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On the Evaluation of Multifold Convolutions of Polynomials Using Difference and Shift Operators

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Abstract Here concerned and further investigated is a certain operator method for the computation of convolutions of polynomials. We provide a general formulation of the method with a refinement for certain old results, and also give some new applications to convolved sums involving several noted special polynomials. The advantage of the method using operators is illustrated with concrete examples. Finally, also presented is a brief investigation on convolution polynomials having two types of summations.

Keywords convolved polynomial sum; difference operator; classical special polynomial; symbolic computation

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1. Introduction

It is known that the problem for the computation of sums of convolved powers of the type

$$S(m;i,j)=\sum_{0\le k\le m}k^i(m-k)^j \tag{1.1}$$

was first investigated by Glaisher [3] and [4] in 1911–1912, and he found a summation formula using Bernoulli numbers. Some further investigations and extensions were given, during the years 1977–1978, by Neumann-Schonbach [12], Carlitz [2] and Gould [5] respectively, in which Eulerian numbers as well as Stirling numbers of the second kind had been utilized. Various numerical examples were also presented in Gould [5].

Actually, the most gegernal formulation of the computational problem for convolved polynomial sums was given in the author's earlier paper [6] (in 1944), and a kind of general summation formula was found via several lemmas. However, [6] contains some notational errors, and all related formulas were given in quite complicated forms. This may be the reason why the general result of [6] could not be used in practice.

Having done some practical computations, we eventually get realized that a kind of symbolic operator approach adopted in another earlier paper [7] (in 1948) should be the most effective way for dealing with general convolved polynomial sums. The object of this paper is to develop the operator method conceived previously. We will present certain general operator summation formulas that could be specialized and applied in various ways (see §4–§5).

2. Summation formulas involving operators

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Throughout the paper $f(x)$ and $f_i(x)$ $(i = 1, \dots, n)$ are assumed to be arbitrary polynomials over the real or complex number field, with degrees denoted by ∂f and ∂f_i, respectively. We are concerned with the problem for the computation of convolved polynomial sums of the type

$$S(m, [f_1]\cdots[f_n]) := \sum_{(m;0;x)} f_1(x_1)\cdots f_n(x_n) \tag{2.1}$$

where m is any given positive integer, and the sum on the RHS of (1) is taken over all the n-compositions of m with non-negative integer components, namely, over the set $(m; 0; x)$ of all the integer solutions of $x_1 + \cdots + x_n = m$ with each $x_i \geq 0$ $(i = 1, \dots, n)$.

Obviously, $S(m; i, j)$ of (1.1) just corresponds to the special case of (2.1) with $n = 2$, $f_1(x) = x^i$ and $f_2(x) = x^j$.

We shall make use of the ordinary difference operator Δ and the shift operator E which are defined by the relations $\Delta f(x) = f(x+1) - f(x)$, $Ef(x) = f(x+1)$. Powers of these operators are defined in the usual way with $\Delta^0 = E^0 = 1$ denoting the identity operator, so that $\Delta^0 f(x) = E^0 f(x) = 1f(x) = f(x)$, and $E = 1 + \Delta$.

Definition 2.1 *For any given polynomial $f(x)$ with degree $\partial f \geq 0$, there are two operator polynomials constructed from $f(x)$ as follows*

$$\Lambda(\Delta, f) := \sum_{\nu=0}^{\partial f} \Delta^\nu f(0)\Delta^\nu, \tag{2.2}$$

$$\Lambda^*(E, f) := \sum_{\nu=0}^{\partial f} \Delta^\nu f(-\nu-1)E^\nu. \tag{2.3}$$

These are called Λ-operators associated with f. In particular, $\Lambda(\Delta, f) \equiv \Lambda^*(E, f) \equiv f(0)\cdot 1$ for the case $\partial f = 0$ (viz $f(x) \equiv f(0)$).

Note that computations of backward differences $\Delta^\nu f(-\nu-1)$ are as easy as that of $\Delta^\nu f(0)$. Thus Λ^* and Λ are equally useful for practical computations. In fact, Λ and Λ^* are the same operator, and the operator identity $\Lambda(\Delta, f) \equiv \Lambda^*(E, f)$ could be verified easily by starting with $E = 1 + \Delta$ (see [8]).

A main proposition to be studied and given applications in this paper is the following

Theorem 2.2 *Let $f_1(x), \dots, f_n(x)$ be any given polynomials. Then there hold a pair of summation formulas as follows*

$$S(m, [f_1]\cdots[f_n]) = \Big(\prod_{i=1}^{n} \Lambda(\Delta, f_i)\Big)\binom{x}{m}_{x=m+n-1}, \tag{2.4}$$

$$S(m, [f_1]\cdots[f_n]) = \Big(\prod_{i=1}^{n} \Lambda^*(E, f_i)\Big)\binom{x}{m}_{x=m+n-1}. \tag{2.5}$$

Proof It suffices to verify (2.4), since (2.5) $\Leftrightarrow$ (2.4). Let us recall that there is a well-known identity in Combinatorics, namely

$$\sum_{(m;0;x)} \binom{x_1}{\nu_1}\cdots\binom{x_n}{\nu_n} = \binom{m+n-1}{m-(\nu_1+\cdots+\nu_n)} \tag{2.6}$$

where $\nu_i \geq 0$ $(1 \leq i \leq n)$, and $m \geq (\nu_1 + \cdots + \nu_n)$. Also there are simple relations

$$\Delta^\nu \binom{x}{m} = \binom{x}{m-\nu},\ 0 \leq \nu \leq m;\quad \Delta^\nu \binom{x}{m} = 0,\ \nu > m.$$

Thus the RHS of (2.6) may be expressed in the form

$$\binom{m+n-1}{m-(\nu_1+\cdots+\nu_n)} = \Delta^{\nu_1}\cdots\Delta^{\nu_n}\binom{x}{m}_{x=m+n-1}. \tag{2.6*}$$

Consequently, employing Newton's formula for $f(x)$ and making use of (2.6)–(2.6)*, one may compute the LHS of (2.4) as follows

$$\begin{aligned}
S(m,[f_1]\cdots[f_n]) &= \sum_{(m;0;x)} \prod_{i=1}^{n} \Big(\sum_{\nu_i=0}^{\partial f_i} \Delta^{\nu_i} f_i(0) \binom{x_i}{\nu_i} \Big) \\
&= \sum_{\nu_1=0}^{\partial f_1} \cdots \sum_{\nu_n=0}^{\partial f_n} \Delta^{\nu_1} f_1(0) \cdots \Delta^{\nu_n} f_n(0) \sum_{(m;0;x)} \prod_{i=1}^{n} \binom{x_i}{\nu_i} \\
&= \sum_{\nu_1=0}^{\partial f_1} \cdots \sum_{\nu_n=0}^{\partial f_n} (\Delta^{\nu_1} f_1(0) \Delta^{\nu_1}) \cdots (\Delta^{\nu_n} f_n(0) \Delta^{\nu_n}) \binom{x}{m}_{x=m+n-1} \\
&= \prod_{i=1}^{n} \Big(\sum_{\nu_i=0}^{\partial f_i} \Delta^{\nu_i} f_i(0) \Delta^{\nu_i} \Big) \binom{x}{m}_{x=m+n-1} \\
&= \Big(\prod_{i=1}^{n} \Lambda(\Delta, f_i) \Big) \binom{x}{m}_{x=m+n-1}.
\end{aligned}$$

Hence (2.4) is proved. $\square$

Corollary 2.3 *For the case $f_1(x) = \cdots = f_n(x) = f(x)$ there are summation formulas for $S(m,[f]^n)$:*

$$\sum_{(m;0;x)} f(x_1)\cdots f(x_n) = (\Lambda(\Delta,f))^n \binom{x}{m}_{x=m+n-1}, \tag{2.7}$$

$$\sum_{(m;0;x)} f(x_1)\cdots f(x_n) = (\Lambda^*(E,f))^n \binom{x}{m}_{x=m+n-1}. \tag{2.8}$$

Corollary 2.4 *For the monomials $f_i(x) = x^{p_i}$ with $p_i \geq 0$ $(i = 1,\dots,n)$, there is a summation formula of the form*

$$\sum_{(m;0;x)} x_1^{p_1}\cdots x_n^{p_n} = \Big(\prod_{i=1}^{n} \Big(\sum_{\nu=0}^{p_i} \nu! \left\{ {p_i \atop \nu} \right\} \Delta^\nu \Big) \Big) \binom{x}{m}_{x=m+n-1} \tag{2.9}$$

where $\left\{ {p_i \atop \nu} \right\}$ are Stirling numbers of the second kind with $\left\{ {0 \atop 0} \right\} = 1$. In particular, for the case $p_i \geq 1$ $(i = 1,\dots,n)$, (2.9) can be replaced by the form

$$\sum_{(m;1;x)} x_1^{p_1}\cdots x_n^{p_n} = \Big(\prod_{i=1}^{n} \Big(\sum_{\nu=1}^{p_i} \nu! \left\{ {p_i \atop \nu} \right\} \Delta^\nu \Big) \Big) \binom{x}{m}_{x=m+n-1} \tag{2.10}$$

where $(m;1;x)$ denotes the set of n-compositions of m with each component $x_i \geq 1$ $(i = 1,\dots,n)$.

Observe that (2.9) with $n = 2$ is an old result given by Gould [5], in which several numerical instances have been displaced. Also, note that (2.7) and (2.8) are even much older results that had been derived and employed in [7] and [8], respectively.

Obviously, the set $(m; 0; x)$ under the summation of (2.1) may be replaced by $(m; 1; x)$ in case $f_1(0) = \cdots = f_n(0) = 0$. In what follows we will present a few examples, requiring a bit of algebraic computations.

Example 2.5 Suppose we want to find a formula for the summation of the form

$$S(m, [x^2][x^3][x^4]) \equiv \sum_{(m;1;x)} x_1^2 \cdot x_2^3 \cdot x_3^4. \tag{2.11}$$

It requires that the summation formula should be consisting of a least number of terms.

In accordance with (2.4) we have to do computations

$$\begin{aligned}\Lambda(\Delta, x^2) &= \Delta + 2\Delta^2,\\ \Lambda(\Delta, x^3) &= \Delta + 6\Delta^2 + 6\Delta^3,\\ \Lambda(\Delta, x^4) &= \Delta + 14\Delta^2 + 36\Delta^3 + 24\Delta^4.\end{aligned}$$

Clearly we may rewrite $\Lambda(\Delta, x^3) = \Delta(1 + 6\Delta E)$. Moreover, using a simple factorization technique, we find

$$\Lambda(\Delta, x^4) = \Delta(1 + 2\Delta)(1 + 12\Delta E). \tag{2.12}$$

Consequently, we obtain

$$\begin{aligned}\Lambda(\Delta, x^2)\Lambda(\Delta, x^3)\Lambda(\Delta, x^4) &= \Delta^3(1 + 2\Delta)^2(1 + 6\Delta E)(1 + 12\Delta E)\\ &= \Delta^3(1 + 4\Delta E)(1 + 6\Delta E)(1 + 12\Delta E)\\ &= \Delta^3(1 + 22\Delta E + 144(\Delta E)^2 + 288(\Delta E)^3).\end{aligned}$$

Hence an application of (2.4) (with $n = 3$) to the sum (2.11) gives a formula as follows

$$\sum_{(m;1;x)} x_1^2 \cdot x_2^3 \cdot x_3^4 = \binom{m+2}{5} + 22\binom{m+3}{7} + 144\binom{m+4}{9} + 288\binom{m+5}{11}. \tag{2.13}$$

Similarly, noting that $\Lambda(\Delta, x) = \Delta$ and using (2.4) with $n = 4$, we may obtain

$$\sum_{(m;1;x)} x_1^1 \cdot x_2^2 \cdot x_3^3 \cdot x_4^4 = \binom{m+3}{7} + 22\binom{m+4}{9} + 144\binom{m+5}{11} + 288\binom{m+6}{13}. \tag{2.14}$$

Surely, (2.13)–(2.14) are the shortest formulas for the sums in question. Obviously (2.13)–(2.14) involve the following asymptotic estimates

$$\sum_{(m;1;x)} x_1^2 \cdot x_2^3 \cdot x_3^4 = 288\binom{m+5}{11}(1 + O(\frac{1}{m^2})), \quad m \to \infty,$$

$$\sum_{(m;1;x)} x_1^1 \cdot x_2^2 \cdot x_3^3 \cdot x_4^4 = 288\binom{m+6}{13}(1 + O(\frac{1}{m^2})), \quad m \to \infty.$$

From the above example we have

$$(\Lambda(\Delta, x^2))^n = \Delta^n(1 + 2\Delta)^n,$$

$$(\Lambda(\Delta, x^3))^n = \Delta^n(1+6\Delta E)^n,$$
$$(\Lambda(\Delta, x^4))^n = \Delta^n(1+2\Delta)^n(1+12\Delta E)^n.$$

Accordingly, as particular consequences of (2.7) of Corollary 2.3, we may state the following

Example 2.6 There are 3 formulas as follows

$$\sum_{(m;1;x)} (x_1 \cdot x_2 \cdots x_n)^2 = \sum_{\nu=0}^{n} 2^\nu \binom{n}{\nu}\binom{m+n-1}{2n+\nu-1}, \tag{2.15}$$

$$\sum_{(m;1;x)} (x_1 \cdot x_2 \cdots x_n)^3 = \sum_{\nu=0}^{n} 6^\nu \binom{n}{\nu}\binom{m+n+\nu-1}{2n+2\nu-1}, \tag{2.16}$$

$$\sum_{(m;1;x)} (x_1 \cdot x_2 \cdots x_n)^4 = \sum_{\nu=0}^{n}\sum_{\mu=0}^{n} 2^\nu \cdot 12^\mu \binom{n}{\nu}\binom{n}{\mu}\binom{m+n+\mu-1}{m-n-\nu-\mu}. \tag{2.17}$$

Certainly these formulas are especially useful when m is much bigger than n. Note that (2.15)–(2.16) have appeared previously [7,8] and that (2.17) could be replaced by a formula of similar nature via the operator $\Lambda(\Delta, x^4) = \Delta^n(\Delta+E)^n(1+12\Delta E)^n$. However it appears to be impossible to get a simpler formula consisting of $(n+1)$ terms for the sum of (2.17).

3. Some remarks on related summation formulae

What is worth commenting is that summation formulas using operators (such as (2.4), (2.7), (2.10) etc.) should be the most available formulas for the practical computation of polynomial convolutions.

Remark 3.1 Usually, a good summation formula is such a formula that consists of a least number of easily computed terms. From this view-point, one may find that (2.13) and (2.14) could be regarded as good formulas, each consisting of only 4 terms. As a matter of fact, if the summation formula (2.10) is replaced by the equivalent formula

$$\sum_{(m;1;x)} x_1^{p_1} \cdots x_n^{p_n} = \sum_{\substack{1\le\nu_i\le p_i \\ 1\le i\le n}} \nu_1! \cdots \nu_n! \left\{ {p_1 \atop \nu_1} \right\} \cdots \left\{ {p_n \atop \nu_n} \right\} \binom{m+n-1}{m-\nu_1-\cdots-\nu_n}$$

without using Δ-operators, and if it is applied to the sums of (2.13)–(2.14), one will get particular formulas, each consisting of $2\times 3\times 4 = 24$ terms (since $p_1 = 1, p_2 = 2, p_3 = 3, p_4 = 4$). Thus, (2.13) and (2.14) just provide examples showing that operator summation formulas such as (2.4) etc., may sometimes lead to much more brief formulas. Here the real reason is that the products of $\Lambda(\Delta, f_1)\cdots\Lambda(\Delta, f_n)$ or the like may sometimes be reduced to rather simple forms via algebraic manipulations.

Remark 3.2 It is known that there is a general formula for expressing a multifold convolution of arbitrary real-valued functions defined on the set of non-negative integers. For details, see a recent paper [9] by Hsu and Ma. As may be observed, the general formula for convolutions is expressed explicitly as a summation over a set of integer partitions, and the expression cannot be simplified any way for the particular case of polynomial convolutions [9, Theorem 5.2]. Thus, by comparing with the results of Theorem 2.2 and Corollary 2.3, one may see that (2.4)–(2.5) and (2.7)–(2.8) should be the most advantageous formulas for the computation of polynomial convolutions.

Remark 3.3 In order to disclose a natural relation between the basic formula (2.4) of Theorem 2.2 and a pair of formulas given by Lemmas 3 and 4 of the earliest paper [6], we have at first to mention that a mistaken denotation occurring in [6] should be corrected, namely, the set of n-compositions $(m;1;x)$ under the summations for expressing $S(m,[f]^n)$ and $S(m,[f_1]\cdots[f_n])$ in §2 of [6] should be replaced by $(m;0;x)$. Otherwise, the results of [6] can only be valid for polynomials $f(x), f_1(x),\dots,f_n(x)$ without constant terms.

Having correctly defined $S(m,[f_1]\cdots[f_n])$ as that of (2.1), one may verify that the following formulas as given by Lemmas 3 and 4 of [6] (viz. $(4)'$ and (6) of [6])

$$S(m,[f]^n) = n!\sum_{(n;0;p)}\binom{m+n-1}{1\cdot p_1+\cdots+k\cdot p_k+n-1}\frac{\beta_0^{p_0}\beta_1^{p_1}\cdots\beta_k^{p_k}}{p_0!p_1!\cdots p_k!} \tag{3.1}$$

$$S(m,[f_1]\cdots[f_n]) = \frac{1}{n!}\sum_{(\nu_1\cdots\nu_k)\in(1\cdots n)}(-1)^{n-k}S(m,[f_{\nu_1}+\cdots+f_{\nu_k}]^n) \tag{3.2}$$

are logically equivalent to (2.7) and (2.4), respectively, wherein $f(x)$ has the degree $\partial f=k$, β_i may be rewritten as $\Delta^i f(0)$ $(i=0,1,\dots,k)$, and the summation on the RHS of (3.1) is taken over all the $(k+1)$-compositions of n, viz. $p_0+\cdots+p_k=n$ with each $p_i\ge 0$; and the RHS summation of (3.2) is over all the different combinations (sub-sets) $\{\nu_1,\dots,\nu_k\}$ out of the set $\{1,\dots,n\}$ $(k=1,\dots,n)$.

Indeed, the RHS of (3.1) may be rewritten as

$$\begin{aligned}
&\sum_{(n;0;p)}\frac{n!}{p_0!p_1!\cdots p_k!}\binom{m+n-1}{m-(p_1+2p_2+\cdots+kp_k)}(f(0))^{p_0}(\Delta f(0))^{p_1}\cdots(\Delta^k f(0))^{p_k}\\
&=\sum_{(n;0;p)}\frac{n!}{p_0!\cdots p_k!}(f(0)\Delta^0)^{p_0}\cdots(\Delta^k f(k)\Delta^k)^{p_k}\binom{x}{m}_{x=m+n-1}\\
&=\Big(\sum_{\nu=0}^{k}\Delta^\nu f(0)\Delta^\nu\Big)^n\binom{x}{m}_{x=m+n-1} = \text{RHS of (2.7)}.
\end{aligned}$$

Also, for every given set $\{\nu_1,\dots,\nu_k\}\subset\{1,\dots,n\}$, it is obvious that the summand within the summation of (3.2) may be expressed in the following forms

$$\begin{aligned}
&(-1)^{n-k}S(m,[f_{\nu_1}+\cdots+f_{\nu_k}]^n)\\
&\quad=(-1)^{n-k}(\Lambda(\Delta,f_{\nu_1}+\cdots+f_{\nu_k}))^n\binom{x}{m}_{x=m+n-1}\\
&\quad=(-1)^{n-k}[\Lambda(\Delta,f_{\nu_1})+\cdots+\Lambda(\Delta,f_{\nu_k})]^n\binom{x}{m}_{x=m+n-1}\\
&\quad=(-1)^{n-k}\sum_{q_1+\cdots+q_t=n}\frac{n!}{q_1!\cdots q_t!}(\Lambda(\Delta,f_{\rho_1}))^{q_1}\cdots(\Lambda(\Delta,f_{\rho_t}))^{q_t}\binom{x}{m}_{x=m+n-1},
\end{aligned}$$

where the last summation is taken over all the subsets $\{\rho_1, \ldots, \rho_t\}$ of $\{\nu_1, \ldots, \nu_k\}$ $(t = 1, \ldots, k)$, and all the compositions $(n; 1; q)$. Certainly the general term (summand) of the last summation is contained in all such terms of the RHS summation of (3.2) that $\{\rho_1, \ldots, \rho_t\} \subset \{\nu_1, \ldots, \nu_k\} \subset \{1, \ldots, n\}$. The number of occurrences is obviously $\binom{n-t}{k-t}$. Thus the total number of occurrences in the RHS summation of (3.2) is given by

$$\sum_{k=t}^{n}(-1)^{n-k}\binom{n-t}{k-t} = (1-1)^{n-t} = \begin{cases} 0. & t < n, \\ 1, & t = n. \end{cases}$$

This means that the general term vanishes except that $t = n$, so that $k = n$, $\{\rho_1, \ldots, \rho_t\} \equiv \{\nu_1, \ldots, \nu_k\} \equiv \{1, \ldots, n\}$, and $q_1 = \cdots = q_n = 1$. Consequently we get

$$\text{RHS of (3.2)} = \Lambda(\Delta, f_1) \cdots \Lambda(\Delta, f_n)\binom{x}{m}_{x=m+n-1}.$$

Hence, as a conclusion we may say that Lemmas 3 and 4 of [6] are implicitly involving (2.4), and the deduction of (2.4) from (3.2) plus (3.1) may be regarded as a different proof for (2.4).

Remark 3.4 Observe that for $n \geq 3$ the summation on the LHS of (2.9) may be rewritten as

$$S(m, [x^{p_1}] \cdots [x^{p_n}]) = \sum x_1^{p_1} \cdots x_{n-1}^{p_{n-1}}(m - x_1 - \cdots - x_{n-1})^{p_n}$$

where the RHS summation extends over all the non-negative integers $x_1, \ldots, x_{n-1}$ such that $x_1 + \cdots + x_{n-1} \leq m$. Apparently, such a sum may be viewed as a discrete analogue of the Dirichlet multiple integral

$$\int \cdots \int_S t_1^{\alpha_1} \cdots t_{n-1}^{\alpha_{n-1}}(1 - t_1 - \cdots - t_{n-1})^{\alpha_n} \mathrm{d}t_1 \cdots \mathrm{d}t_{n-1} = \frac{\Gamma(\alpha_1 + 1) \cdots \Gamma(\alpha_n + 1)}{\Gamma(\alpha_1 + \cdots + \alpha_n + n)}$$

where the domain of integration is defined by the $(n-1)$-dimensional set

$$S : t_1 \geq 0, \ldots, t_{n-1} \geq 0, \ t_1 + \cdots + t_{n-1} \leq 1,$$

and α_i $(i = 1, \ldots, n-1)$ are real numbers such that $\alpha_i + 1 > 0$. Certainly, the formula (2.9) is much more complicated than the integration formula displayed above. However, it should be possible to verify that (2.9) implies the following limit

$$\lim_{m \to \infty} S(m, [x^{p_1}] \cdots [x^{p_n}])/m^{p_1 + \cdots + p_n + n - 1} = \frac{p_1! \cdots p_n!}{(p_1 + \cdots + p_n + n - 1)!}$$

which is consistent with the integral formula when taking $\alpha_i = p_i$.

4. Some special convolved polynomial sums

Here we will present several new examples showing how to make use of the summation formulas (2.4) and (2.7) to evaluate some convolution sums that consist of certain classical polynomials. Evidently, in order to get explicit results for $S(m, [f]^n)$ and $S(m, [f_1] \cdots [f_n])$ by using (2.7) and (2.4), it requires firstly to find explicit expressions for $\Delta^\nu f(0)$ and $\Delta^\nu f_i(0)$ $(\nu = 1, 2, \ldots; i = 1, \ldots, n)$. In particular, related computations could be greatly shortend, in cases $f(x)$ and $f_i(x)$ are known to have explicit expressions in Newton interpolation series.

Example 4.1 We wish to evaluate the convolution

$$S(m,[B_k]^n)=\sum_{(m;0;x)} B_k(x_1)\cdots B_k(x_n),\quad k\ge 1$$

where $B_k(x)$ is the k-th degree Bernoulli polynomial defined by the expansion

$$\frac{te^{xt}}{e^t-1}=\sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!},\quad |t|<2\pi \tag{4.1}$$

$B_k(0)=B_k$ $(k=0,1,2,\ldots)$, are known as Bernoulli numbers.

It is known that $B_k(x)$ can be expanded in terms of Newton's polynomials with Stirling numbers as coefficient [10, §78], namely

$$B_k(x)=B_k+k\sum_{\nu=1}^{k}(\nu-1)!\left\{ {k-1 \atop \nu-1} \right\}\binom{x}{\nu}. \tag{4.2}$$

This implies that $\Delta^\nu B_k(0)=k\cdot(\nu-1)!\left\{ {k-1 \atop \nu-1} \right\}$. Consequently, using (2.7), we get

$$S(m,[B_k]^n)=\Big(B_k+k\sum_{\nu=1}^{k}(\nu-1)!\left\{ {k-1 \atop \nu-1} \right\}\Delta^\nu\Big)^n\binom{x}{m}_{x=m+n-1}. \tag{4.3}$$

Certainly, this is a useful formula when m is much bigger than k and n. Also, an application of (2.4) with $n=2$ yields the formula

$$\sum_{(m;0;x)} B_p(x_1)B_q(x_2)=\Big(B_p+p\sum_{\nu=1}^{p}(\nu-1)!\left\{ {p-1 \atop \nu-1} \right\}\Delta^\nu\Big)$$

$$\Big(B_q+q\sum_{\mu=1}^{q}(\mu-1)!\left\{ {q-1 \atop \mu-1} \right\}\Delta^\mu\Big)\binom{x}{m}_{x=m+1} \tag{4.4}$$

where p and q are given positive integers.

Example 4.2 It is known that the Bernoulli polynomial of the second kind of degree k may be written as [10, §89]

$$\Psi_k(x)=\int_0^1\binom{x+t}{k}\mathrm{d}t. \tag{4.5}$$

Accordingly, $b_k=\Psi_k(0)$ may be called Bernoulli numbers of the second kind, viz.

$$b_k=\int_0^1\binom{t}{k}\mathrm{d}t,\quad k=0,1,2,\ldots \tag{4.6}$$

where $b_0=1$, $b_1=1/2$, $b_2=-1/12$, etc. A table of b_k's for $k\le 10$ may be found in [10, §89]. Note that $\Delta^\nu f_k(0)=\Psi_{k-\nu}(0)=B_{k-\nu}$, $0\le\nu\le k$, so that $\Lambda(\Delta,\Psi_k)=\sum_0^k b_{k-\nu}\Delta^\nu=\sum_0^k b_\nu\Delta^{k-\nu}$. Consequently, (2.7) and (2.4) imply the following special formulas

$$S(m,[\Psi_k]^n)=\Big(\sum_{\nu=0}^{k} b_\nu\Delta^{k-\nu}\Big)^n\binom{x}{m}_{x=m+n-1} \tag{4.7}$$

and

$$\sum_{(m;0;x)} \Psi_p(x_1)\Psi_q(x_2) = \Big(\sum_{\nu=0}^{p} b_\nu \Delta^{p-\nu}\Big)\Big(\sum_{\mu=0}^{q} b_\mu \Delta^{q-\mu}\Big)\binom{x}{m}_{x=m+1}. \tag{4.8}$$

Example 4.3 As is known, Boole's polynomial of degree k may be expressed in the form [10, §113]

$$\xi_k(x) = \sum_{\nu=0}^{k} (-1/2)^{k-\nu}\binom{x}{\nu}. \tag{4.9}$$

This implies that $\Delta^\nu \xi_k(0) = (-1/2)^{k-\nu}$. Consequently (2.7) and (2.4) with $n = 2$ yield the following special formulas

$$S(m, [\xi_k]^n) = \Big(\sum_{\nu=0}^{k} (-1/2)^\nu \Delta^{k-\nu}\Big)^n \binom{x}{m}_{x=m+n-1} \tag{4.10}$$

and

$$\sum_{(m;0;x)} \xi_p(x_1)\xi_q(x_2) = \Big(\sum_{\nu=0}^{p} (-1/2)^\nu \Delta^{p-\nu}\Big)\Big(\sum_{\mu=0}^{q} (-1/2)^\mu \Delta^{q-\mu}\Big)\binom{x}{m}_{x=m+1}. \tag{4.11}$$

Example 4.4 Let us consider the Mittag-Leffler polynomials defined by the power-type generating function [1]

$$(\frac{1+t}{1-t})^x = \Big(1 + \sum_{n=1}^{\infty} 2t^n\Big)^x = \sum_{k=0}^{\infty} (ML)_k(x) \cdot t^k \tag{4.12}$$

where $(ML)_0(x) \equiv 1$ and $(ML)_k(x)$ is of degree k.

Instead of $(m; 0; x)$, we shall use the set $(m; 0; \nu)$ consisting of all the non-negative integer solutions of the equation $\nu_1 + \cdots + \nu_n = m$. It is easily seen that for any given set of real numbers $\{x_1, \ldots, x_n\}$, there holds the rather simple convolution sum

$$\sum_{(m;0;\nu)} (ML)_{\nu_1}(x_1) \cdots (ML)_{\nu_n}(x_n) = (ML)_m(x_1 + \cdots + x_n). \tag{4.13}$$

Actually this follows from the expansion of $((1+t)/(1-t))^{x_1+\cdots+x_n}$ in terms of t^m, and may be called convolution "in degrees".

On the other hand, the summation (with fixed $k \geq 1$)

$$S(m, [(ML)_k]^n) = \sum_{(m;0;x)} (ML)_k(x_1) \cdots (ML)_k(x_n) \tag{4.14}$$

should be properly called the convolution "in arguments". Let us now evaluate (4.14) and the following sum

$$S(m, [(ML)_p][(ML)_q]) = \sum_{(m;0;x)} (ML)_p(x_1) \cdot [(ML)_q](x_2) \tag{4.15}$$

by means of (2.7) and (2.4) with $n = 2$. First, using the extracting-coefficient operator $[t^k]$, we find

$$(ML)_k(x) = [t^k](\frac{1+t}{1-t})^x = [t^k](1 + \frac{2t}{1-t})^x = \sum_{\nu\geq 0} \binom{x}{\nu} [t^k](\frac{2t}{1-t})^\nu$$

$$= \sum_{\nu\ge 0} 2^\nu \binom{x}{\nu} [t^{k-\nu}] \sum_{j\ge 0} \binom{\nu+j-1}{j} t^j = \sum_{\nu=1}^{k} 2^\nu \binom{k-1}{\nu-1}\binom{x}{\nu}.$$

Consequently we have $\Delta^\nu (ML)_k(0) = 2^\nu \binom{k-1}{\nu-1}$ and we get

$$\text{RHS of (4.14)} = \Big(\sum_{\nu=1}^{k} 2^\nu \binom{k-1}{\nu-1}\Delta^\nu\Big)^n \binom{x}{m}_{x=m+n-1} \tag{4.16}$$

and

$$\text{RHS of (4.15)} = \Big(\sum_{\nu=1}^{p} 2^\nu \binom{p-1}{\nu-1}\Delta^\nu\Big)\Big(\sum_{\mu=1}^{q} 2^\mu \binom{q-1}{\mu-1}\Delta^\mu\Big)\binom{x}{m}_{x=m+1}. \tag{4.17}$$

5. Convolution polynomials and two types of summations

Generally, a sequence $\{f_k(x)\}_0^\infty$ of polynomials with $f_0(x) \equiv 1$ and $\partial f_k(x) = k$ $(k = 0, 1, 2, \ldots)$, is called a convolution polynomial sequence, if there holds the convolution identity

$$\sum_{k=0}^{n} f_k(x) f_{n-k}(y) = f_n(x+y), \quad n = 0, 1, 2, \ldots.$$

Of course, this identity implies the multifold convolution in degrees

$$\sum_{(m;0;\nu)} f_{\nu_1}(x_1)\cdots f_{\nu_n}(x_n) = f_m(x_1+\cdots+x_n). \tag{5.1}$$

Apparently, the sequence $\{(ML)_k(x)\}$ gives a special example.

It is known that there are various noticeable properties enjoyed by convolution polynomials. For details the reader is referred to D. E. Knuth's fundamental paper "convolution polynomials" appearing in Math. J., 24(1992), 67–78. In what follows we will show that, for any given convolution polynomial sequence, there exist summation formulas for multifold convolutions in arguments.

Note that convolution polynomials can always be generated by power-type generating functions. Let $\phi(t) = 1 + a_1 t + a_2 t^2 + \cdots$ be a formal power series over the real or complex number field. Then the formal series expansion

$$(\phi(t))^x = \sum_{k=0}^{\infty} f_k(x) t^k \overset{\text{def}}{=} \sum_{k=0}^{\infty} \begin{bmatrix} x \\ k \end{bmatrix}_\phi \cdot t^k \tag{5.2}$$

yields the convolution polynomials $f_k(x) = \begin{bmatrix} x \\ k \end{bmatrix}_\phi$ $(k = 0, 1, 2, \ldots)$. Here we adopt the notation $\begin{bmatrix} x \\ k \end{bmatrix}_\phi$ just for expressiveness. Thus for instances we have the special convolution polynomials:

$$\begin{bmatrix} x \\ k \end{bmatrix}_{1+t} = \binom{x}{k}, \quad \begin{bmatrix} x \\ k \end{bmatrix}_{(1-t)^{-1}} = \binom{x+k-1}{k}, \quad \begin{bmatrix} x \\ k \end{bmatrix}_{e^t} = \frac{x^k}{k!},$$

$$\begin{bmatrix} x \\ k \end{bmatrix}_{(1+t)/(1-t)} = (ML)_k(x), \quad \begin{bmatrix} x \\ k \end{bmatrix}_{\exp(e^t-1)} = T_k(x),$$

where $T_k(x)$ are known as Touchard's polynomials.

Certainly (5.1) may be rewritten in the form

$$\sum_{(m;0;\nu)} \begin{bmatrix} x_1 \\ \nu_1 \end{bmatrix}_\phi \cdots \begin{bmatrix} x_n \\ \nu_n \end{bmatrix}_\phi = \begin{bmatrix} x_1 + \cdots + x_n \\ m \end{bmatrix}_\phi . \tag{5.1*}$$

For given non-negative integers $k_1, \ldots, k_n$, we want to evaluate the multifold convolution in arguments:

$$S\Big(m, \begin{bmatrix} x \\ k_1 \end{bmatrix}_\phi \cdots \begin{bmatrix} x \\ k_n \end{bmatrix}_\phi\Big) = \sum_{(m;0;x)} \begin{bmatrix} x_1 \\ k_1 \end{bmatrix}_\phi \cdots \begin{bmatrix} x_n \\ k_n \end{bmatrix}_\phi .$$

Denote $\phi_1(t) = \phi(t) - 1 = \sum_{i\ge 1} a_i t^i$ with $a_1 \ne 0$. Then we have

$$\begin{aligned} \begin{bmatrix} x \\ k \end{bmatrix}_\phi &= [t^k](\phi(t_1))^x = [t^k](1+\phi_1(t))^x \\ &= \sum_{j=0}^{k} \binom{x}{j} [t^k](\phi_1(t))^j \overset{\text{def}}{=} \sum_{j=0}^{k} s(k,j,\phi_1) \binom{x}{j}. \end{aligned} \tag{5.3}$$

Here the numbers $s(k,j,\phi_1)$ defined by

$$s(k,j,\phi_1) = [t^k](\phi_1(t))^j, \quad 0 \le j \le k \tag{5.4}$$

form a simple special Riordan matrix whose elements may be called modified Stirling-type numbers, since $(k!/j!)s(k,j,\phi_1)$ just give the two kinds of ordinary Stirling numbers by taking $\phi_1(t) = \log(1+t)$ and $\phi_1(t) = e^t - 1$, viz.

$$\frac{k!}{j!} s(k,j,\log(1+t)) = S_1(k,j), \quad \frac{k!}{j!} s(k,j,e^t-1) = S_2(k,j) = \left\{ \begin{matrix} k \\ j \end{matrix} \right\}.$$

We may now state the following

Theorem 5.1 *There holds a summation formula for the convolution in arguments of the form*

$$S\Big(m, \begin{bmatrix} x \\ k_1 \end{bmatrix}_\phi \cdots \begin{bmatrix} x \\ k_n \end{bmatrix}_\phi\Big) = \Big(\prod_{i=1}^{n} \Big(\sum_{j=0}^{k_i} s(k_i,j,\phi_1)\Delta^j\Big)\Big) \binom{x}{m}_{x=m+n-1} \tag{5.5}$$

where $s(k,j,\phi_1)$ are given by (5.4).

Proof From (5.3) we see that

$$\Big(\Delta^j \begin{bmatrix} x \\ k \end{bmatrix}_\phi \Big)_{x=0} = s(k,j,\phi_1).$$

Thus (5.5) follows from (2.4) as a consequence. □

Example 5.2 As is known, Touchard's polynomials are given by the generating function [1]

$$e^{x(e^t-1)} = \sum_{k=0}^{\infty} T_k(x) t^k. \tag{5.6}$$

Moreover, $T_k(x)$ has an explicit expression

$$T_k(x) = \frac{1}{k!}\sum_{j=0}^{k}\left\{ {k \atop j} \right\} x^j. \tag{5.7}$$

Accordingly, we get the values of differences at zero

$$\Delta^\nu T_k(0) = (\Delta^\nu T_k(x))_{x=0} = \frac{\nu!}{k!}\sum_{j=\nu}^{k}\left\{ {k \atop j} \right\}\left\{ {j \atop \nu} \right\}. \tag{5.8}$$

Let us denote $\delta(k,\nu) := \Delta^\nu T_k(0)$. Then, using (5.5) we obtain

$$S(m,[T_{k_1}]\cdots[T_{k_n}]) = \Big(\prod_{i=1}^{n}\Big(\sum_{\nu=0}^{k_i}\delta(k_i,\nu)\Delta^\nu\Big)\Big)\binom{x}{m}_{x=m+n-1}. \tag{5.9}$$

In particular we have

$$S(m,[T_k]^n) = \Big(\sum_{\nu=0}^{k}\delta(k,\nu)\Delta^\nu\Big)^n\binom{x}{m}_{x=m+n-1}. \tag{5.10}$$

Certainly, (5.9) and (5.10) could be used to get exact numerical results whenever $k_1,\ldots,k_n$, k and m are given concretely.

Remark 5.3 Convolutions of polynomials "in degrees" and "in arguments" may be called two types of convolutions. Note that convolution in degrees can only be obtained from convolution polynomials (by definition). Thus one may infer from Theorem 5.1 that only the class of convolution polynomials could lead to the two types of convolutions which are both computable with really available summation formulas. Also, one may guess that both (2.4) and (5.5) could be extended to the cases of q-polynomials.

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