{p}}{(\alpha-u)^3}\langle\nabla u,\nabla\varphi\rangle-\end{aligned}$$

$$\begin{aligned}&(p-2)\frac{|\nabla u|^{p-2}}{(\alpha-u)^2}u_iu_j\varphi_{ij}\\ =&\mathrm{III}_1+\mathrm{III}_2+\mathrm{III}_3.\end{aligned}\tag{3.7}$$

For the first and the second term on the right-hand side of (3.7), using Young's inequality, Schwarz's inequality and $0=\nabla(\omega\varphi)=\varphi\nabla\omega+\omega\nabla\varphi$, we can get

$$\begin{aligned}\mathrm{III}_1=&\frac{2-p}{2}|\nabla u|^{p-2}\langle\nabla\omega,\nabla\varphi\rangle\le\frac{|p-2|}{2}\frac{|\nabla u|^p}{(\alpha-u)^2}\frac{|\nabla\varphi|^2}{\varphi}\\ =&\frac{|p-2|}{2}\left[\frac{|\nabla u|^2}{(\alpha-u)^2}\right]^{\frac{p}{2}}\varphi^{\frac{p}{p+2}}\frac{|\nabla\varphi|^2}{\varphi^{\frac{2p+2}{p+2}}}(\alpha-u)^{p-2}\\ \le&[\varepsilon\omega^{\frac{p+2}{2}}\varphi+C(p,\varepsilon)\frac{|\nabla\varphi|^{p+2}}{\varphi^{p+1}}](\alpha-u)^{p-2}\\ \le&[\varepsilon\omega^{\frac{p+2}{2}}\varphi+C(p,\varepsilon)\frac{1}{R^{p+2}}](\alpha-u)^{p-2},\\ \mathrm{III}_2=&(p-2)\frac{|\nabla u|^p}{(\alpha-u)^3}\langle\nabla u,\nabla\varphi\rangle\le|p-2|\frac{|\nabla u|^{p+1}}{(\alpha-u)^3}|\nabla\varphi|\\ =&|p-2|[\frac{|\nabla u|^2}{(\alpha-u)^2}]^{\frac{p+1}{2}}\varphi^{\frac{p+1}{p+2}}\frac{|\nabla\varphi|}{\varphi^{\frac{p+1}{p+2}}}(\alpha-u)^{p-2}\\ \le&[\varepsilon\omega^{\frac{p+2}{2}}\varphi+C(p,\varepsilon)\frac{|\nabla\varphi|^{p+2}}{\varphi^{p+1}}](\alpha-u)^{p-2}\\ \le&[\varepsilon\omega^{\frac{p+2}{2}}\varphi+C(p,\varepsilon)\frac{1}{R^{p+2}}](\alpha-u)^{p-2}.\end{aligned}$$

For the fourth term on the right-hand side of (3.6), using

$$\langle\nabla f,\nabla\varphi\rangle=(\alpha-u)^2\langle\nabla\varphi,\nabla\omega\rangle-\frac{2|\nabla u|^2}{(\alpha-u)}\langle\nabla\varphi,\nabla u\rangle\text{ and }0=\nabla(\omega\varphi)=\varphi\nabla\omega+\omega\nabla\varphi,$$

and Schwarz's inequality and Young's inequality, we obtain

$$\begin{aligned}\mathrm{IV}=&-\frac{1}{2}(p-2)\frac{|\nabla u|^{p-2}}{(\alpha-u)^2}\langle\nabla f,\nabla\varphi\rangle\\ =&-\frac{1}{2}(p-2)\frac{|\nabla u|^{p-2}}{(\alpha-u)^2}[(\alpha-u)^2\langle\nabla\omega,\nabla\varphi\rangle-\frac{2|\nabla u|^2}{(\alpha-u)}\langle\nabla u,\nabla\varphi\rangle]\\ \le&[\varepsilon\omega^{\frac{p+2}{2}}\varphi+C(p,\varepsilon)\frac{|\nabla\varphi|^{p+2}}{\varphi^{p+1}}](\alpha-u)^{p-2}+\\ &[\varepsilon\omega^{\frac{p+2}{2}}\varphi+C(p,\varepsilon)\frac{|\nabla\varphi|^{p+2}}{\varphi^{p+1}}](\alpha-u)^{p-2}\\ \le&[2\varepsilon\omega^{\frac{p+2}{2}}\varphi+C(p,\varepsilon)\frac{1}{R^{p+2}}](\alpha-u)^{p-2}.\end{aligned}$$

For the fifth term on the right-hand side of (3.6), using Young's inequality and Schwarz's inequality, we obtain

$$\begin{aligned}\mathrm{V}=&-(p-2)\frac{|\nabla u|^{p-2}}{(\alpha-u)^2}\langle\nabla u,\nabla\varphi\rangle\triangle u\\ \le&\frac{n}{2}(p-2)^2\frac{|\nabla u|^p}{(\alpha-u)^2}\frac{|\nabla\varphi|^2}{\varphi}+\frac{1}{2n}\frac{|\nabla u|^{p-2}}{(\alpha-u)^2}|\triangle u|^2\varphi\end{aligned}$$

$$\le \frac{n}{2}(p-2)^2\frac{|\nabla u|^p}{(\alpha-u)^2}\frac{|\nabla\varphi|^2}{\varphi}+\frac{|\nabla u|^{p-2}}{2(\alpha-u)^2}\varphi u_{ij}^2$$
$$\le [\varepsilon\omega^{\frac{p+2}{2}}\varphi+C(n,p,\varepsilon)\frac{1}{R^{p+2}}](\alpha-u)^{p-2}+\frac{|\nabla u|^{p-2}}{2(\alpha-u)^2}\varphi u_{ij}^2.$$

For the seventh term on the right-hand side of (3.6), using Schwarz's inequality and Young's inequality, we can get

$$\mathrm{VII}=-\frac{p|\nabla u|^p}{(\alpha-u)^2}\langle\nabla u,\nabla\varphi\rangle\le[\varepsilon\omega^{\frac{p+2}{2}}\varphi+C(p,\varepsilon)\frac{1}{R^{p+2}}](\alpha-u)^{p-1}.$$

Choose a local orthonormal frame $\{e_i\}$ near any such given point such that $\nabla u=|\nabla u|e_1$. As in [3], using the Hessian Comparison Theorem [1,6], which states that $r_{ij}\le\frac{1+kr}{r}g_{ij}$, noting that $|\nabla r|\le 1$, and $\varphi_r'=0$ if $r\le\frac{R}{2}$, we have that

$$\triangle\varphi+(p-2)\frac{u_iu_j}{|\nabla u|^2}\varphi_{ij}=\triangle\varphi+(p-2)\varphi_{11}$$
$$\ge-(n+p-2)\frac{2+kR}{R}|\partial_r\varphi|-\max\{p-1,1\}|\partial_r^2\varphi|. \tag{3.8}$$

Using the inequality (3.8), for the sixth term on the right-hand side of (3.6) and the third term on the right-hand side of (3.7), we can get

$$\mathrm{VI}+\mathrm{III}_3=-\frac{|\nabla u|^p}{(\alpha-u)^2}\triangle\varphi-(p-2)\frac{|\nabla u|^{p-2}}{(\alpha-u)^2}u_iu_j\varphi_{ij}$$
$$\le\frac{|\nabla u|^p}{(\alpha-u)^2}((n+p-2)\frac{2+kR}{R}|\partial_r\varphi|+\max\{p-1,1\}|\partial_r^2\varphi|)$$
$$\le t[3\varepsilon\omega^{\frac{p+2}{2}}\varphi+C(n,p,\varepsilon)\frac{1}{R^{p+2}}+C(n,p,\varepsilon)k^{p+2}](\alpha-u)^{p-2}.$$

Combining the estimates above and (3.6), we get (3.5), where ε will be chosen later and $C(n,p,\varepsilon)$, $C(p,\varepsilon)$ are constants. □

Proof of Theorem 1.2 If $|\nabla v|=0$, the result is obvious. Now we assume $|\nabla v|>0$, so $f=|\nabla u|^2>0$. Using the linear operation $\mathcal{L}$, we can obtain

$$\frac{\partial(\varphi\omega)}{\partial t}-\mathcal{L}(\varphi\omega)=[\frac{\partial\varphi}{\partial t}-\mathcal{L}(\varphi)]\omega+[\frac{\partial\omega}{\partial t}-\mathcal{L}(\omega)]\varphi-2|\nabla u|^{p-2}\langle\nabla\omega,\nabla\varphi\rangle-$$
$$2(p-2)|\nabla u|^{p-4}\langle\nabla\omega,\nabla u\rangle\langle\nabla\varphi,\nabla u\rangle. \tag{3.9}$$

Suppose the maximum of $\omega\varphi$ is reached at (x_1,t_1). By [6], we can assume, without loss of generality that x_1 is not in the cut-locus of M. Then at this point, one has, $\mathcal{L}(\omega\varphi)\le 0$, $(\omega\varphi)_t\ge 0$ and $\nabla(\omega\varphi)=0$.

Using Young's inequality, Schwarz's inequality and $0=\nabla(\omega\varphi)=\varphi\nabla\omega+\omega\nabla\varphi$, we obtain

$$-2|\nabla u|^{p-2}\langle\nabla\omega,\nabla\varphi\rangle\le 2\frac{|\nabla u|^p}{(\alpha-u)^2}\frac{|\nabla\varphi|^2}{\varphi}\le[\varepsilon\omega^{\frac{p+2}{2}}\varphi+\frac{C(p,\varepsilon)}{R^{p+2}}](\alpha-u)^{p-2}. \tag{3.10}$$

Similarly, we obtain

$$-2(p-2)|\nabla u|^{p-4}\langle\nabla\omega,\nabla u\rangle\langle\nabla\varphi,\nabla u\rangle\le 2|p-2|\frac{|\nabla u|^p}{(\alpha-u)^2}\frac{|\nabla\varphi|^2}{\varphi}$$
$$\le[\varepsilon\omega^{\frac{p+2}{2}}\varphi+\frac{C(p,\varepsilon)}{R^{p+2}}](\alpha-u)^{p-2}. \tag{3.11}$$

Then, using Lemmas 3.1, 3.2 and Equation (3.9), Inequality (3.10) and (3.11), we obtain at maximum point (x_1, t_1)

$$\begin{aligned}0 \leq & [(2n+2)(p-2)^2+1]\omega^{\frac{p+2}{2}}\varphi(\alpha-u)^{p-2}+2(1-p)\omega^{\frac{p+2}{2}}\varphi(\alpha-u)^{p-1}+\\ & [3\varepsilon\omega^{\frac{p+2}{2}}\varphi+C(n,\ p,\ \varepsilon)k^{p+2}+C(p,\ \varepsilon)\frac{1}{R^{p+2}}](\alpha-u)^{p-2}+\\ & \varepsilon\omega^{\frac{p+2}{2}}\varphi+\frac{1}{T^{\frac{p+2}{p}}}+[\varepsilon\omega^{\frac{p+2}{2}}\varphi+\frac{c(p,\varepsilon)}{R^{p+2}}](\alpha-u)^{p-1}+\\ & [10\varepsilon\omega^{\frac{p+2}{2}}\varphi+\frac{c(n,p,\varepsilon)}{R^{p+2}}+C(n,p,\varepsilon)k^{p+2}](\alpha-u)^{p-2}+\\ & [2\varepsilon\omega^{\frac{p+2}{2}}\varphi+\frac{C(p,\varepsilon)}{R^{p+2}}](\alpha-u)^{p-2}.\end{aligned}$$

Using $\alpha-u=1+(p-1)\ln A-(p-1)\ln v\geq 1$, and let $\beta=2(p-1)-(2n+2)(p-2)^2-1$, we obtain

$$\beta\omega^{\frac{p+2}{2}}\varphi\leq 17\varepsilon\omega^{\frac{p+2}{2}}\varphi+C(n,p,\varepsilon)\frac{1}{R^{p+2}}+C(p,\varepsilon)\frac{1}{T^{\frac{p+2}{p}}}+C(n,p,\varepsilon)k^{p+2}.$$

If $2<p<2+\frac{1+\sqrt{2n+3}}{2n+2}$, we can get $\beta>0$. Taking $\varepsilon=\frac{\beta}{34}$, we can get

$$\omega^{\frac{p+2}{2}}\varphi\leq C(n,p)(\frac{1}{R^{p+2}}+\frac{1}{T^{\frac{p+2}{p}}}+k^{p+2}).$$

For all (x,t) in $Q_{R,T}$,

$$\begin{aligned}\omega^{\frac{p+2}{2}}(x,t)\varphi^{\frac{p+2}{2}}(x,t) & \leq \omega^{\frac{p+2}{2}}(x_1,t_1)\varphi^{\frac{p+2}{2}}(x_1,t_1)\leq\omega^{\frac{p+2}{2}}(x_1,t_1)\varphi(x_1,t_1)\\ & \leq C(n,p)(\frac{1}{R^{p+2}}+\frac{1}{T^{\frac{p+2}{p}}}+k^{p+2}).\end{aligned}$$

Notice that $\varphi(x,t)=1$ in $Q_{R/2,T/2}$ and $\omega=|\nabla\ln(\alpha-u)|^2$, we finally get

$$\frac{|\nabla v(x,t)|}{v(x,t)}\leq C(n,p)(\frac{1}{R}+\frac{1}{T^{\frac{1}{p}}}+k)(1+(p-1)\ln\frac{A}{v(x,t)}).$$

So we arrive at (2.2). □

Proof of Corollary 1.3 Let $\gamma(s)$ be a minimal geodesic joining x_1 and x_2 in M, $\gamma(s):[0,1]\to M$, $\gamma(0)=x_1,\gamma(1)=x_2$. Using $u(x,t)=(p-1)\ln v$, $\alpha=1+(p-1)\ln A$, and $u(\gamma(s),t)$, we obtain

$$\begin{aligned}\ln\frac{\alpha-u(x_2,t)}{\alpha-u(x_1,t)} & =\int_0^1\frac{\mathrm{d}(\ln(\alpha-u(\gamma(s),t)))}{\mathrm{d}s}\mathrm{d}s\leq\int_0^1|\dot{\gamma}|\cdot\frac{(p-1)|\nabla v|}{v(\alpha-(p-1)\ln v)}\mathrm{d}s\\ & \leq C(n,p)\rho(x_1,x_2)(\frac{1}{t^{\frac{1}{p}}}+k).\end{aligned}$$

Then,

$$\frac{\alpha-u(x_2,t)}{\alpha-u(x_1,t)}\leq\exp\{C(n,p)\rho(x_1,x_2)(\frac{1}{t^{1/p}}+k)\}. \tag{3.12}$$

Let $\gamma=\exp\{-C(n,p)\rho(\frac{1}{t^{1/p}}+k)\}$. Then (3.12) implies that

$$\frac{\alpha-(p-1)\ln v(x_2,t)}{\alpha-(p-1)\ln v(x_1,t)}\leq\frac{1}{\gamma}.$$

Then, we get

$$v(x_1,t) \le v^{\gamma}(x_2,t)e^{\frac{\alpha}{(p-1)}(1-\gamma)},$$

where $\gamma = \exp\{-C(n,p)\rho(\frac{1}{t^{1/p}}+k)\}$, and $\rho=\rho(x_1,x_2)$ denotes the geodesic distance between x_1 and x_2. □

Proof of Corollary 1.4 Fixing $(x_0,\ t_0)$ in space-time and using Theorem 1.2 for v on the cube $B(x_0,\ R)\times[t_0-R^p,\ t_0]$, we obtain

$$\frac{|\nabla v(x_0,t_0)|}{v(x_0,t_0)} \le C(n,p)(\frac{o(R)}{R}).$$

Suppose that $R\to\infty$, it follows that $|\nabla v(x_0,t_0)|=0$. Since (x_0,t_0) is arbitrary, we get $v=c$. But $|\nabla v|>0$, so it is a contradiction. □

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