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Trace Formulae for the Eigenfunctions of Periodic Finite-Bands Dirac Operator

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Abstract In this paper, we deal with a Dirac operator with periodic and finite-bands potentials. Taking advantage of the commutativity of the monodromy operator and the Dirac operator, we define the Bloch functions and multiplicator curve. Further, we obtain the formulae of Dubrovin-Novikov's type, which illustrate the inherent relations between the Bloch functions and potentials. Finally, we get the trace formulae of eigenfunctions corresponding to the left end-points, right end-points and all end-points of the spectral bands by calculation of residues on the complex sphere, respectively.

Keywords trace formulae; eigenfunctions; periodic N-bands Dirac operator

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1. Introduction

Trace formulae play a prominent role in spectral theory of differential operator, spectral geometry, integrable systems and in quantum chaos [1–4]. Dirac operator and Shrödinger operator are two important operators in quantum theory. Several trace formulae of Dirac operator have already been known [4–7]. However, almost all of them are trace formulae for eigenvalues, and the trace characterization of eigenfunctions has few report. It is well-known that Hill's operator $H=-\partial^2+u(x)$, $\partial=\partial/\partial_x$, possesses band spectra $\bigcup_{j=0}^{\infty}[E_{2j},E_{2j+1}]$ (see [8,9]), where all end-points are eigenvalues of H under $2T$ periodic boundary conditions. In case that all spectral bands change into one spectrum $[E_{2N},+\infty)$ as $j\geq N$, while the remaining N spectral bands still separate each other, $u(x)$ is called periodic N-bands potential. The eigenfunctions $\Psi_{2j}(x)$, corresponding to the left end-points of the spectral bands, satisfy the famous Mckean-Trubowitz's identity [10,11]

$$\sum_{j=0}^{N}\Psi_{2j}^2(x)=1; \tag{1.1}$$

while the eigenfunctions $\Psi_{2j+1}(x)$, corresponding to the right end-points of the spectral bands,

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satisfy Cao's identity [12]

$$u(x) = -2\sum_{j=1}^{N} \Psi_{2j-1}^2(x) + \sigma, \tag{1.2}$$

where

$$\sigma = E_0 + \sum_{j=1}^{N}(E_{2j} - E_{2j-1}).$$

These two identities play dramatic roles in theory of integrable systems: Eq. (1.1) corresponds to the famous Neumann system [13], while Eq. (1.2) yields the Bargmann constraint of the restricted KdV system [12].

It hints that the periodic finite-bands Dirac operator must possess properties that parallel with Hill's operator due to the similarity between the known facts of them. In this paper, we treat the periodic Dirac operator L:

$$L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \partial_x + \begin{pmatrix} -p & 0 \\ 0 & -r \end{pmatrix}, \tag{1.3}$$

where $p(x)$ and $r(x)$ are real smooth functions with period T. In Section 2, we define the monodromy operator M, which is commutative with Dirac operator L. Their common eigenfunctions are Bloch functions. In Section 3, we prove the formulae of Duborovin-Novikov's type which illustrate the inherent relation between the Bloch functions and the periodic N-bands potentials. In the last section, we derive several trace formulae of eigenfunctions by calculation of residues on the complex sphere.

2. The monodromy operator and Bloch functions

Define the translation (monodromy) operator $M : f(x) \mapsto f(x+T)$, where

$$f(x) = (f_1(x), f_2(x))^T, \ f_j(x) \in C^\infty(\mathbb{R}), \quad j = 1, 2.$$

It is easy to see that M and L are commutative. Therefore, the kernel of $L-\lambda$ is invariant under the action of M for any complex number λ. Denote $\ker(L-\lambda)$ as D_λ and take arbitrarily a real number x_0 as the reference point. Let $\theta_0(x,\lambda) = (\theta_{01}, \theta_{02})^T$ and $\varphi_0(x,\lambda) = (\varphi_{01}, \varphi_{02})^T$ be the solutions of the following two initial value problems, respectively:

$$\begin{aligned} L\theta = \lambda\theta, &\quad \theta_1(x_0) = 1, \quad \theta_2(x_0) = 0; \\ L\varphi = \lambda\varphi, &\quad \varphi_1(x_0) = 0, \quad \varphi_2(x_0) = 1, \end{aligned}$$

where θ_0 and φ_0 constitute a base of D_λ. The matrix of M for this base is

$$M = \begin{pmatrix} \theta_{01}(x_0+T,\lambda) & \varphi_{01}(x_0+T,\lambda) \\ \theta_{02}(x_0+T,\lambda) & \varphi_{02}(x_0+T,\lambda) \end{pmatrix},$$

whose eigenpolynomial is independent of the choice of fundamental solutions. Hence, it is also independent of the choice of the reference point x_0. Noting that

$$\det M = \begin{vmatrix} \theta_{01}(x_0+T,\lambda) & \varphi_{01}(x_0+T,\lambda) \\ \theta_{02}(x_0+T,\lambda) & \varphi_{02}(x_0+T,\lambda) \end{vmatrix} = \begin{vmatrix} \theta_{01}(x_0,\lambda) & \varphi_{01}(x_0,\lambda) \\ \theta_{02}(x_0,\lambda) & \varphi_{02}(x_0,\lambda) \end{vmatrix} = 1,$$

$$\mathrm{tr}M = \theta_{01}(x_0 + T, \lambda) + \varphi_{02}(x_0 + T, \lambda) =: \Delta(\lambda),$$

we have

$$\det(\mu I - M) = \mu^2 - \Delta(\lambda)\mu + 1 = 0. \tag{2.1}$$

The roots μ_+, μ_- of Eq. (2.1) are the eigenvalues of M, called Floquet multiplicator. For $\Delta^2(\lambda) \neq 4$, $\mu_1 \neq \mu_2$, the corresponding eigenfunctions $\Psi_+(x, \lambda)$, $\Psi_+(x, \lambda)$ are called Bloch functions, which are the common eigenfunctions of L and M:

$$L\Psi_\pm = \lambda\Psi_\pm, \quad M\Psi_\pm = \Psi_\pm(x + T) = \mu_\pm\Psi_\pm(x).$$

The equation (2.1) determines a complex curve in $(\mu, \lambda) \in \mathbb{C}^2$. Its extension, an analytic variety in $P^2\mathbb{C}$, is called multiplicator curve, denoted as $\mathscr{C}$, a double leaves Riemann surface.

Suppose that $\Psi_{01}(x_0) = 1$. For $\Delta^2(\lambda) \neq 4$, the Bloch-Floquet solution is uniquely determined. Denote $\Psi_{0\pm} = (\Psi_{01\pm},\ \Psi_{02\pm})^T$ for short. It reads

$$\Psi_{0\pm} = \theta_0(x, \lambda) + m_\pm(x_0, \lambda)\varphi_0(x, \lambda),$$

where $m_\pm$ satisfies the following equations:

$$\begin{pmatrix} \mu_\pm - \theta_{01}(x_0 + T, \lambda) & -\varphi_{01}(x_0 + T, \lambda) \\ -\theta_{02}(x_0 + T, \lambda) & \mu_\pm - \varphi_{02}(x_0 + T, \lambda) \end{pmatrix} \begin{pmatrix} 1 \\ m_\pm \end{pmatrix} = 0,$$

$$m_\pm(x_0, \lambda) = \frac{\mu_\pm - \theta_{01}(x_0 + T, \lambda)}{\varphi_{01}(x_0 + T, \lambda)} = \Psi_{02\pm}(x_0, \lambda). \tag{2.2}$$

Consider an eigenvalue problem:

$$LY = \lambda Y, \quad y_1(x_0) = y_1(x_0 + T) = 0, \quad Y = (y_1, y_2)^T. \tag{2.3}$$

The zeros $\lambda = \alpha_j(x_0)$ of $\varphi_{01}(x_0+T, \lambda)$, dependent on the choice of x_0, are eigenvalues of Eq. (2.3), which constitute an auxiliary spectral problem (I).

Let $\Phi_{02}(x_0) = 1$. We have similar results as above. For $\Delta^2 \neq 4$, the Bloch-Floquet solution is also completely determined. Denote $\Phi_{0\pm} = (\Phi_{01\pm}, \Phi_{02\pm})^T$ for short. It reads

$$\Phi_{0\pm}(x, \lambda) = n_\pm\theta_0(x, \lambda) + \varphi_0(x, \lambda),$$

$$n_\pm(x_0, \lambda) = \frac{\mu_\pm - \varphi_{02}(x_0 + T, \lambda)}{\theta_{02}(x_0 + T, \lambda)} = \Phi_{01\pm}(x_0, \lambda). \tag{2.4}$$

The zeros $\lambda = \beta_j(x_0)$ of $\theta_{02}(x_0 + T, \lambda)$, dependent on the choice of x_0, are eigenvalues of the following eigenvalue problem:

$$LY = \lambda Y, \quad y_2(x_0) = y_2(x_0 + T) = 0, \quad Y = (y_1, y_2)^T. \tag{2.5}$$

They constitute the auxiliary spectral problem (II).

Define meromorphic functions on the multiplicator curve $\mathscr{C}$ (remove the infinite point):

$$m(x_0, \mathfrak{p}) = \frac{\mu - \theta_{01}(x_0 + T, \lambda)}{\varphi_{01}(x_0 + T, \lambda)},$$

$$n(x_0, \mathfrak{p}) = \frac{\mu - \varphi_{02}(x_0 + T, \lambda)}{\theta_{02}(x_0 + T, \lambda)},$$

$$\Psi_0(x,\mathfrak{p}) = \theta_0(x,\lambda) + m(x_0,\mathfrak{p})\varphi_0(x,\lambda),$$
$$\Phi_0(x,\mathfrak{p}) = n(x_0,\mathfrak{p})\theta_0(x,\lambda) + \varphi_0(x,\lambda),$$

where $\mathfrak{p} = (\mu,\lambda)$, Ψ_0 and Φ_0 are Bloch functions. Both Ψ_{0+} and Ψ_{0-} are values of Ψ_0 at points $\mathfrak{p}_+(\mu_+,\lambda)$ and $\mathfrak{p}_-(\mu_-,\lambda)$ respectively. Similarly, Φ_{0+} and Φ_{0-} are values of Φ_0 at points $\mathfrak{p}_+(\mu_+,\lambda)$ and $\mathfrak{p}_-(\mu_-,\lambda)$, respectively.

It can be proved that $\Delta^2(\lambda)-4$ has only real zeros [14,15], which are independent of x_0 and can be numbered in increasing order as

$$\cdots\lambda_{-2} \le \lambda_{-1} < \mu_{-2} \le \mu_{-1} < \lambda_0 \le \lambda_1 <$$
$$\mu_0 \le \mu_1 < \lambda_2 \le \lambda_3 < \mu_2 \le \mu_3 < \cdots$$

where $\lambda_0, \lambda_{\pm1}, \lambda_{\pm2}, \ldots$ are eigenvalues under the periodic condition $Y(x+T) = Y(x)$, while $\mu_0, \mu_{\pm1}, \mu_{\pm1}, \ldots$ are eigenvalues under the semi-periodic condition $Y(x+T) = -Y(x)$. Both (μ_{2j}, μ_{2j+1}) and $(\lambda_{2j}, \lambda_{2j+1})$ are unstable intervals, in which $|\Delta| > 2$; while (μ_j, λ_{j+1}) and (λ_j, μ_{j-1}) are stable intervals, in which $|\Delta| > 2$. Both $\alpha_j(x_0)$ and $\beta_j(x_0)$ fall into the unstable intervals [15].

3. Dubrovin-Novikov formulae

Define a new meromorphic function χ on the multiplicator curve $\mathscr{C}$, the values of χ at $\mathfrak{p}_\pm = (\mu_\pm, \lambda)^T$ reads

$$\chi_\pm = -i\frac{\Psi'_{01\pm}}{\Psi_{01\pm}} = -i\frac{\partial}{\partial x}\ln\Psi_{01\pm}. \tag{3.1}$$

Then, χ is independent of the choice of x_0, since the role of x_0 is only to adjust the coefficients of Ψ_{0+} and Ψ_{0-}. And, $\chi_\pm$ satisfy the Riccati equation

$$i\chi'_\pm - i\frac{r_x}{\lambda+r}\chi_\pm = \chi^2_\pm + (\lambda+p)(\lambda+r) = 0. \tag{3.2}$$

Proposition 3.1 *Let λ be a real number belonging to the stable intervals and $\chi_+ = \chi_R + i\chi_I$. Then the following equalities hold:*

$$\Psi_{01-}(x,\lambda) = \bar{\Psi}_{0+}(x,\lambda), \tag{3.3}$$
$$\chi_- = -\chi_R + i\chi_I, \tag{3.4}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\chi_R = \frac{r_x}{\lambda+r}\chi_R + 2\chi_R\chi_I, \tag{3.5}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\chi_I = \frac{r_x}{\lambda+r}\chi_I + (\lambda+p)(\lambda+r) - \chi_R^2 + \chi_I^2. \tag{3.6}$$

Proof By definition, $L\Psi_{01+} = \lambda\Psi_{01+}, \Psi_{01+}(x+T) = \mu_+\Psi_{01+}(x+T)$. Taking complex conjugation of the former two equalities and noting that $\mu_- = \bar{\mu}_+$, we get Eq. (3.3). Direct calculation verifies Eq. (3.4). Substituting Eq. (3.4) into Eq. (3.1) yields Eqs. (3.5) and (3.6). □

Corollary 3.2 *Let λ be a real number belonging to the stable intervals. We have*

$$\chi_R(x_0,\lambda) = -\frac{(\lambda + r(x_0))\sqrt{4-\Delta^2}}{2\varphi_{01}(x_0+T,\lambda)}, \tag{3.7}$$

$$\chi_I = \frac{[\lambda + r(x_0)][\varphi_{02}(x_0+T,\lambda) - \theta_{01}(x_0+T,\lambda)]}{2\varphi_{01}(x_0+T,\lambda)}. \tag{3.8}$$

Proof By Eq. (2.1), we have

$$\mu_+ = \frac{1}{2}(\Delta + i\sqrt{4-\Delta^2}) = \frac{1}{2}[\theta_{01}(x_0+T,\lambda) + \varphi_{02}(x_0+T,\lambda) + i\sqrt{4-\Delta^2}].$$

Remembering $\Psi_{01}(x_0) = 1$ and Eq. (2.2), we see that

$$\begin{aligned} i\chi_+(x_0) &= \Psi'_{01+}(x_0) = -[\lambda + r(x_0)]\Psi_{02+}(x_0) \\ &= -[\lambda + r(x_0)]\frac{\varphi_{02}(x_0+T,\lambda) - \theta_{01}(x_0+T,\lambda) + i\sqrt{4-\Delta^2}}{2\varphi_{01}(x_0+T,\lambda)}. \end{aligned}$$

Separating the real and imaginary parts, we get Eqs. (3.7) and (3.8) immediately.

In a similar way, we define another meromorphic function κ on the multiplicator curve $\mathscr{C}$, the values of κ at $\mathfrak{p}_\pm = (\mu_\pm, \lambda)$ read

$$\kappa_\pm = -i\frac{\Phi'_{02\pm}}{\Phi_{02\pm}} = -i\frac{\partial}{\partial x}\ln \Phi_{02\pm},$$

which satisfy the Riccati equation

$$i\kappa'_\pm - i\frac{p_x}{\lambda + p}\kappa_\pm - \kappa_\pm^2 + (\lambda + p)(\lambda + r) = 0.$$

Claims that parallel to Proposition 3.1 are true. □

Proposition 3.3 *Let λ be a real number, which belongs to the stable intervals and $\kappa_+ = \kappa_R + i\kappa_I$. Then the following facts hold:*

$$\Phi_{01-}(x,\lambda) = \bar{\Phi}_{01+}(x,\lambda), \tag{3.9}$$

$$\kappa_- = -\kappa_R + i\kappa_I, \tag{3.10}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\kappa_R = \frac{p_x}{\lambda + p}\kappa_R + 2\kappa_R\kappa_I, \tag{3.11}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\kappa_I = \frac{p_x}{\lambda + p}\kappa_I + (\lambda + p)(\lambda + r) - \kappa_R^2 + \kappa_I^2, \tag{3.12}$$

$$\kappa_R(x_0,\lambda) = \frac{[\lambda + p(x_0)]\sqrt{4-\Delta^2}}{2\theta_{02}(x_0+T,\lambda)}, \tag{3.13}$$

$$\kappa_I(x_0,\lambda) = \frac{-[\lambda + p(x_0)][\theta_{01}(x_0+T,\lambda) - \varphi_{02}(x_0+T,\lambda)]}{2\theta_{02}(x_0+T,\lambda)}. \tag{3.14}$$

Proposition 3.4 *Let λ be a real number, which belongs to the stable intervals. Then the following expressions hold:*

$$\Psi_{01+}\Psi_{01-} = \frac{\varphi_{01}(x+T,\lambda)}{\varphi_{01}(x_0+T,\lambda)}, \tag{3.15}$$

$$\Phi_{02+}\Phi_{02-} = \frac{\theta_{02}(x+T,\lambda)}{\theta_{02}(x_0+T,\lambda)}. \tag{3.16}$$

Proof Due to Eqs. (3.1) and (3.5), it follows that

$$\ln \Psi_{01+} = \int_{x_0}^{x} \big[i\chi_R - \frac{\mathrm{d}}{\mathrm{d}x}\ln\sqrt{\chi_R} + \frac{r_x}{2(\lambda + r)}\big]\mathrm{d}x.$$

Therefore,

$$\Psi_{01+} = \sqrt{\frac{\chi_R(x_0)(\lambda + r(x))}{\chi_R(x)(\lambda + r(x_0))}} \exp \int_{x_0}^{x} i\chi_R \mathrm{d}x. \tag{3.17}$$

Similar result holds for Ψ_{01-} :

$$\Psi_{01-} = \sqrt{\frac{\chi_R(x_0)(\lambda + r(x))}{\chi_R(x)(\lambda + r(x_0))}} \exp \Big\{ - \int_{x_0}^{x} i\chi_R \mathrm{d}x \Big\}. \tag{3.18}$$

Noting Corollary 3.2, we get Eq. (3.15). The proof of Eq.(3.16) is analogous to the proof of Eq. (3.15).

Now we consider the case of $(N+1)$-bands potential, $N \geq 1$. In this case there are only N unstable intervals. Assume that the eigenvalues under periodic and anti-periodic conditions are numbered as follows

$$\begin{aligned} &\cdots < \lambda_{-2j-1} = \lambda_{-2j} < \cdots < \lambda_{-1} = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \\ &\cdots < \lambda_{2N} < \lambda_{2N+1} = \lambda_{2N+2} < \cdots < \lambda_{2j-1} = \lambda_{2j} < \cdots , \end{aligned} \tag{3.19}$$

where $\lambda_1, \lambda_3, \ldots, \lambda_{2N-1}$ are the left end-points of unstable intervals, and $\lambda_2, \lambda_4, \ldots, \lambda_{2N}$ are the right end-points. It is easy to show that $\alpha_j(x_0) = \beta_j(x_0) = \lambda_{2j-1} = \lambda_{2j}$ hold for $j \leq 0$ and $j \geq 2N+1$, since $\alpha_j(x_0)$ and $\beta_j(x_0)$ belong to the unstable intervals. Expanding $\varphi_{01}(x+T, \lambda)$ and $\varphi_{01}(x_0+T, \lambda)$ in infinite product, respectively, we have

$$\frac{\varphi_{01}(x+T,\lambda)}{\varphi_{01}(x_0+T,\lambda)} = k(x) \frac{\prod_{j=1}^{N}(\lambda - \alpha_j(x))}{\prod_{j=1}^{N}(\lambda - \alpha_j(x_0))}.$$

From the asymptotic formula of $\varphi_{01}(x+T, \lambda)$ at $|\lambda| \to \infty$, it follows that $k(x) = 1$. Similarly, we have

$$\frac{\theta_{02}(x+T,\lambda)}{\theta_{02}(x_0+T,\lambda)} = \frac{\prod_{j=1}^{N}(\lambda - \beta_j(x))}{\prod_{j=1}^{N}(\lambda - \beta_j(x_0))}. \ \Box$$

Theorem 3.5 (formulae of Dubrovin-Novikov's type) *Suppose the periodic Dirac operator have N unstable bands. Then*

$$\Psi_{01+}\Psi_{01-} = \frac{\prod_{j=1}^{N}(\lambda - \alpha_j(x))}{\prod_{j=1}^{N}(\lambda - \alpha_j(x_0))}, \quad \Phi_{02+}\Phi_{02-} = \frac{\prod_{j=1}^{N}(\lambda - \beta_j(x))}{\prod_{j=1}^{N}(\lambda - \beta_j(x_0))}. \tag{3.20}$$

Proof The formulae are true when λ belongs to stable bands. Noting that stable bands are $N+1$ intervals and carrying out analytic continuation, we see that Eq. (3.20) holds for all λ, immediately. $\Box$

4. The trace formulae

Lemma 4.1 *Let ω be a meromorphic differential on the complex sphere $\mathcal{S} = \mathbb{C} \cup \{\infty\}$:*

$$\omega = \frac{\prod_{j=1}^{N}(\lambda - a_j)}{\prod_{j=1}^{N}(\lambda - b_j)} \mathrm{d}\lambda. \tag{4.1}$$

Then the residue of ω is equal to $\sum_{j=1}^{N}(a_j - b_j)$ at infinite point.

Proof Taking local coordinate $\zeta = \lambda^{-1}$ in the neighborhood of infinite point, we may get what is supposed by direct calculation. $\square$

Lemma 4.2 *The eigenvalues and end-points of spectral bands expressions for the difference between the two potentials read*

$$r(x) - p(x) = \sum_{j=1}^{2N} \lambda_j - 2\sum_{j=1}^{N} \alpha_j(x) = 2\sum_{j=1}^{N} \beta_j(x) - \sum_{j=1}^{2N} \lambda_j. \tag{4.2}$$

Proof Substituting $\chi_I = a_1\lambda + a_0 + a_{-1}\lambda^{-1} + \cdots$ and $\chi_R = b_1\lambda + b_0 + b_{-1}\lambda^{-1} + \cdots$ into Eqs. (3.5) and (3.6), we get that

$$\chi_R^2 = \lambda^2 + rp + (p + r)\lambda + O(\lambda^{-1}). \tag{4.3}$$

On the other hand, we derive from Eq. (3.7) that

$$\chi_R^2 = (\lambda + r)^2 \prod_{j=1}^{2N} \big(1 - \frac{\lambda_j}{\lambda}\big) \prod_{j=1}^{2N} \big(1 - \frac{\alpha_j(x)}{\lambda}\big)^{-2}. \tag{4.4}$$

Comparing the coefficients of the same power of λ, we get the lemma immediately. $\square$

Theorem 4.3 *Let the periodic Dirac operator have N unstable bands and the eigenvalues be numbered as Eq.(3.19). Then the standardized eigenfunctions $Y(x, \lambda_j) = (Y_1(x, \lambda_j), Y_2(x, \lambda_j))^T$ satisfy the following identities:*

$$\sum_{j=1}^{N} Y_1^2(x, \lambda_{2j-1}) = \frac{1}{2}[p(x) - r(x)] + \frac{1}{2}\sigma, \tag{4.5}$$

$$\sum_{j=1}^{N} Y_1^2(x, \lambda_{2j}) = \frac{1}{2}[r(x) - p(x)] + \frac{1}{2}\sigma, \tag{4.6}$$

$$\sum_{j=1}^{N} \gamma_{2j-1}^2 Y_2^2(x, \lambda_{2j-1}) = \frac{1}{2}[r(x) - p(x)] + \frac{1}{2}\sigma, \tag{4.7}$$

$$\sum_{j=1}^{N} \gamma_{2j}^2 Y_2^2(x, \lambda_{2j}) = \frac{1}{2}[p(x) - r(x)] + \frac{1}{2}\sigma, \tag{4.8}$$

$$\sum_{j=1}^{2N} Y_1^2(x, \lambda_j) = \sigma, \quad \sum_{j=1}^{2N} \gamma_j^2 Y_2^2(x, \lambda_j) = \sigma, \tag{4.9}$$

where σ is the summation of width of spectral gaps:

$$\sigma = \sum_{j=1}^{2N} (\lambda_{2j} - \lambda_{2j-1}),$$

and γ_j's are some constants.

Proof Consider the meromorphic differential on the complex sphere $\mathcal{S}$:

$$\omega_1 = \frac{\prod_{j=1}^{N}(\lambda - \alpha_j(x))}{\prod_{j=1}^{N}(\lambda - \lambda_{2j-1})} \mathrm{d}\lambda = \frac{\prod_{j=1}^{N}(\lambda - \alpha_j(x_0))}{\prod_{j=1}^{N}(\lambda - \lambda_{2j-1})} \Psi_{01+}(x, \lambda)\Psi_{01-}(x, \lambda)\mathrm{d}\lambda.$$

At λ_{2j-1}, $\Psi_{01+}=\Psi_{01-}$, we denote it as $y_1(x,\lambda_{2j-1})$. By the residue theorem, we have

$$\sum_{j=1}^{N}\rho_{2j-1}y_1^2(x,\lambda_{2j-1})+\mathrm{Res}_{\infty}\omega_1=0, \tag{4.10}$$

where

$$\rho_{2j-1}=\frac{\prod_{k=1}^{N}(\lambda_{2j-1}-\alpha_k(x_0))}{\prod_{k\neq j}(\lambda_{2j-1}-\lambda_{2k-1})}.$$

The symbol of the numerator is $(-1)^{N-j+1}$ because $\alpha_j(x)$ belongs to $(\lambda_{2j-1},\lambda_{2j})$, $1\le j\le N$, and the symbol of the denominator is $(-1)^{N-j}$. Hence $\rho_{2j-1}<0$. Let $Y_1(x,\lambda_{2j-1})=\sqrt{-\rho_{2j-1}}y_1(x,\lambda_{2j-1})$. On the other hand, from Lemmas 4.1 and 4.2, we have

$$\mathrm{Res}_{\infty}\omega_1=-\frac{1}{2}[r(x)-p(x)]+\frac{1}{2}\sigma,$$

where $\sigma=\sum_{j=1}^{N}(\lambda_{2j}-\lambda_{2j-1})$. Substituting it into Eq. (4.10), we get Eq. (4.5).

Similarly, considering

$$\omega_2=\frac{\prod_{j=1}^{N}(\lambda-\alpha_j(x))}{\prod_{j=1}^{N}(\lambda-\lambda_{2j})}\mathrm{d}\lambda,$$

and denoting

$$Y_1(x,\lambda_{2j})=\sqrt{\rho_{2j}}y_1(x,\lambda_{2j}),\quad \rho_{2j}=\frac{\prod_{k=1}^{N}(\lambda_{2j}-\alpha_k(x_0))}{\prod_{k\neq j}(\lambda_{2j}-\lambda_{2k})}\mathrm{d}\lambda,$$

we can derive Eq. (4.6).

Adding Eqs. (4.5) and (4.6), we obtain the first formula of Eq. (4.9). Consider

$$\omega_3=\frac{\prod_{j=1}^{N}(\lambda-\beta_j(x))}{\prod_{j=1}^{N}(\lambda-\lambda_{2j-1})}\mathrm{d}\lambda,\quad \omega_4=\frac{\prod_{j=1}^{N}(\lambda-\beta_j(x))}{\prod_{j=1}^{N}(\lambda-\lambda_{2j})}\mathrm{d}\lambda,$$

and denote $z_2(x,\lambda_j)=\Phi_{02+}(x,\lambda)=\Phi_{02-}(x,\lambda)$. We obtain

$$-\sum_{j=1}^{N}[\sqrt{-\delta_{2j-1}}z_2(x,\lambda_{2j-1})]^2=\frac{1}{2}[r(x)-p(x)]+\frac{1}{2}\sigma,$$

$$\sum_{j=1}^{N}[\sqrt{\delta_{2j}}z_2(x,\lambda_{2j})]^2=\frac{1}{2}[p(x)-r(x)]+\frac{1}{2}\sigma,$$

where

$$\delta_{2j-1}=\frac{\prod_{k=1}^{N}(\lambda_{2j-1}-\beta_k(x_0))}{\prod_{k\neq j}(\lambda_{2j-1}-\lambda_{2k-1})},\quad \delta_{2j}=\frac{\prod_{k=1}^{N}(\lambda_{2j}-\beta_k(x_0))}{\prod_{k\neq j}(\lambda_{2j}-\lambda_{2k})}.$$

It is easy to show that $z_2(x,\lambda_j)=n(x_0,\lambda_j)y_2(x,\lambda_j)$. Denoting

$$\gamma_j=\sqrt{\delta_j/\rho_j}\,n(x_0,\lambda),$$

we have Eqs. (4.7), (4.8) and the second formula of Eq. (4.9). □

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